Integr. equ. oper. theory 99 (9999), 1–5
0378-620X/99000-0, DOI 10.1007/s00020-003-0000

Integral Equations and Operator Theory

Stability theorems for linear combinations of idempotents

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Abstract. We prove a stability theorem for the nullity of a linear combination $c_1P_1 + c_2P_2$ of two idempotent operators P_1, P_2 on a Banach space provided c_1, c_2 and c_1+c_2 are nonzero. We then show that for $c_1P_1+c_2P_2$ the property of being upper semi-Fredholm, lower semi-Fredholm and Fredholm, respectively, is independent of the choice of c_1, c_2, and that the nullity, defect and index of $c_1P_1 + c_2P_2$ are stable.

Mathematics Subject Classification (2000). 47A53, 47B99, 46H99, 15A99.

Keywords. linear combinations of idempotents, nullity, stability theorems.

1. Introduction and preliminaries

In [5] we studied the nonsingularity of the difference and sum of two idempotent matrices. Baksalary and Baksalary [1] then proved that, for idempotent matrices P_1, P_2, the nonsingularity of $P_1 + P_2$ is equivalent to the nonsingularity of any linear combination $c_1P_1 + c_2P_2$, where $c_1, c_2 \neq 0$ and $c_1 + c_2 \neq 0$. Recently, Du et al. [3] gave a rather complicated proof of this result for two idempotent operators on a Hilbert space. In [7] we extended the Baksalary and Baksalary result [1] by proving the stability of the nullity and rank of $c_1P_1 + c_2P_2$ under the choice of c_1 and c_2, and posed the following question motivated by results of [1] and [6]:

If P_1, P_2 are idempotent operators in a Hilbert space, is it true that $P_1 + P_2$ is Fredholm if and only if any linear combinations $c_1P_1 + c_2P_2$ is Fredholm, where $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ and $c_1 + c_2 \neq 0$?

In this note we give an affirmative answer to this problem extended to Banach space operators using simple arguments based on the stability of the nullity of linear combinations of two idempotent operators. The main result of [3] then follows as a special case.

Let X be an infinite-dimensional complex Banach space and let $\mathcal{B}(X)$ be the set of all bounded linear operators on X. An operator $P \in \mathcal{B}(X)$ is *idempotent* if $P^2 = P$. Throughout this paper $\mathcal{N}(T)$ and $\mathcal{R}(T)$ will denote the nullspace and the range of $T \in \mathcal{B}(X)$, respectively. Set $\alpha(T) = \dim \mathcal{N}(T)$, the *nullity* of T, and $\beta(T) = \dim X/\mathcal{R}(T)$, the *defect* of T. An operator $T \in \mathcal{B}(X)$ is *semi-Fredholm* if $\mathcal{R}(T)$ is closed and at least one of $\alpha(T)$ and $\beta(T)$ is finite. For such an operator we define the *index* of T by $i(T) = \alpha(T) - \beta(T)$. Let $\Phi_+(X)$ ($\Phi_-(X)$) denote the set of *upper* (*lower*) *semi-Fredholm* operators, that is, the set of all semi-Fredholm operators with $\alpha(T) < \infty$ ($\beta(T) < \infty$). An operator $T \in \mathcal{B}(X)$ is *Fredholm* if $T \in \Phi(X) := \Phi_+(X) \cap \Phi_-(X)$.

If $T \in \mathcal{B}(X)$, we write $T' \in \mathcal{B}(X')$ for the adjoint of T. Recall that $\mathcal{R}(T)$ is closed if and only if $\mathcal{R}(T')$ is closed, and that in this case $\alpha(T') = \beta(T)$ and $\beta(T') = \alpha(T)$. Furthermore, $T \in \Phi_+(X)$ if and only if $T' \in \Phi_-(X')$, and $T \in \Phi_-(X)$ if and only if $T' \in \Phi_+(X')$ (see [4]).

