Further integral inequalities related to Hardy's inequality

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Abstract

The paper gives generalisations of Hardy's integral inequality that further extend results of Mohapatra and Russell (*Aequationes Math.* **28** (1985), 199–207) and other authors by using α-submultiplicative or α-supermultiplicative functions.

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1 Two main theorems

The well known Hardy's inequality [5, Theorem 327] has been generalized in many directions by a number of mathematicians (see, for instance, [2, 4, 6, 7, 8]). The purpose of the present paper is to derive inequalities of Hardy type which generalise results of Mohapatra and Russell [9] as well as results of Copson [3], Beesack [1] and others using α-submultiplicative and α-supermultiplicative functions.

Hypotheses. Throughout we assume that all the functions considered in the paper are nonnegative and measurable on their domains of definition, which is usually $\mathbb{R}_+ := [0, \infty)$. In particular, a is a nonnegative measurable function defined on $\mathbb{R}_+ \times \mathbb{R}_+$.

Let f, g be nonnegative measurable functions defined on $\mathbb{R}_+$. For any x, $t \in \mathbb{R}_+$, we set

$$F_1(x) = \int_0^x a(x,u)f(u)\,du, \qquad F_2(x) = \int_x^\infty a(x,u)f(u)\,du, \tag{1.1}$$

$$G_1(t) = \int_0^t g(u)a(u,t)\,du, \qquad G_2(t) = \int_t^\infty g(u)a(u,t)\,du \tag{1.2}$$

assuming that the integrals are finite. The weight function w is measurable and positive. The function φ is always assumed nonnegative and continuously differentiable in $\mathbb{R}_+$. In appropriate places in the paper we will further assume that φ' is nonnegative, increasing and α-submultiplicative, or nonnegative, decreasing and α-supermultiplicative. In general, α-submultiplicativity and α-supermultiplicativity cannot be replaced by mere submultiplicativity and supermultiplicativity.

Definition 1.1. Let $\alpha > 0$. A function $f : J \subset \mathbb{R}_+ \to \mathbb{R}_+$ is *α-submultiplicative* if the function αf is submultiplicative; equivalently,

$$f(xy) \le \alpha f(x) f(y) \text{ for all } x, y \in J;$$

f is *α-supermultiplicative* if the reverse inequality holds, and *α-multiplicative* if it is both α-submultiplicative and α-supermultiplicative.

We give an example of typical functions in these classes.

Example 1.2. Let $\alpha > 0$, $c \ge 0$, $p > 0$, and define

$$f(x) = \alpha^{-1}(c+x)^{p-1}, \quad x \ge 0.$$

We can verify the following properties of f:

(i) If $c = 0$, then f is α-multiplicative.

(ii) If $c \ge 1$ and $p > 1$, then f is α-submultiplicative.

(iii) If $c \ge 1$ and $0 < p < 1$, then f is α-supermultiplicative.

Theorem 1.3. *Assume that the function a satisfies*

$$a(x,t) \begin{cases} = 0 & \text{if} \quad t > x, \\ \le k_1 a(y,t) & \text{if} \quad x > y > t > 0 \quad (\text{for some constant } k_1 \ge 1). \end{cases} \tag{1.3}$$

(i) *Suppose that $1 < p < \infty$, $p' = p/(p-1)$, $0 < m \le \infty$, $g(x) > 0$ on $(0,m)$, and that φ' is nonnegative, increasing and α-submultiplicative for some $\alpha > 0$. Then*

$$\int_0^m g(x)\varphi(F_1(x))\,dx \le \alpha\varphi'(k_1)\left\{\int_0^m [w(x)f(x)G_2(x)]^p\,dx\right\}^{1/p}\left\{\int_0^m \frac{\{\varphi'(F_1(x))\}^{p'}}{w(x)^{p'}}\,dx\right\}^{1/p'}. \tag{1.4}$$

(ii) *Suppose that $0 < p < 1$, $p' = p/(p-1)$, $0 \le r < \infty$, $F_1(x) > 0$ on $\mathbb{R}_+$, and that φ' is nonnegative, decreasing and α-supermultiplicative for some $\alpha > 0$. Then*

$$\int_r^\infty g(x)\varphi(F_1(x))\,dx$$
$$\geq \alpha\varphi'(k_1)\left\{\int_r^\infty [w(x)f(x)G_2(x)]^p\,dx\right\}^{1/p}\left\{\int_r^\infty \frac{\{\varphi'(F_1(x))\}^{p'}}{w(x)^{p'}}\,dx\right\}^{1/p'}. \quad (1.5)$$

