

# HILBERT–PACHPATTE TYPE MULTIDIMENSIONAL INTEGRAL INEQUALITIES

G. D. HANDLEY, J. J. KOLIHA, AND J. PEČARIĆ

ABSTRACT. In this paper we use a new approach to obtain a class of multivariable integral inequalities of Hilbert type from which we can recover as special cases integral inequalities obtained recently by Pachpatte and the present authors.

## 1. INTRODUCTION

The integral version of Hilbert’s inequality [7, Theorem 316] has been generalized in several directions (see [1, 3, 4, 7, 8, 9, 20, 21, 22]). Recently, inequalities similar to those of Hilbert were considered by Pachpatte in [12, 13, 14, 15, 16, 19]. The present authors in [5, 6] established a new class of related inequalities, which were further extended by Dragomir and Kim [2]. Two and higher dimensional variants were treated by Pachpatte in [17, 18]. In the present paper we use a new systematic approach to these inequalities based on Theorem 3.1, which serves as an abstract springboard to classes of concrete inequalities.

To motivate our investigation, we give a typical result of [17]. In this theorem,  $H(I \times J)$  denotes the class of functions  $u \in C^{(n-1, m-1)}(I \times J)$  such that  $D_1^i u(0, t) = 0$ ,  $0 \leq i \leq n-1$ ,  $t \in J$ ,  $D_2^j u(s, 0) = 0$ ,  $0 \leq j \leq m-1$ ,  $s \in I$ , and  $D_1^n D_2^{m-1} u(s, t)$  and  $D_1^{n-1} D_2^m u(s, t)$  are absolutely continuous on  $I \times J$ . Here  $I, J$  are intervals of the type  $I_\xi = [0, \xi)$  for some real  $\xi > 0$ .

**Theorem 1.1.** (Pachpatte [17, Theorem 1].) *Let  $u(s, t) \in H(I_x \times I_y)$  and  $v(k, r) \in H(I_z \times I_w)$ . Then, for  $0 \leq i \leq n-1$ ,  $0 \leq j \leq m-1$ , the following inequality holds:*

$$\begin{aligned} & \int_0^x \int_0^y \left( \int_0^z \int_0^w \frac{|D_1^i D_2^j u(s, t) D_1^i D_2^j v(k, r)|}{s^{2n-2i-1} t^{2m-2j-1} + k^{2n-2i-1} r^{2m-2j-1}} dk dr \right) ds dt \\ & \leq \frac{1}{2} [A_{i,j} B_{i,j}]^2 \sqrt{xyzw} \left( \int_0^x \int_0^y (x-s)(y-t) |D_1^n D_2^m u(s, t)|^2 ds dt \right)^{1/2} \\ & \quad \cdot \left( \int_0^z \int_0^w (z-k)(w-r) |D_1^n D_2^m v(k, r)|^2 dk dr \right)^{1/2}, \end{aligned}$$

where

$$A_{ij} = \frac{1}{(n-i-1)!(m-j-1)!}, \quad B_{ij} = \frac{1}{(2n-2i-1)(2m-2j-1)}.$$

The purpose of the present paper is to obtain a simultaneous generalization of Pachpatte’s multivariable results [17], and of the results [5, 6] of the present authors. The

---

1991 *Mathematics Subject Classification.* 26D15.

*Key words and phrases.* Hilbert’s inequality, Hilbert-Pachpatte integral inequalities, Hölder’s inequality.

single variable results [14, 15, 16, 19] follow as special cases of our theorems. Our treatment is based on Theorem 3.1, in particular on the abstract inequality (3.1), which yields a variety of special cases when the functions  $\Phi_i$  are specified.