Recall that Sadovskii [8] and later (independently) Buoni, Harte and Wickstead [2] introduced the following useful functorial construction known as the *essential enlargement* of a Banach space. For any Banach space X we set

$$\ell^\infty(X) = \{\boldsymbol{x} = (x_n) : x_n \in X, \quad \sup_n \|x_n\| < \infty\}.$$

Clearly, $\ell^\infty(X)$ is a Banach space equipped with the supremum norm, and

$$m(X) = \{\boldsymbol{x} = (x_n) \in \ell^\infty(X) : \{x_n : n \in \mathbb{N}\} \text{ is totally bounded in } X\}$$

is a closed subspace of $\ell^\infty(X)$. Hence the quotient space

$$\widetilde{X} = \ell^\infty(X)/m(X)$$

is a Banach space. Any $T \in \mathcal{B}(X)$ determines an operator $\widetilde{T} \in \mathcal{B}(\widetilde{X})$ defined by

$$\widetilde{T}((x_n) + m(X)) = (Tx_n) + m(X), \quad (x_n) \in \ell^\infty(X). \tag{1.1}$$

The mapping $T \mapsto \widetilde{T}$ is a continuous algebra homomorphism of $\mathcal{B}(X)$ to $\mathcal{B}(\widetilde{X})$.

The following result whose proof can be found in [2], [4] or [8], will play a crucial role in the proof of our main Theorem 3.1.

Theorem 1.1. *If $T \in \mathcal{B}(X)$, then T is upper semi-Fredholm if and only if $\widetilde{T}$ is injective.*

Let $\mathcal{A}$ be a Banach algebra. For any $a \in \mathcal{A}$ we define the left regular representation of a by

$$L_a(x) = ax, \qquad \text{for all } x \in \mathcal{A}.$$

Then $L_a \in \mathcal{B}(\mathcal{A})$, the mapping $a \mapsto L_a$ is an algebra monomorphism of $\mathcal{A}$ to $\mathcal{B}(\mathcal{A})$ with $\|L_a\| = \|a\|$.

We will need the following known fact whose proof we include for completeness.

Lemma 1.2. *Let $a \in \mathcal{A}$. Then a is invertible in $\mathcal{A}$ if and only if L_a is invertible in $\mathcal{B}(\mathcal{A})$.*

Proof. Let a be invertible with the inverse $b \in \mathcal{A}$. Then L_b is the inverse of L_a. Conversely, if L_a is invertible and $b = L_a^{-1}(1)$, then $ab = L_a L_a^{-1}(1) = 1$, and $L_a L_b = L_{ab} = L_1 = I$, that is, $L_b = L_a^{-1}$. Thus a is invertible in $\mathcal{A}$ with the inverse b. □

2. The nullity of $c_1P_1 + c_2P_2$

We start our observations with the following result which for matrices was proved in [5, Theorem 2.2]. For convenience, we define a subset Γ of $\mathbb{C}^2$ by

$$\Gamma = \{(c_1, c_2) \in \mathbb{C} : c_1 \neq 0, c_2 \neq 0, c_1 + c_2 \neq 0\}.$$

Theorem 2.1. *Let P_1, P_2 be two idempotents in $\mathcal{B}(X)$ and let $(c_1, c_2) \in \Gamma$. Then*

$$\dim [\mathcal{N}(c_1P_1 + c_2P_2)] = \dim [\mathcal{N}((I - P_1)P_2) \cap \mathcal{N}(P_1)]. \tag{2.1}$$

Proof. First we prove that (2.1) holds with $\leq$ in place of equality. For this suppose that $x \in \mathcal{N}(c_1P_1 + c_2P_2)$. Then

$$\begin{aligned}[(I - P_1)P_2](I - P_1)x &= c_1^{-1}[(I - P_1)P_2](c_1I - c_1P_1 - c_2P_2 + c_2P_2)x \\ &= c_1^{-1}[(I - P_1)P_2](c_1I + c_2P_2)x \\ &= (c_1 + c_2)(c_1c_2)^{-1}(I - P_1)(c_1P_1 + c_2P_2)x = 0.\end{aligned}$$