Proof. (i) Let φ' be nonnegative, increasing and α-submultiplicative. In order to apply the hypotheses and Hölder's inequality, we need to express $\varphi(F_1(x))$ in a suitable form. In the following argument we need to differentiate the integral $F_1(x) = \int_0^x a(x,s)f(s)\,ds$ as a function of the upper terminal. To this end we stipulate that the variable x in $a(x,s)$ is not coupled with the variable x in the integral terminal. We have

$$\begin{aligned}\varphi(F_1(x)) &= \int_0^x \left(\frac{d}{dt}\varphi(F_1(t))\right)dt = \int_0^x \varphi'(F_1(t))\frac{d}{dt}F_1(t)\,dt\\ &= \int_0^x \varphi'\left(\int_0^t a(x,u)f(u)\,du\right)\left(\frac{d}{dt}\int_0^t a(x,u)f(u)\,du\right)dt\\ &= \int_0^x \varphi'\left(\int_0^t a(x,u)f(u)\,du\right)a(x,t)f(t)\,dt.\end{aligned}$$

Hence

$$\varphi(F_1(x)) = \int_0^x \varphi'\left(\int_0^t a(x,u)f(u)\,du\right)a(x,t)f(t)\,dt. \quad (1.6)$$

Equation (1.6) is a generalization of [4, Lemma 2] which is recovered when we set $a(x,t) = 1$ for $x > t > 0$ and $\varphi(x) = x^p$, $p > 1$.

Since $\varphi'(x)$ is increasing and $a(x,u) \leq k_1 a(t,u)$ in (1.6), we have

$$\varphi(F_1(x)) \leq \int_0^x \varphi'\left(k_1\int_0^t a(t,u)f(u)\,du\right)a(x,t)f(t)\,dt. \quad (1.7)$$

Since $\varphi'(x)$ is α-submultiplicative,

$$\varphi(F_1(x)) \leq \alpha\varphi'(k_1)\int_0^x \varphi'(F_1(t))a(x,t)f(t)\,dt \quad (1.8)$$

and

$$\begin{aligned}\int_0^m g(x)\varphi(F_1(x))\,dx &\leq \alpha\varphi'(k_1)\int_0^m g(x)\left\{\int_0^x a(x,t)f(t)\varphi'(F_1(t))\,dt\right\}dx\\ &= \alpha\varphi'(k_1)\int_0^m f(t)\varphi'(F_1(t))\left\{\int_t^m g(x)a(x,t)\,dx\right\}dt\\ &\leq \alpha\varphi'(k_1)\int_0^m f(t)\varphi'(F_1(t))G_2(t)\,dt\end{aligned}$$

by changing the order of integration. Hence

$$\int_0^m g(x)\varphi(F_1(x))\,dx \le \alpha\varphi'(k_1)\int_0^m \{w(t)f(t)G_2(t)\}\left\{\frac{\varphi'(F_1(t))}{w(t)}\right\}dt. \tag{1.9}$$

Applying Hölder's inequality with the conjugate indices p and p', we obtain (1.4). (Recall that all functions are measurable and nonnegative.)

(ii) Proceeding analogously as in (i), we obtain (1.7) and (1.8) with the inequalities reversed since φ' is decreasing and α-supermultiplicative.

If $g = 0$ a.e. on $\mathbb{R}_+$ then (1.5) holds trivially. Hence suppose that $g > 0$ on a set A of positive measure and $g = 0$ on $\mathbb{R}_+\backslash A$. Then multiplying the reverse inequality (1.8) by $g(x)$ and integrating over $E = A \cap (r,\infty)$, we obtain

$$\begin{aligned}\int_E g(x)\varphi(F_1(x))\,dx &\ge \alpha\varphi'(k_1)\int_E g(x)\left\{\int_0^x a(x,t)f(t)\varphi'(F_1(t))\,dt\right\}dx\\ &= \alpha\varphi'(k_1)\int_0^\infty f(t)\varphi'(F_1(t))\left\{\int_{E\cap(t,\infty)} g(x)a(x,t)\,dx\right\}dt\\ &\ge \alpha\varphi'(k_1)\int_E f(t)\varphi'(F_1(t))G_2(t)\,dt.\end{aligned}$$

Since $E \subset (r,\infty)$ and $g = 0$ on $(r,\infty)\backslash E$, we have

$$\int_r^\infty g(x)\varphi(F_1(x))\,dx \ge \alpha\varphi'(k_1)\int_r^\infty \{w(t)f(t)G_2(t)\}\left\{\frac{\varphi'(F_1(t))}{w(t)}\right\}dt. \tag{1.10}$$

Applying Hölder's reverse inequality for $0 < p < 1$ (see [5, 6.9.3]) to the right hand side of this inequality with the conjugate indices p and p', we obtain (1.5). □

The following theorem is proved in an analogous fashion.