## 2. NOTATION AND PRELIMINARIES

By  $\mathbb{Z}$  ( $\mathbb{Z}_+$ ) and  $\mathbb{R}$  ( $\mathbb{R}_+$ ) we denote the sets of all (nonnegative) integers and (nonnegative) real numbers. We will be working with functions of  $d$  variables, where  $d$  is a fixed positive integer, writing the variable as a vector  $s = (s^1, \dots, s^d) \in \mathbb{R}^d$ . A multiindex  $m$  is an element  $m = (m^1, \dots, m^d)$  of  $\mathbb{Z}_+^d$ . As usual, the factorial of a multiindex  $m$  is defined by  $m! = m^1! \cdots m^d!$ . An integer  $j$  may be regarded as the multiindex  $(j, \dots, j)$  depending on the context. For vectors in  $\mathbb{R}^d$  and multiindices we use the usual operations of vector addition and multiplication of vectors by scalars. We write  $s \leq \tau$  ( $s < \tau$ ) if  $s^j \leq \tau^j$  ( $s^j < \tau^j$ ) for  $1 \leq j \leq d$ . The same convention will apply to multiindices. In particular,  $s \geq 0$  ( $s > 0$ ) will mean  $s^j \geq 0$  ( $s^j > 0$ ) for  $1 \leq j \leq d$ .

If  $s = (s^1, \dots, s^d) \in \mathbb{R}^d$  and  $s > 0$ , we define the *cell*

$$Q(s) = [0, s^1] \times \cdots \times [0, s^j] \times \cdots \times [0, s^d];$$

replacing the factor  $[0, s^j]$  by  $\{0\}$  in this product, we get the *face*  $\partial_j Q(s)$  of  $Q(s)$ .

Let  $s = (s^1, \dots, s^d)$ ,  $\tau = (\tau^1, \dots, \tau^d) \in \mathbb{R}^d$ ,  $s, \tau > 0$ , let  $k = (k^1, \dots, k^d)$  be a multiindex and let  $u : Q(s) \rightarrow \mathbb{R}$ . Write  $D_j = (\partial/\partial s^j)$ . We use the following notation:

$$\begin{aligned} s^\tau &= (s^1)^{\tau^1} \cdots (s^d)^{\tau^d}, \\ D^k u(s) &= D_1^{k^1} \cdots D_d^{k^d} u(s), \\ \int_0^s u(\tau) d\tau &= \int_0^{s^1} \cdots \int_0^{s^d} u(\tau) d\tau^1 \cdots d\tau^d. \end{aligned}$$

An exponent  $\alpha \in \mathbb{R}$  in the expression  $s^\alpha$ , where  $s \in \mathbb{R}^d$ , will be regarded as a multiexponent, that is,  $s^\alpha = s^{(\alpha, \dots, \alpha)}$ .

Another positive integer  $n$  will be fixed throughout.

The following notation and hypotheses will be used throughout the paper.

$$\begin{array}{ll} I = \{1, \dots, n\} & n \in \mathbb{N} \\ m_i, i \in I & m_i = (m_i^1, \dots, m_i^d) \in \mathbb{Z}_+^d \\ x_i, i \in I & x_i = (x_i^1, \dots, x_i^d) \in \mathbb{R}^d, x_i > 0 \\ p_i, q_i, i \in I & p_i, q_i \in \mathbb{R}_+, 1/p_i + 1/q_i = 1 \\ p, q & 1/p = \sum_{i=1}^n (1/p_i), 1/q = \sum_{i=1}^n (1/q_i) \\ a_i, b_i, i \in I & a_i, b_i \in \mathbb{R}_+, a_i + b_i = 1 \\ w_i, i \in I & w_i \in \mathbb{R}, w_i > 0, \sum_{i=1}^n w_i = 1 \end{array}$$

Throughout the paper,  $u_i, v_i, \Phi$  will denote functions from  $[0, x_i]$  to  $\mathbb{R}$  of sufficient smoothness. If  $m$  is a multiindex and  $x \in \mathbb{R}^d$ ,  $x > 0$ , then  $C^m[0, x]$  will denote the set of all functions  $u : [0, x] \rightarrow \mathbb{R}$  which possess continuous derivatives  $D^k u$ , where  $0 \leq k \leq m$ .

The coefficients  $p_i, q_i$  are conjugate Hölder exponents used in applications of Hölder's inequality, and the coefficients  $a_i, b_i$  are used in exponents to factorize integrands. The coefficients  $w_i$  act as weights in applications of the geometric-arithmetic mean inequality; this enables us to pass from products to sums of terms.

## 3. THE MAIN RESULT

First we present a theorem that can be regarded as a template for concrete inequalities obtained by selecting suitable functions  $\Phi_i$  in (3.1). A special case of this theorem is given in [6, Theorem 3.1].

