Fundamental Theorem of Calculus for Lebesgue Integration

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The existing proofs of the Fundamental theorem of calculus for Lebesgue integration typically rely either on the Vitali–Carathéodory theorem on approximation of Lebesgue integrable functions by semi-continuous functions (as in [3, 9, 12]), or on the theorem characterizing increasing functions in terms of the four Dini derivates (as in [6, 10]). Alternatively, the theorem is derived using the Perron or the Kurzweil–Henstock integral and its relation to the Lebesgue integral (see [5, 8]).

In this note we give a proof of the theorem which uses only standard results of the Lebesgue measure and integration without resorting to any extraneous material. Two of these results, the theorem that an absolutely continuous function with derivative equal to zero almost everywhere is constant, and Lebesgue's theorem on differentiation of monotonic functions, have received an elegant treatment by elementary means in this Monthly in the hands of Michael Botsko [1, 2].

To simplify formulations we employ the following often used terminology. A statement is true *nearly everywhere* in $S \subset \mathbb{R}$ if it is true in S except for a countable subset of S. The idea for the proof of the following key lemma comes from [7].

Lemma 1. *Let $F : [a, b] \to \mathbb{C}$ be continuous on $[a, b]$, let $f : [a, b] \to \mathbb{C}$ be Lebesgue integrable on $[a, b]$, and let $F'(t) = f(t)$ nearly everywhere in $[a, b]$. Then*

$$|F(b) - F(a)| \le \int_a^b |f(t)|\, dt. \tag{1}$$

Proof. Let $F'(t) = f(t)$ for all $t \in A = [a, b] \setminus D$, where D is countable. We may assume that the (one sided) derivatives exist at the end points of $[a, b]$, otherwise we consider intervals $[a_n, b_n]$ with this property, and satisfying $[a_n, b_n] \nearrow [a, b]$.

Let $\varepsilon > 0$ be given. Set $c_i = i\varepsilon/(b - a)$, $i = 0, 1, 2, \dots$, and define

$$E_i = \{t \in A : c_{i-1} \le |f(t)| < c_i\},\ i \in \mathbb{N}.$$

Since f and $|f|$ are Lebesgue integrable on $[a, b]$, the sets E_i are Lebesgue measurable, and A is the disjoint union of the E_i. Hence the Lebesgue measure of A is $m(A) = b - a = \sum_{i=1}^{\infty} m(E_i)$, and

$$c_{i-1} m(E_i) \le \int_{E_i} |f(t)|\, dt \le c_i m(E_i), \quad i \in \mathbb{N},$$

which gives

$$0 \le c_i m(E_i) - \int_{E_i} |f(t)|\, dt \le \frac{\varepsilon}{b-a}\, m(E_i).$$

From the countable additivity of the Lebesgue integral we conclude that

$$\sum_{i=1}^{\infty} c_i m(E_i) \le \int_a^b |f(t)|\, dt + \varepsilon. \tag{2}$$

For each $i \in \mathbb{N}$ there exists a bounded open set $G_i \subset \mathbb{R}$ such that

$$G_i \supset E_i \text{ and } m(G_i) \le m(E_i) + c_i^{-1}(\tfrac{1}{2})^i \varepsilon, \quad i \in \mathbb{N}. \tag{3}$$

Define functions $H, M : [a,b] \to \mathbb{R}$ by $H(a) = M(a) = 0$ and

$$H(t) = \sum_{i=1}^{\infty} c_i m(G_i \cap [a,t]), \quad M(t) = \sum_{u_j \in [a,t)} (\tfrac{1}{2})^j \varepsilon, \quad a < t \le b, \tag{4}$$

where $\{u_j : j \in \mathbb{N}\}$ is an enumeration of D. Both H and M are increasing.

Let x be the supremum of all $t \in [a,b]$ such that $|F(t) - F(a)| \le H(t) + M(t)$. For a proof by contradiction assume that $x < b$. Suppose first that $x \in A$. Then $x \in E_k$ for some $k \in \mathbb{N}$, and from $|f(x)| < c_k$ it follows that there exists $x_1 \in (x,b)$ such that

$$[x, x_1] \subset G_k \text{ and } |F(x_1) - F(x)| < c_k(x_1 - x).$$

Since F is continuous, $|F(x) - F(a)| \le H(x) + M(x)$. Then

$$|F(x_1) - F(a)| \le |F(x_1) - F(x)| + |F(x) - F(a)| \le c_k(x_1 - x) + H(x) + M(x),$$

while $H(x) + c_k(x_1 - x) \le H(x_1)$. Then $|F(x_1) - F(a)| \le H(x_1) + M(x_1)$, which contradicts the definition of x.