Theorem 1.4. *Assume that the function a satisfies*

$$a(x,t)\begin{cases} = 0 & \text{if} \quad t < x,\\ \le k_2 a(y,t) & \text{if} \quad 0 < x < y < t \quad (\text{for some constant } k_2 \ge 1).\end{cases} \tag{1.11}$$

(i) *Suppose that $1 < p < \infty$, $p' = p/(p-1)$, $0 \le r < \infty$, and that φ' is nonnegative, increasing and α-submultiplicative for some $\alpha > 0$. Then*

$$\begin{aligned}&\int_r^\infty g(x)\varphi(F_2(x))\,dx\\ &\le \alpha\varphi'(k_2)\left\{\int_r^\infty [w(x)f(x)G_1(x)]^p\,dx\right\}^{1/p}\left\{\int_r^\infty \frac{\{\varphi'(F_2(x))\}^{p'}}{w(x)^{p'}}\,dx\right\}^{1/p'}.\end{aligned} \tag{1.12}$$

(ii) *Suppose that* $0 < p < 1$, $p' = p/(p-1)$, $0 < m \leq \infty$, *and that* φ' *is nonnegative, decreasing and* α*-supermultiplicative for some* $\alpha > 0$. *Then*

$$\int_0^m g(x)\varphi(F_2(x))\,dx \geq \alpha\varphi'(k_2)\left\{\int_0^m [w(x)f(x)G_1(x)]^p\,dx\right\}^{1/p}\left\{\int_0^m \frac{\{\varphi'(F_2(x))\}^{p'}}{w(x)^{p'}}\,dx\right\}^{1/p'}. \quad (1.13)$$

2 Further inequalities

In this section we consider a situation when the weight function w is chosen in such a way that (2.1) is satisfied, and the factor $\int_0^m (w^{-1}\varphi'(F_1))^{1/p'}\,dx$ can be eliminated from the right hand side of (1.4) and (1.5). (Similarly for (1.12) and (1.13).)

Theorem 2.1. *Assume that the function a satisfies* (1.3).

(i) *Suppose that* $1 < p < \infty$, $0 < m \leq \infty$, *and* $g(x) > 0$ *on* $(0, m)$, *and that the solution* φ *to the differential equation*

$$\left(\frac{\varphi'(F_1(x))}{w(x)}\right)^{p/(p-1)} = g(x)\varphi(F_1(x)), \quad x > 0, \quad (2.1)$$

is such that φ' *is increasing and* α*-submultiplicative for some* $\alpha > 0$. *Then*

$$\int_0^m g(x)\varphi(F_1(x))\,dx \leq (\alpha\varphi'(k_1))^p \int_0^m [w(x)f(x)G_2(x)]^p\,dx. \quad (2.2)$$

(ii) *Suppose that* $0 < p < 1$, $0 \leq r < \infty$, *and* $F_1(x) > 0$ *on* $\mathbb{R}_+$, *and that the solution* φ *to the differential equation* (2.1) *is such that* φ' *is decreasing and* α*-supermultiplicative for some* $\alpha > 0$. *Then*

$$\int_r^\infty g(x)\varphi(F_1(x))\,dx \geq (\alpha\varphi'(k_1))^p \int_r^\infty [w(x)f(x)G_2(x)]^p\,dx. \quad (2.3)$$

Proof. (i) Let $p' = p/(p-1)$. When we differentiate the integral $\int_0^x a(x,t)t(f)\,dt$ as a function of the upper terminal, we again assume that the variable x in $a(x,t)$ is not coupled with the terminal x.