**Theorem 3.1.** *Let  $v_i, \Phi_i \in C(Q(x_i))$  and let  $c_i$  be multiindices for  $i \in I$ . If*

$$(3.1) \quad |v_i(s_i)| \leq \int_0^{s_i} (s_i - \tau_i)^{c_i} \Phi_i(\tau_i) d\tau_i, \quad s_i \in Q(x_i), \quad i \in I,$$

then

$$(3.2) \quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |v_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)}} ds_1 \cdots ds_n \\ \leq U \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i)^{\beta_i+1} \Phi_i(s_i)^{p_i} ds_i \right)^{1/p_i}$$

where  $\alpha_i = (a_i + b_i q_i) c_i$ ,  $\beta_i = a_i c_i$ , and

$$(3.3) \quad U = \frac{1}{\prod_{i=1}^n [(\alpha_i + 1)^{1/q_i} (\beta_i + 1)^{1/p_i]}.$$

*Remark 3.1.* Remembering our conventions, we observe that, for example,

$$x_i^{1/q_i} = (x_i^1)^{1/q_i} \cdots (x_i^d)^{1/q_i}, \quad \prod_{i=1}^n (\alpha_i + 1)^{1/q_i} = \prod_{i=1}^n \prod_{j=1}^d (\alpha_i^j + 1)^{1/q_i}.$$

*Proof.* Factorize the integrand on the right side of (3.1) as

$$(s_i - \tau_i)^{(a_i/q_i + b_i) c_i} \cdot (s_i - \tau_i)^{a_i c_i} \Phi_i(\tau_i)$$

and apply Hölder's inequality [10, p. 106] and Fubini's theorem. Then

$$|v_i(s_i)| \leq \left( \int_0^{s_i} (s_i - \tau_i)^{(a_i + b_i q_i) c_i} d\tau_i \right)^{1/q_i} \\ \cdot \left( \int_0^{s_i} (s_i - \tau_i)^{a_i c_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{1/p_i} \\ = \frac{s_i^{(\alpha_i+1)/q_i}}{(\alpha_i + 1)^{1/q_i}} \left( \int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{1/p_i}.$$

Using the inequality of means [10, p. 15]

$$\prod_{i=1}^n s_i^{(\alpha_i+1)/q_i} \leq \sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)},$$

we get

$$\prod_{i=1}^n |v_i(s_i)| \leq W \sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)} \prod_{i=1}^n \left( \int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{1/p_i}$$

where

$$W = \frac{1}{\prod_{i=1}^n (\alpha_i + 1)^{1/q_i}}.$$

In the following estimate we apply Hölder's inequality, Fubini's theorem, and, at the end, change the order of integration:

$$\begin{aligned}
& \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |v_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)}} ds_1 \cdots ds_n \\
& \leq W \prod_{i=1}^n \left[ \int_0^{x_i} \left( \int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{1/p_i} ds_i \right] \\
& \leq W \prod_{i=1}^n x_i^{1/q_i} \left( \int_0^{x_i} \left( \int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right) ds_i \right)^{1/p_i} \\
& = \frac{W}{\prod_{i=1}^n (\beta_i + 1)^{1/p_i}} \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left( \int_0^{x_i} (x_i - \tau_i)^{\beta_i+1} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{1/p_i}.
\end{aligned}$$

This proves the theorem.  $\square$

If  $d = 1$  and  $v_i$  are replaced by the derivatives  $u_i^{(k)}$ , the preceding theorem reduces to [6, Theorem 3.1].

**Corollary 3.2.** *Under the assumptions of Theorem 3.1,*

$$\begin{aligned}
& \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |v_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)}} ds_1 \cdots ds_n \\
(3.4) \quad & \leq p^{1/p} U \prod_{i=1}^n x_i^{1/q_i} \left( \sum_{i=1}^n \frac{1}{p_i} \int_0^{x_i} (x_i - s_i)^{\beta_i+1} \Phi_i(\tau_i)^{p_i} ds_i \right)^{1/p},
\end{aligned}$$

where  $U$  is given by (3.2).