Thus, since $(I - P_1)x \in \mathcal{N}(P_1)$, we conclude that

$$(I - P_1)\mathcal{N}(c_1P_1 + c_2P_2) \subset \mathcal{N}((I - P_1)P_2) \cap \mathcal{N}(P_1). \tag{2.2}$$

Suppose that that $x \in \mathcal{N}(c_1P_1 + c_2P_2)$ and $(I - P_1)x = 0$. Then $x = P_1x$ and $(c_1+c_2)P_2x = P_2(c_1P_1x+c_2P_2x) = 0$. So $P_2x = 0$, and $x = P_1x = -c_1^{-1}c_2P_2x = 0$. Thus $I - P_1$ embeds $\mathcal{N}(c_1P_1 + c_2P_2)$ injectively into $\mathcal{N}((I - P_1)P_2) \cap \mathcal{N}(P_1)$, and the inequality $\leq$ in (2.1) is proved.

To complete the proof of the theorem we prove the reverse inequality in (2.1). Towards this end we set $c = c_1c_2^{-1}$ and prove that

$$((1 + c)I - P_2)[\mathcal{N}((I - P_1)P_2) \cap \mathcal{N}(P_1)] \subset \mathcal{N}(c_1P_1 + c_2P_2). \tag{2.3}$$

Suppose that $x \in \mathcal{N}((I - P_1)P_2) \cap \mathcal{N}(P_1)$. Then $P_1x = 0$ and $P_2x = P_1P_2x$. Thus

$$\begin{aligned}&(c_1P_1 + c_2P_2)\big((1 + c)I - P_2\big)x \\ &\quad = (c_1(1 + c)P_1 - c_1P_1P_2 + (c_1 + c_2)P_2 - c_2P_2)x \\ &\quad = -c_1P_1P_2x + (c_1 + c_2)P_2x - c_2P_2x = 0,\end{aligned}$$

and we obtain (2.3).

Let $x \in \mathcal{N}((I - P_1)P_2) \cap \mathcal{N}(P_1)$ and $((1 + c)I - P_2)x = 0$. Then $P_1x = 0$, $P_2x = P_1P_2x = (1 + c)x$. Thus $(1 + c)x = P_2x = P_1(1 + c)x = 0$, that is, $x = 0$. Hence $(1+c)I - P_2$ embeds $\mathcal{N}((I - P_1)P_2) \cap \mathcal{N}(P_1)$ injectively into $\mathcal{N}(c_1P_1 + c_2P_2)$, and (2.1) holds with $\geq$ in place of equality. This completes the proof. □

Corollary 2.2. *For any two idempotents $P_1, P_2 \in \mathcal{B}(X)$ the nullity of $c_1P_1 + c_2P_2$ is constant on Γ, that is,*

$$\alpha(c_1P_1 + c_2P_2) = \alpha(P_1 + P_2) \quad \textit{for all } (c_1, c_2) \in \Gamma.$$

3. Fredholm properties of $c_1P_1 + c_2P_2$

The main result of this note is the following stability theorem.

Theorem 3.1. *Let $P_1, P_2 \in \mathcal{B}(X)$ be idempotents. Then*:

(i) *If $c_1P_1+c_2P_2$ is upper semi-Fredholm for some $(c_1, c_2) \in \Gamma$, then it is upper semi-Fredholm for all $(c_1, c_2) \in \Gamma$, and $\alpha(c_1P_1 + c_2P_2)$ is constant on Γ.*

(ii) *If $c_1P_1+c_2P_2$ is lower semi-Fredholm for some $(c_1, c_2) \in \Gamma$, then it is lower semi-Fredholm for all $(c_1, c_2) \in \Gamma$, and $\beta(c_1P_1 + c_2P_2)$ is constant on Γ.*