Let φ be a solution to (2.1). Then $(\varphi'(F_1)/w)^{p'} = g\varphi(F_1)$ implies

$$\frac{d}{dx}\varphi(F_1)^{1/p} = \frac{1}{p}wg^{1/p'}F_1',$$

and

$$\varphi(F_1(x)) = \left\{\frac{1}{p}\int_0^x w(t)g(t)^{1/p'}\, a(x,t)f(t)\,dt\right\}^p. \tag{2.4}$$

From $\varphi'(F_1) = wg^{1/p'}\varphi(F_1)^{1/p'}$ we get

$$\varphi'(F_1(x)) = w(x)g(x)^{1/p'}\left\{\frac{1}{p}\int_0^x w(t)g(t)^{1/p'} a(x,t)f(t)\,dt\right\}^{p-1}. \tag{2.5}$$

We can check that the differential equation (2.1) has a solution φ given by (2.4) whose derivative is given by (2.5). From these equations it follows that the composite functions $\varphi\circ F_1$ and $\varphi'\circ F_1$ are nonnegative. If, in addition, φ' is increasing and α-submultiplicative, we can examine the proof of Theorem 1.3 (i) to conclude that the theorem applies also in this situation. Then

$$\int_0^m g\varphi(F_1)\,dx \le \alpha\varphi'(k_1)\left(\int_0^m (wfG_2)^p\,dx\right)^{1/p}\left(\int_0^m g\varphi(F_1)\,dx\right)^{1/p'}.$$

Dividing by $(\int_0^m g\varphi(F_1)\,dx)^{1/p'}$ and raising both sides to the power p we obtain (2.2).

Part (ii) is proved similarly. □

We observe that we can eliminate φ from the inequalities (2.2) and (2.3) on substituting (2.4):

Corollary 2.2. *Let the function a satisfy condition* (1.3).

(i) *Under the hypotheses of Theorem* 2.1 (i),

$$\int_0^m g(x)\left\{\int_0^x w(t)g(t)^{(p-1)/p}\, a(x,t)f(t)\,dt\right\}^p dx \le (\alpha\varphi'(k_1))^p\int_0^m [w(x)f(x)G_2(x)]^p dx. \tag{2.6}$$

(ii) *Under the hypotheses of Theorem* 2.1 (ii),

$$\int_0^m g(x)\left\{\int_0^x w(t)g(t)^{(p-1)/p}\, a(x,t)f(t)\,dt\right\}^p dx \ge (\alpha\varphi'(k_1))^p\int_0^m [w(x)f(x)G_2(x)]^p dx. \tag{2.7}$$

Remark 2.3. In the proof of Theorem 2.1 we derived an explicit formula for the composite function $\varphi'\circ F_1$ without assuming anything special about the nature of

φ'. In the case that $\lim_{x\to\infty} F_1(x) = \infty$, we can obtain an explicit formula for φ', and test whether it has the properties required by Theorem 2.1. Indeed, if $\lim_{x\to\infty} F_1(x) = \infty$, then F_1 is surjective as a function on $\mathbb{R}_+$ to $\mathbb{R}_+$, and there exists a right inverse $H : \mathbb{R}_+ \to \mathbb{R}_+$ for F_1. According to (2.5),

$$\varphi'(u) = w(H(u))g(H(u))^{1/p'} \left\{ \frac{1}{p} \int_0^{H(u)} w(t)g(t)^{1/p'} a(H(u), t) f(t)\, dt \right\}^{p-1}. \tag{2.8}$$

We can then decide whether the right hand side of this equation considered as a function of u is increasing and α-submultiplicative (or decreasing and α-supermultiplicative).

If the functions involved in Theorem 2.1 are continuous and $F_1(x_1) = k_1$ for some $x_1 \in \mathbb{R}_+$, then φ and φ' can be eliminated from (2.2) and (2.3) altogether:

Corollary 2.4. *Suppose that a satisfies condition* (1.3)*, that the functions a, f, g and w are continuous, and that there exists $x_1 \in \mathbb{R}_+$ such that $F_1(x_1) = k_1$.*

(i) *If the hypotheses of Theorem* 2.1 (i) *hold, then*

$$\int_0^m g(x) \left\{ \int_0^x w(t)g(t)^{(p-1)/p} a(x,t) f(t)\, dt \right\}^p dx$$
$$\leq \alpha^p w(x_1)^p g(x_1)^{p-1} \left\{ \int_0^{x_1} w(t)g(t)^{(p-1)/p} a(x_1,t) f(t)\, dt \right\}^{p(p-1)}$$
$$\cdot \int_0^m [w(x)f(x)G_2(x)]^p\, dx.$$