*Proof.* By the inequality of means, for any  $A_i \geq 0$ ,

$$\prod_{i=1}^n A_i^{1/p_i} \leq p^{1/p} \left( \sum_{i=1}^n \frac{1}{p_i} A_i \right)^{1/p}.$$

The corollary then follows from the preceding theorem.  $\square$

The preceding corollary reduces to [6, Corollary 3.2] in the special case when  $d = 1$  and  $v_i$  are replaced by  $u_i^{(k)}$ .

#### 4. APPLICATIONS TO DERIVATIVES

**Convention 4.1.** In this section we shall assume that  $m_i, k_i$  are multiindices satisfying  $0 \leq k_i \leq m_i - 1$ , and write

$$(4.1) \quad \alpha_i = (a_i + b_i q_i)(m_i - k_i - 1), \quad \beta_i = a_i(m_i - k_i - 1).$$

Recall that according to our conventions,  $m_i - k_i - 1 = (m_i^1 - k_i^1 - 1, \dots, m_i^d - k_i^d - 1)$ .

**Theorem 4.2.** *Let  $u_i \in C^{m_i}(Q(x_i))$  be such that  $D_j^r u_i(s_i) = 0$  for  $s_i \in \partial_j Q(x_i)$ ,  $0 \leq r \leq m_i^j - 1$ ,  $1 \leq j \leq d$ ,  $i \in I$ . Then*

$$\begin{aligned}
& \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{k_i} u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)}} ds_1 \cdots ds_n \\
(4.2) \quad & \leq U_1 \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i)^{\beta_i+1} |D^{m_i} u_i(s_i)|^{p_i} ds_i \right)^{1/p_i}
\end{aligned}$$

where

$$(4.3) \quad U_1 = \frac{1}{\prod_{i=1}^n [(m_i - k_i - 1)! (\alpha_i + 1)^{1/q_i} (\beta_i + 1)^{1/p_i}]}.$$

*Proof.* Under the hypotheses of the theorem we have the following multivariable identities established in [11],

$$D^{k_i} u_i(s) = \frac{1}{(m_i - k_i - 1)!} \int_0^{s_i} (s_i - \tau_i)^{m_i - k_i - 1} D^{m_i} u_i(\tau_i) d\tau_i, \quad i \in I.$$

Inequality (4.2) is proved when we set  $v_i(s_i) = D^{k_i} u_i(s_i)$ ,  $c_i = m_i - k_i - 1$ , and

$$(4.4) \quad \Phi_i(s_i) = \frac{|D^{m_i} u_i(s_i)|}{(m_i - k_i - 1)!}$$

in Theorem 3.1. □

**Corollary 4.3.** *Under the hypotheses of Theorem 4.2,*

$$(4.5) \quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{k_i} u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i + 1)/(q_i w_i)}} ds_1 \cdots ds_n \leq p^{1/p} U_1 \prod_{i=1}^n x_i^{1/q_i} \left( \sum_{i=1}^n \frac{1}{p_i} \int_0^{x_i} (x_i - s_i)^{\beta_i + 1} |D^{m_i} u_i(s_i)|^{p_i} ds_i \right)^{1/p}$$

where  $U_1$  is given by (4.3).

*Proof.* The result follows by applying the inequality of means to the preceding theorem. □

A single variable analogues of the preceding two results were obtained in [6, Theorem 4.1] and [6, Corollary 4.2].

We discuss a number of special cases of Theorem 4.2 with similar examples applying also to Corollary 4.3.

*Example 4.1.* If  $a_i = 0$  and  $b_i = 1$  for  $i \in I$ , then (4.2) becomes

$$(4.6) \quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{k_i} u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(q_i m_i - q_i k_i - q_i + 1)/(q_i w_i)}} ds_1 \cdots ds_n \leq \bar{U}_1 \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i) |D^{m_i} u_i(s_i)|^{p_i} ds_i \right)^{1/p_i}$$

where

$$(4.7) \quad \bar{U}_1 = \frac{1}{\prod_{i=1}^n [(m_i - k_i - 1)! (q_i m_i - q_i k_i - q_i + 1)^{1/q_i}]}.$$