(iii) *If $c_1P_1 + c_2P_2$ is Fredholm for some $(c_1, c_2) \in \Gamma$, then it is Fredholm for all $(c_1, c_2) \in \Gamma$, and $\alpha(c_1P_1 + c_2P_2), \beta(c_1P_1 + c_2P_2)$ and $i(c_1P_1 + c_2P_2)$ are constant on Γ.*

Proof. (i) Let $c_1P_1 + c_2P_2 \in \Phi_+(X)$ for some $(c_1, c_2) \in \Gamma$, and let $(\lambda_1, \lambda_2) \in \Gamma$. Under the algebra homomorphism $T \mapsto \widetilde{T}$ defined by (1.1), $(c_1P_1+c_2P_2)^{\sim} = c_1\widetilde{P}_1+ c_2\widetilde{P}_2$, and the operators $\widetilde{P}_1$ and $\widetilde{P}_2$ are idempotents in $\mathcal{B}(\widetilde{X})$. By Theorem 1.1, $\mathcal{N}(c_1\widetilde{P}_1 + c_2\widetilde{P}_2) = \{0\}$, and then by Corollary 2.2 (in the space $\widetilde{X}$), $\mathcal{N}(\lambda_1\widetilde{P}_1 + \lambda_2\widetilde{P}_2) = \{0\}$. Thus $\lambda_1P_1 + \lambda_2P_2$ is upper semi-Fredholm by Theorem 1.1. Finally, by Theorem 2.1, we have $\alpha(c_1P_1 + c_2P_2) = \alpha(\lambda_1P_1 + \lambda_2P_2)$, and (i) is proved.

(ii) Let $c_1P_1 + c_2P_2 \in \Phi_-(X)$ for some $(c_1, c_2) \in \Gamma$. This implies that $c_1P_1' + c_2P_2' \in \Phi_+(X)$ and $\beta(c_1P_1 + c_2P_2) = \alpha(c_1P_1' + c_2P_2')$. Further, P_1' and P_2' are idempotents in $\mathcal{B}(X')$. Thus (ii) follows from (i).

(iii) This follows from (i) and (ii). □

As a corollary to Theorem 3.1 we obtain the following result.

Corollary 3.2. *Let P_1 and P_2 be two idempotents in $\mathcal{B}(X)$. Then the invertibility of $c_1P_1 + c_2P_2$ is independent of the choice of $(c_1, c_2) \in \Gamma$.*

Proof. Let $c_1P_1 + c_2P_2$ be invertible for some choice of $(c_1, c_2) \in \Gamma$. Then $c_1P_1 + c_2P_2$ is Fredholm with the nullity and defect equal to zero. By Theorem 3.1 (iii), $\lambda_1P_1 + \lambda_2P_2$ is invertible for any choice of $(\lambda_1, \lambda_2) \in \Gamma$. □

Remark 3.3. Corollary 3.2 was recently proved for Hilbert space operators as the main result in [3, Theorem 1] by Du et al. In constrast with our arguments, their proof is applicable only in a Hilbert space, and is rather long and complicated.

Our final application is to idempotent elements in a Banach algebra.

Corollary 3.4. *Let p_1, p_2 be two idempotents in a Banach algebra $\mathcal{A}$. Then the invertibility of $c_1p_1 + c_2p_2$ is independent of the choice of $(c_1, c_2) \in \Gamma$.*

Proof. Suppose that the element $c_1p_1+c_2p_2$ is invertible for some pair $(c_1,c_2)\in\Gamma$. According to Lemma 1.2, the operator $c_1L_{p_1}+c_2L_{p_2}$ is invertible in $\mathcal{B}(\mathcal{A})$ with L_{p_1} and L_{p_2} idempotent. By Corollary 3.2, $\lambda_1L_{p_1}+\lambda_2L_{p_2}$ is invertible for any choice of $(\lambda_1,\lambda_2)\in\Gamma$. Then by Lemma 1.2, $\lambda_1p_1+\lambda_2p_2$ is invertible in $\mathcal{A}$. □

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