(ii) *If the hypotheses of Theorem* 2.1 (ii) *hold, then*

$$\int_0^m g(x) \left\{ \int_0^x w(t)g(t)^{(p-1)/p} a(x,t) f(t)\, dt \right\}^p dx$$
$$\geq \alpha^p w(x_1)^p g(x_1)^{p-1} \left\{ \int_0^{x_1} w(t)g(t)^{(p-1)/p} a(x_1,t) f(t)\, dt \right\}^{p(p-1)}$$
$$\cdot \int_0^m [w(x)f(x)G_2(x)]^p\, dx.$$

Proof. It is enough to observe that, under the assumptions of continuity, equation (2.5) holds pointwise, rather than almost everywhere. □

For certain special choices of the weight function w, the differential equation (2.1) has a required type of solution independently of a, f and g. Such an example is illustrated in the following corollary of Theorem 2.1 which yields the inequalities obtained by Mohapatra and Russell in [9, Theorem 1].

Corollary 2.5. *Let the function a satisfy* (1.3)*, let* $g(x) > 0$ *on* $(0, m)$ *and let* $w = g^{(1-p)/p}$.

(i) *Suppose that* $1 < p < \infty$ *and* $0 < m \le \infty$*. Then the differential equation* (2.1) *admits a solution* $\varphi(x) = p^{-p}x^p$ *with* $\varphi'(x) = p^{1-p}x^{p-1}$*, which is nonnegative, increasing and* α*-submultiplicative with* $\alpha = p^{p-1}$*, and*

$$\int_0^m g(x)F_1(x)^p\,dx \le (pk_1^{p-1})^p \int_0^m g(x)^{1-p}[f(x)G_2(x)]^p\,dx. \tag{2.9}$$

(ii) *Suppose that* $0 < p < 1$*,* $0 \le r < \infty$*, and* $F_1(x) > 0$ *on* $\mathbb{R}_+$*. Then the differential equation* (2.1) *admits a solution* $\varphi(x) = p^{-p}x^p$ *with* $\varphi'(x) = p^{1-p}x^{p-1}$*, which is nonnegative, decreasing and* α*-submultiplicative with* $\alpha = p^{p-1}$*, and*

$$\int_r^\infty g(x)F_1(x)^p\,dx \ge (pk_1^{p-1})^p \int_r^\infty g(x)^{1-p}[f(x)G_2(x)]^p\,dx. \tag{2.10}$$

Proof. (i) The differential equation (2.1) becomes $(d/dx)\varphi(F_1)^{1/p} = p^{-1}F_1'$ with the solution $\varphi(F_1) = (p^{-1}F_1)^p$. Hence $\varphi(x) = p^{-p}x^p$ and $\varphi'(x) = p^{1-p}x^{p-1}$. In view of Example 1.2, φ' is p^{p-1}-(sub)multiplicative. Substituting into (2.2), we get

$$\int_0^m g\left(\frac{F_1}{p}\right)^p dx \le (p^{p-1})^p\left(\left(\frac{k_1}{p}\right)^{p-1}\right)^p \int_0^m g^{1-p}f^pG_2^p\,dx,$$

from which (2.9) follows.

The proof of (ii) is analogous. □

We have also the following counterpart of Theorem 2.1.

Theorem 2.6. *Suppose that the function a satisfies condition* (1.11)*.*

(i) *If* $1 < p < \infty$*,* $0 < m \le \infty$*,* $g(x) > 0$ *on* $(0, m)$*, and the solution* φ *to the differential equation* (2.1) *is such that* φ' *is increasing and* α*-submultiplicative, then*

$$\int_r^\infty g(x)\varphi(F_2(x))\,dx \le (\alpha\varphi'(k_2))^p \int_r^\infty [w(x)f(x)G_1(x)]^p\,dx. \tag{2.11}$$

(ii) *If* $0 < p < 1$*,* $0 \le r < \infty$*,* $F_1(x) > 0$ *on* $\mathbb{R}_+$*, and the solution* φ *to the differential equation* (2.1) *is such that* φ' *is decreasing and* α*-supermultiplicative, then*

$$\int_r^\infty g(x)\varphi(F_2(x))\,dx \ge (\alpha\varphi'(k_2))^p \int_r^\infty [w(x)f(x)G_1(x)]^p\,dx. \tag{2.12}$$

The preceding theorem admits corollaries analogous to Corollaries 2.2–2.4. We leave the formulation to the reader. Specializing w in the preceding theorem to $w = g^{(1-p)/p}$ as in Corollary 2.5, we recover [9, Theorem 2].