*Example 4.2.* If  $a_i = 0$ ,  $b_i = 1$ ,  $q_i = n$ ,  $w_i = 1/n$ ,  $p_i = n/(n-1)$ ,  $m_i = m$  and  $k_i = k$  for  $i \in I$ , then

$$(4.8) \quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^k u_i(s_i)|}{\sum_{i=1}^n s_i^{nm - nk - n + 1}} ds_1 \cdots ds_n \leq \frac{1}{n [(m - k - 1)!]^n (n(m - k - 1) + 1)} \times \frac{\sqrt[n]{x_1 \cdots x_n}}{(m - k - 1)!} \times \prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i) |D^m u_i(s_i)|^{n/(n-1)} ds_i \right)^{(n-1)/n}.$$

For  $d = 2$  and  $q = p = n = 2$  this is Pachpatte's theorem [17, Theorem 1] cited in the Introduction; if  $d = 1$  and  $q = p = n = 2$ , we obtain [14, Theorem 1].

*Example 4.3.* Let  $a_i = 1$  and  $b_i = 0$  for  $i \in I$ . Then (4.2) becomes

$$(4.9) \quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{k_i} u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(m_i - k_i)/(q_i w_i)}} ds_1 \cdots ds_n \\ \leq \tilde{U}_1 \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i)^{m_i - k_i} |D^{m_i} u_i(s_i)|^{p_i} ds_i \right)^{1/p_i}$$

where

$$(4.10) \quad \tilde{U}_1 = \frac{1}{\prod_{i=1}^n [(m_i - k_i - 1)!(m_i - k_i)]}.$$

*Example 4.4.* Set  $a_i = 0$ ,  $b_i = 1$ ,  $q_i = n$ ,  $w_i = 1/n$ ,  $p_i = n/(n-1)$ ,  $m_i = m$  and  $k_i = k$  for  $i \in I$ . Then (4.2) becomes

$$(4.11) \quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^k u_i(s_i)|}{\sum_{i=1}^n s_i^{m-k}} ds_1 \cdots ds_n \\ \leq \frac{1}{n} \frac{\sqrt[n]{x_1 \cdots x_n}}{[(m-k-1)!]^n (m-k)^n} \prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i)^{m-k} |D^m u_i(s_i)|^{n/(n-1)} ds_i \right)^{(n-1)/n}.$$

In the following theorem we establish another inequality similar to the integral analogue of Hilbert's inequality.

**Theorem 4.4.** *Let  $u_i \in C^{m_i+1}(Q(x_i))$  be such that  $D^{m_i} u_i(s_i) = 0$  for  $s_i \in \partial_j Q(s_i)$ ,  $1 \leq j \leq d$ ,  $i \in I$ . Then*

$$(4.12) \quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{m_i} u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{1/(q_i w_i)}} ds_1 \cdots ds_n \\ \leq \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i) |D^{m_i+1} u_i(s_i)|^{p_i} ds_i \right)^{1/p_i}.$$

*Proof.* Under the hypotheses of the theorem we have the following multivariable identities established in [11] for  $m_i = (0, \dots, 0)$ :

$$(4.13) \quad D^{m_i} u_i(s_i) = \int_0^{s_i} D^{m_i+1} u_i(\tau_i) d\tau_i, \quad i \in I.$$

In Theorem 3.1 set  $v_i(s_i) = D^{m_i} u_i(s_i)$ ,  $c_i = 0$ ,  $\Phi_i(s_i) = |D^{m_i+1} u_i(s_i)|$ , and the result follows.  $\square$

In the special case that  $d = 2$ ,  $m_i = (0, 0)$ ,  $p = q = n = 2$ , and  $w_i = 1/2$ , the preceding theorem reduces to [17, Theorem 2].

When we apply the inequality of means to the preceding theorem, we get the following corollary which generalizes the inequality obtained in [17, Remark 3].