Suppose that $x \in D$. Then $x = u_m$ for some $m \in \mathbb{N}$. Since F is continuous, there exits $x_2 \in (x,b)$ such that $|F(x_2) - F(x)| < (\frac{1}{2})^m \varepsilon$, and

$$|F(x_2) - F(a)| \le |F(x_2) - F(x)| + |F(x) - F(a)| \le (\tfrac{1}{2})^m \varepsilon + H(x) + M(x);$$

since $M(x) + (\frac{1}{2})^m \varepsilon \le M(x_2)$, we have $|F(x_2) - F(a)| \le H(x_2) + M(x_2)$, which again contradicts the definition of x. This proves that $x = b$. Hence, by (2), (3) and (4),

$$|F(b) - F(a)| \le H(b) + M(b) \le \int_a^b |f(t)|\, dt + 3\varepsilon.$$

Since ε was arbitrary, (1) holds. □

Theorem 1. (Fundamental theorem of calculus.) *Let* $F : [a, b] \to \mathbb{C}$ *be continuous on* $[a, b]$*, let* $f : [a, b] \to \mathbb{C}$ *be Lebesgue integrable on* $[a, b]$*, and let* $F'(t) = f(t)$ *nearly everywhere in* $[a, b]$*. Then* F *is absolutely continuous on* $[a, b]$*, and*

$$\int_a^b f(t)\,dt = F(b) - F(a). \tag{5}$$

Proof. By Lemma 1, $|F(v) - F(u)| \le \int_u^v |f(t)|\,dt$ for any subinterval $[u, v]$ of $[a, b]$. Since the Lebesgue integral is absolutely continuous, so is F, and (5) holds by Lebesgue's theorem on integration of derivatives of absolutely continuous functions. □

Let $-\infty \le a < b \le \infty$. We say that a complex valued function f is *Newton integrable* on (a, b) if there exists a complex valued function F (a *generalized primitive of* f) continuous on (a, b) such the $F'(t) = f(t)$ nearly everywhere in (a, b), and such that the one sided limits $F(a+)$, $F(b-)$ exist. The function f is *absolutely Newton integrable* on (a, b) if both f and $|f|$ are Newton integrable on (a, b). The complex number $F(b-) - F(a+)$ is the *Newton integral* of f on (a, b), written

$$(\mathcal{N})\int_a^b f = F(b-) - F(a+).$$

The definition of the Newton integral is independent of the choice of a generalized primitive: This is guaranteed by the following well known result (see, for instance, (8.5.1) in [4]), from which it follows that two generalized primitives to f differ by a constant.

Lemma 2. *Let* $-\infty \le a < b \le \infty$*, let* $F : (a, b) \to \mathbb{R}$ *be continuous, and let* $F'(t) \ge 0$ *nearly everywhere on* (a, b)*. Then* F *is increasing on* (a, b)*.*

An elementary proof of this lemma in the spirit of Thomson [11] and Botsko [1] can be based on properties of full covers of $[a, b]$.

Theorem 2. *Let* $-\infty \le a < b \le \infty$*. If* $f : (a, b) \to \mathbb{C}$ *is both Newton and Lebesgue integrable on* (a, b)*, then*

$$\int_a^b f(t)\,dt = (\mathcal{N})\int_a^b f. \tag{6}$$

Proof. Choose a sequence $[a_n, b_n]$ of subintervals of (a, b) such that $[a_n, b_n] \nearrow (a, b)$, and set $f_n = f\chi_{[a_n, b_n]}$, where $\chi_{[a_n, b_n]}$ is the characteristic function of $[a_n, b_n]$. Then $f_n \to f$ pointwise on (a, b), and $|f_n| \le |f|$ for all $n \in \mathbb{N}$. By Lebesgue's dominated convergence theorem and by Theorem 1,

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \int_{a_n}^{b_n} f(t)\,dt = \lim_{n\to\infty} (F(b_n) - F(a_n)) = F(b-) - F(a+). \qquad \square$$

We show that an absolutely Newton integrable complex valued function is also Lebesgue integrable, and the two integrals are consistent.

Theorem 3. *Let $-\infty \le a < b \le \infty$ and let $f : (a,b) \to \mathbb{C}$ be absolutely Newton integrable on (a,b). Then f is Lebesgue integrable, and*

$$\int_a^b f(t)\,dt = (\mathcal{N})\int_a^b f. \tag{7}$$

Proof. Assume first that f is nonnegative. By Lemma 2, a generalized primitive F to f is an increasing function on (a,b) and so, by Lebesgue's theorem on differentiation of monotonic functions, f is Lebesgue integrable on any compact subinterval of (a,b). Choose a sequence $[a_n,b_n]$ of subintervals of (a,b) such that $[a_n,b_n] \nearrow (a,b)$. By Theorem 1,

$$0 \le \int_{a_n}^{b_n} f(t)\,dt = F(b_n) - F(a_n) \le F(b-) - F(a+).$$

Writing $f_n = f\chi_{[a_n,b_n]}$, we have $f_n \nearrow f$, and the monotonic convergence theorem ensures that f is Lebesgue integrable on (a,b).

For the general case we observe that the complex valued function f is Lebesgue measurable as it is the limit of continuous functions

$$F_n(t) = n(F(t+\tfrac{1}{n}) - F(t))$$

convergent nearly (and therefore almost) everywhere in (a,b). By the first part of the proof, $|f|$ is Lebesgue integrable on (a,b). Then so is f, and Theorem 2 applies to complete the proof. □

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