3 Special cases

As our first result we obtain a convolution inequality which generalizes [9, Theorem 3].

Corollary 3.1. *Assume that $s(t) \geq 0$ if $t > 0$ and $s(t) = 0$ if $t < 0$, that for some constant $k \geq 1$, $s(x) \leq ks(y)$ for $x > y > 0$, and $1 < p < \infty$. Let $S(x) = \int_0^x s(t)\, dt$ and $F(x) = \int_0^x s(x-t)f(t)\, dt$ for $x > 0$. Suppose that the solution φ to the differential equation*

$$\left(\frac{\varphi'(F(x))}{w(x)}\right)^{p/(p-1)} = S(x)^{-p}\varphi(F(x)), \quad x > 0, \tag{3.1}$$

is such that φ' is increasing and α-submultiplicative for some $\alpha > 0$. Then

$$\int_0^\infty S(x)^{-p}\varphi\left(\int_0^x s(x-t)f(t)\, dt\right) dx \leq \left(\frac{p\alpha\varphi'(k)}{p-1}\right)^p \int_0^\infty [w(x)S(x)^{1-p}f(x)]^p\, dx. \tag{3.2}$$

Proof. Write $G(t) = \int_t^\infty S(x)^{-p}\, s(x-t)\, dx$ for $t > 0$, and take $a(x,t) = s(x-t)$, $g(x) = S(x)^{-p}$, $r = 0$, $m = \infty$ in Theorem 2.1 (i). As in [9, Theorem 3],

$$G(x) \leq p(p-1)^{-1}S(x)^{1-p}.$$

The result then follows when we substitute in (2.2). □

An analogous result is obtained for $0 < p < 1$, in which case φ' is assumed to be α-supermultiplicative, and the inequality in (3.2) is reversed.

Remark 3.2. When we set $w(x) = S(x)^{p-1}$ in the preceding theorem, the differential equation (3.1) has the solution $\varphi(x) = p^{-p}x^p$ for which $\varphi'(x) = p^{1-p}x^{p-1}$ is nonnegative, increasing and α-multiplicative with $\alpha = p^{p-1}$. We thus recover Theorem 3 of Mohapatra and Russell [9],

$$\int_0^\infty S(x)^{-p}\left(\int_0^x s(x-t)t(f)\, dt\right)^p dx \leq \left(\frac{p^2k^{p-1}}{p-1}\right)^p \int_0^\infty [f(x)]^p\, dx$$

(correcting a misprint in [9], where the constant is given as $[p^2k/(p-1)]^p$).

Theorems 1 and 2 of Mohapatra and Russell [9] were seen as special cases of our Theorem 2.1 and Theorem 2.6. In that case we made a special choice of the weight function w, and obtained φ as a solution of the differential equation (2.1)—there was no freedom of choice for φ. In the following example we apply a reverse choice in Theorem 1.3: We first select φ, and still have a freedom of choice for the weight function w.

The following is a new inequality of Hardy type from which we recover Theorem 1 of [9] by setting $c = q = 0$.

Corollary 3.3. *Let a satisfy* (1.3), *let* $0 < p < \infty$, $p \neq 1$, $q > 0$, $c \in \{0\} \cup [1, \infty)$, $F_1(x) > 0$ *and* $g(x) > 0$.

(i) *If* $1 < p < \infty$ *and* $0 < m \le \infty$, *then*

$$\int_0^m g(c + F_1)^{p+q}\, dx \le \big[(p+q)(c+k_1)^{p+q-1}\big]^p \int_0^m g^{1-p}(c+F_1)^q (fG_2)^p\, dx. \quad (3.3)$$

(ii) *If* $0 < p < 1$, $q < 1 - p$ *and* $0 \le r < \infty$, *then*

$$\int_r^\infty g(c + F_1)^{p+q}\, dx \ge \big[(p+q)(c+k_1)^{p+q-1}\big]^p \int_r^\infty g^{1-p}(c+F_1)^q (fG_2)^p\, dx. \quad (3.4)$$

Proof. (i) Write $p' = p/(p-1)$. Set $\varphi(x) = (p+q)^{-1}(c+x)^{p+q}$ in Theorem 1.3 (i). Then $\varphi'(x) = (c+x)^{p+q-1}$ is submultiplicative (multiplicative for $c = 0$), and (1.4) specializes to

$$\int_0^m g(c+F_1)^{p+q}\, dx \le (p+q)(c+k_1)^{p+q-1} \left(\int_0^m [wfG_2]^p\, dx\right)^{1/p} \cdot \left(\int_0^m (w^{-1}(c+F_1)^{p+q-1})^{p'}\, dx\right)^{1/p'}.$$