**Corollary 4.5.** *Under the hypotheses of Theorem 4.4,*

$$(4.14) \quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{m_i} u_i(s_i)|}{\sum_{i=1}^n w_i s_i^{1/(q_i w_i)}} ds_1 \cdots ds_n \\ \leq p^{1/p} \prod_{i=1}^n x_i^{1/q_i} \left( \sum_{i=1}^n \frac{1}{p_i} \int_0^{x_i} (x_i - s_i) |D^{m_i+1} u_i(s_i)|^{p_i} ds_i \right)^{1/p}.$$

## REFERENCES

- [1] Y. C. CHOW, On inequalities of Hilbert and Widder, *J. London Math. Soc.* **14** (1939), 151–154.
- [2] S. S. DRAGOMIR AND YOUNG-HO KIM, Hilbert–Pachpatte type integral inequalities and their improvement, *JIPAM. J. Inequal. Pure Appl. Math.* **4** (2003), no. 1, Article 16, 9 pp. (electronic).
- [3] M. GAO, An improvement of Hardy–Riesz’s extension of the Hilbert inequality, *J. Math. Res. Exp.* **14** (1994), 255–259.
- [4] M. GAO, On Hilbert’s inequality and its applications, *J. Math. Anal. Appl.* **212** (1997)
- [5] G. D. HANDLEY, J. J. KOLIHA AND J. PEČARIĆ, A Hilbert type inequality, *Tamkang J. Math.* **31** (2000), 311–315.
- [6] G. D. HANDLEY, J. J. KOLIHA AND J. PEČARIĆ, New Hilbert–Pachpatte type integral inequalities, *J. Math. Anal. Appl.* **257** (2001), 238–250.
- [7] G. H. HARDY, J. E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge University Press, 1934.
- [8] L. HE, M. GAO AND S. WEI, A note on Hilbert’s inequality, *Math. Inequal. Appl.* **6** (2003), 283–288.
- [9] D. S. MITRINOVIĆ AND J. E. PEČARIĆ, On inequalities of Hilbert and Widder, *Proc. Edinburgh Math. Soc.* **34** (1991), 411–414.
- [10] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Acad. Publ., Dordrecht, 1993.
- [11] B. G. PACHPATTE, Existence and uniqueness of solutions of higher order hyperbolic partial differential equations, *Chinese J. Math.* **17** (1989), 181–189.
- [12] B. G. PACHPATTE, A note on Hilbert type inequality, *Tamkang J. Math.* **29** (1998), 293–298.
- [13] B. G. PACHPATTE, On some new inequalities similar to Hilbert’s inequality, *J. Math. Anal. Appl.* **226** (1998), 166–179.
- [14] B. G. PACHPATTE, Inequalities similar to the integral analogue of Hilbert’s inequality, *Tamkang J. Math.* **30** (1999), 139–146.
- [15] B. G. PACHPATTE, On a new inequality analogous to Hilbert’s inequality, *Rad. Mat.* **9** (1999), 5–11.
- [16] B. G. PACHPATTE, Inequalities similar to certain extensions of Hilbert’s inequality, *J. Math. Anal. Appl.* **243** (2000), 217–227.
- [17] B. G. PACHPATTE, On two new multidimensional integral inequalities of the Hilbert type, *Tamkang J. Math.* **31** (2000), 123–129.
- [18] B. G. PACHPATTE, On Hilbert type inequality in several variables. *An. Științ. Univ. Al. I. Cuza Iași. Mat. (N.S.)* **46** (2000), 245–250 (2001).
- [19] B. G. PACHPATTE, On an inequality similar to Hilbert’s inequality, *Bul. Inst. Politeh. Iași. Sect. I. Mat. Mec. Teor. Fiz.* **46** (50) (2000), 31–36 (2002).
- [20] BICHENG YANG, On Hilbert’s integral inequality, *J. Math. Anal. Appl.* **220** (1988), 778–785.
- [21] BICHENG YANG, On a new inequality similar to Hardy–Hilbert’s inequality, *Math. Inequal. Appl.* **6** (2003), 37–44.
- [22] BICHENG YANG AND LOKENATH DEBNATH, On the extended Hardy–Hilbert’s inequality, *J. Math. Anal. Appl.* **272** (2002), 187–199.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MELBOURNE, VIC 3010, AUSTRALIA

*E-mail address:* `g.handley@pgrad.unimelb.edu.au`

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MELBOURNE, VIC 3010, AUSTRALIA

*E-mail address:* `j.koliha@ms.unimelb.edu.au`

FACULTY OF TEXTILE ENGINEERING, UNIVERSITY OF ZAGREB, 10 000 ZAGREB, CROATIA

*E-mail address:* `pecaric@juda.element.hr`