g-Kato operators form a regularity

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This announcement is to introduce the concept of generalised Kato operators on a complex Banach space and some of their properties.

1 Notation

We will denote the set of bounded linear operators on a complex Banach space, X, by $\mathcal{B}(X)$. For an operator $T \in \mathcal{B}(X)$ we denote the nullspace, $\mathcal{N}(T)$, range, $\mathcal{R}(T)$, analytic core, $\mathcal{K}(T)$ and quasinilpotent part, $\mathcal{H}_0(T)$. The spectrum will be denoted by $\sigma(T)$. Given a closed invariant subspace $M \subseteq X$ we denote the restrictions $T_M : M \to M, T_M m = Tm$ for $m \in M$ and $T_{X/M} : X/M \to X/M, T_{X/M}(x + M) = Tx + M$.

Recall that an operator is called bounded below if $\mathcal{N}(T) = \{0\}$ and $\mathcal{R}(T)$ is closed and an operator is called Kato, or semi-regular, if among many equivalent conditions $\mathcal{R}(T)$ is closed and $\mathcal{H}_0(T) \subseteq \mathcal{K}(T)$. This is equivalent to the existence of a closed invariant subspace $M \subseteq X$ such that $T_{X/M}$ is bounded below and T_M is surjective. The analytic core, $M = \mathcal{K}(T)$ suffices for this purpose. The set of $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not Kato, the Kato spectrum, will be denoted by $\sigma_K(T)$.

An operator $T \in \mathcal{B}(X)$ is called Koliha-Drazin invertible if $0 \notin \operatorname{acc} \sigma(T)$, the accumulation points of the spectrum. Racall, finally, that the set of quasi-Fredholm operators, or operators of Kato-type, can be characterised by the existence of a pair of closed T-invariant subspaces, $N \subseteq M \subseteq X$ such that $T_{X/M}$ is bounded below, $T_{M/N}$ nilpotent and T_N surjective.

2 Generalised Kato Operators

The following is a definition for a generalised Kato operator requiring only that three distinguished sets be closed. There is an equivalent, more familiar setting however, this method seems much cleaner and clearer to state.

Definition 2.1. An operator, $T \in \mathcal{B}(X)$ will be called a generalised Kato operator, or g-Kato, if the sets $\mathcal{R}(T) + \mathcal{H}_0(T), \mathcal{K}(T) + \mathcal{H}_0(T)$ and $\mathcal{K}(T)$ are closed.

The following lemma is interesting in its own right but is very useful for the later equivalent definition.

Lemma 2.2. *Let* $T \in \mathcal{B}(X), 0 \notin \operatorname{acc} \sigma_K(T)$ *and* G *be the component of* $\mathbb{C} \setminus \sigma_K(T)$ *such that* $0 \in \overline{G}$. *Then for any* $\lambda \in G$, $\mathcal{K}(T - \lambda I) = \mathcal{K}(T) + \mathcal{H}_0(T)$.

Theorem 2.3. *An operator* $T \in \mathcal{B}(X)$ *is g-Kato if and only if there exist invariant subspaces* $N \subseteq M \subseteq X$ *such that* $T_{X/M}$ *is bounded below,* $T_{M/N}$ *is Koliha-Drazin invertible and* T_N *is surjective. We may choose* $M = \mathcal{K}(T) + \overline{\mathcal{H}_0(T)}, N = \mathcal{K}(T)$ *or* $M = \overline{\mathcal{H}_0(T)}, N = \mathcal{K}(T) \cap \overline{\mathcal{H}_0(T)}$.

Corollary 2.4. *An operator* $T \in \mathcal{B}(X)$ *is g-Kato if and only if the dual operator* $T^* \in \mathcal{B}(X^*)$ *is g-Kato.*

Corollary 2.5. *The set of generalised Kato operators is stable under commuting quasinilpotent operators.*

3 Spectral mapping theorem

We conclude this announcement with the theorem that the set of generalised Kato operators forms a regularity (in the sense of Müller) and hence satisfies the spectral mapping theorem.

Lemma 3.1. *Let* $T \in \mathcal{B}(X)$. *Then* T *is a g-Kato operator if and only if* T^n *is a g-Kato operator for all* $n \in \mathbb{N}$.

Lemma 3.2. *Let* $A, B \in \mathcal{B}(X)$ *be a relative prime pair of operators. That is, there exist* $C, D \in \mathcal{B}(X)$ *such that* A, B, C, D *pairwise commute and* $AC + BD = I$. *Then* A, B *are g-Kato if and only if* AB *is g-Kato.*

Theorem 3.3 (Spectral mapping theorem)**.** *Let* $T \in \mathcal{B}(X)$ *and*

$$\sigma_{gK}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not g-Kato } \}$$

denote the g-Kato spectrum of T. *Then for any function* f, *holomorphic on a neighbourhood* $\sigma(T)$ *and non-constant on any component,*

$$f(\sigma_{gK}(T)) = \sigma_{gK}(f(T)).$$