Choosing the weight function $w = g^{-1/p'}(c+F_1)^{q/p}$, we obtain

$$\begin{aligned}
&\int_0^m g(c+F_1)^{p+q}\, dx \\
&\quad\le (p+q)(c+k_1)^{p+q-1} \left(\int_0^m [g^{-1/p'}(c+F_1)^{q/p} fG_2]^p\, dx\right)^{1/p} \\
&\qquad \cdot \left(\int_0^m g((c+F_1)^{p+q-1-q/p})^{p'}\, dx\right)^{1/p'} \\
&\quad= (p+q)(c+k_1)^{p+q-1} \left(\int_0^m g^{1-p}(c+F_1)^q (fG_2)^p\, dx\right)^{1/p}
\end{aligned}$$

$$\cdot\left(\int_0^m g(c+F_1)^{p+q}\,dx\right)^{1/p'}.$$

Dividing by $(\int_0^m g(c+F_1)^{p+q}\,dx)^{1/p'}$ and raising both sides to the power p, we get (3.3).

For the proof of part (ii) we note that under the hypotheses, $p+q-1<0$, and $(c+x)^{p+q-1}$ is supermultiplicative. The rest is proved similarly as in part (i). □

We also have the following counterpart to the preceding corollary. For $q=c=0$ this reduces to [9, Theorem 2].

Corollary 3.4. *Let a satisfy* (1.11), *let* $0<p<\infty$, $p\neq 1$, $p'=p/(p-1)$, $q>0$, $c\in\{0\}\cup[1,\infty)$, $F_2(x)>0$ *and* $g(x)>0$.

(i) *If* $1<p<\infty$ *and* $0\le r<\infty$, *then*

$$\int_r^\infty g(c+F_2)^{p+q}\,dx \le \left[(p+q)(c+k_2)^{p+q-1}\right]^p \int_r^\infty g^{1-p}(c+F_2)^q(fG_1)^p\,dx. \quad (3.5)$$

(ii) *If* $0<p<1$, $q<1-p$ *and* $<m\le\infty$, *then*

$$\int_0^m g(c+F_2)^{p+q}\,dx \ge \left[(p+q)(c+k_2)^{p+q-1}\right]^p \int_0^m g^{1-p}(c+F_2)^q(fG_1)^p\,dx. \quad (3.6)$$

We now derive several new inequalities of Hardy type that will yield results of Hardy et al. [5], Beesack [1] and Copson [3] when we make a special choice of parameters.

Example 3.5. In Corollary 3.3 set $g(x)=x^{-\beta}$ for some $\beta>1$, $a(x,t)=1$ if $0<t<x$, $a(x,t)=0$ if $x<t$, $k_1=1$, $c\in\{0\}\cup[1,\infty)$ and $m=\infty$. We note that $G_2(x)=(\beta-1)^{-1}x^{1-\beta}$. If $1<p<\infty$, then

$$\int_0^\infty x^{-\beta}(c+F_1(x))^{p+q}\,dx \le \left(\frac{(p+q)(c+1)^{p+q-1}}{\beta-1}\right)^p \int_0^\infty x^{-\beta}(c+F_1(x))^q(xf(x))^p\,dx. \quad (3.7)$$

If $0<p<1$, $\beta>1$ and $q<1-p$, the inequality is reversed. This generalizes [5, Theorem 330] which is obtained when $c=0$ and $q=0$. When in addition $\beta=p$, we obtain the original Hardy's inequality [5, Theorem 327].

Example 3.6. In Corollary 3.4 set $g(x) = x^{-\beta p}$ for some $\beta > 0$, $a(x,t) = t^{\beta-1}$ for $0 < x < t$, $k_2 = 1$, $c \in \{0\} \cup [1,\infty)$ and $r = 0$. If $1 < p < \infty$ and $\beta < 1/p$, then

$$\int_0^\infty x^{-\beta p}\left(c + \int_x^\infty t^{\beta-1} f(t)\,dt\right)^{p+q} dx \leq \left(\frac{(p+q)(c+1)^{p+q-1}}{1-\beta p}\right)^p \int_0^\infty \left(c + \int_x^\infty t^{\beta-1} f(t)\,dt\right)^q f(x)^p\,dx. \quad (3.8)$$

When we set $c = q = 0$, we obtain [5, Equation (9.9.9)].

In the last three examples we assume that ψ, f are positive and measurable on $\mathbb{R}_+$, and that

$$\Psi(x) = \int_0^x \psi(t)\,dt, \quad F(x) = \int_0^x f(t)\psi(t)\,dt, \quad H(x) = \int_x^\infty f(t)\psi(t)\,dt.$$

Example 3.7. Suppose that $1 < p < \infty$, $q > 0$, $c \in \{0\} \cup [1,\infty)$ and $0 < m < \infty$. Then

$$\int_0^m (c+F)^{p+q}\psi\Psi^{-1}\,dx \leq [(p+q)(c+1)^{p+q-1}]^p \int_0^m (c+F)^q f^p \Psi^{p-1}\left(\log\frac{\Psi(m)}{\Psi(x)}\right)^p dx. \quad (3.9)$$

If $0 < p < 1$ and $q < 1-p$, this inequality is reversed.

Inequality (3.9) is obtained from Corollary 3.3 by setting $a(x,t) = \psi(t)$ if $0 < t < x$, $k_1 = 1$, $g(x) = \psi(x)\Psi(x)^{-1}$ if $0 < x < m$ and $g(x) = 0$ if $x > m$.

The choice $q = c = 0$ yields [1, (28), (32)] and [3, Theorem 5].

Example 3.8. Suppose that $1 < p < \infty$, $q > 0$, $c \in \{0\} \cup [1,\infty)$ and $0 < r < \infty$. Then

$$\int_r^\infty (c+H)^{p+q}\psi\Psi^{-1}\,dx \leq [(p+q)(c+1)^{p+q-1}]^p \int_r^\infty (c+H)^q f^p \Psi^{p-1}\left(\log\frac{\Psi(x)}{\Psi(r)}\right)^p dx. \quad (3.10)$$

If $0 < p < 1$ and $q < 1-p$, this inequality is reversed.

Inequality (3.10) is obtained from Corollary 3.4 by setting $a(x,t) = \psi(t)$ if $0 < x < t$, $k_2 = 1$, $g(x) = 0$ if $0 < x < r$ and $g(x) = \psi(x)\Psi(x)^{-1}$ if $x > r$.

The choice $q = c = 0$ yields [1, (29), (33)] and [3, Theorem 6].

Example 3.9. (i) Suppose that $1 < p < \infty$, $q > 0$, $c \in \{0\} \cup [1, \infty)$, $\beta > 1$ and $0 < m \leq \infty$. Then

$$\int_0^m (c+F)^{p+q} \psi \Psi^{-\beta}\, dx \leq \left(\frac{(p+q)(c+1)^{p+q-1}}{\beta - 1} \right)^p \int_0^m (c+F)^q \psi \Psi^{p-\beta} f^p\, dx. \tag{3.11}$$

(ii) Suppose that $0 < p < 1$, $q > 0$, $q < 1-p$, $c \in \{0\} \cup [1, \infty)$, $\beta > 1$, $0 \leq r < \infty$ and $\lim_{s\to\infty} \Psi(s) = \infty$. Then

$$\int_r^\infty (c+F)^{p+q} \psi \Psi^{-\beta}\, dx \geq \left(\frac{(p+q)(c+1)^{p+q-1}}{\beta - 1} \right)^p \int_r^\infty (c+F)^q \psi \Psi^{p-\beta} f^p\, dx. \tag{3.12}$$

The inequalities are obtained from Corollary 3.3 on setting $a(x,t) = \psi(t)$ if $0 < t < x$, $k_1 = 1$, $g(x) = \psi(x)\Psi(x)^{-\beta}$. We have

$$G_2(t) = (\beta - 1)^{-1} \psi(t) [\Psi(t)^{1-\beta} - A], \qquad A = \lim_{s\to\infty} \Psi(s)^{1-\beta}.$$

For part (i) we use the inequality $G_2(t) \leq (\beta - 1)^{-1} \psi(t) \Psi(t)^{1-\beta}$, in part (ii) we need $A = 0$. When we set $q = c = 0$, we obtain Theorem 1 and 2 of [3].

The preceding example has a companion result for $\beta < 1$ obtained by an analogous procedure from Corollary 3.4. Setting $q = c = 0$, we recover Theorem 3 and 4 of [3].

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