Perturbation of the Drazin inverse for matrices with equal eigenprojections at zero

N. Castro González [1]

Departamento de Matemática Aplicada, Facultad de Informática, Universidad Politécnica de Madrid, Spain

J. J. Koliha [2]

Department of Mathematics and Statistics, University of Melbourne, Australia

Yimin Wei [3] [4]

Department of Mathematics, Fudan University, Shanghai, 200433, P. R. of China

Abstract

Let A^π denote the eigenprojection of a matrix A corresponding to the eigenvalue 0. We characterize matrices B such that $B^\pi = A^\pi$, and derive from the results (1) error bounds for the Drazin inverse of a perturbation, (2) improvement of error bounds given recently by Wei; Wei and Wang; Castro, Koliha and Straškraba, and (3) a new characterization of EP matrices.

Key words: Eigenprojection; Drazin inverse; Perturbation bounds; EP matrices

1 Introduction

Let $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\sigma(A)$ and $r(A)$ denote the range, nullspace, spectrum and spectral radius of $A \in \mathbb{C}^{d\times d}$. The *index* of $A \in \mathbb{C}^{d\times d}$ is the least nonnegative integer $k = \operatorname{ind}(A)$ such that $\mathcal{N}(A^k) = \mathcal{N}(A^{k+1})$; then $\mathbb{C}^d = \mathcal{N}(A^k) \oplus \mathcal{R}(A^k)$ [2, Chap. 7], and the *eigenprojection* A^π of A (corresponding to the eigenvalue 0) is the uniquely determined idempotent matrix with

$$\mathcal{R}(A^\pi) = \mathcal{N}(A^k) \quad \text{and} \quad \mathcal{N}(A^\pi) = \mathcal{R}(A^k). \tag{1.1}$$

[1] E-mail: nieves@fi.upm.es
[2] E-mail: j.koliha@ms.unimelb.edu.au
[3] E-mail: ymwei@fudan.edu.cn
[4] Project 19901006 supported by National Natural Science Foundation of China.

The following characterization of eigenprojections is a consequence of [6, Theorem 2.3].

Lemma 1.1 *Let A, $P \in \mathbb{C}^{d\times d}$. Then $P = A^\pi$ if and only if*

$$P^2 = P, \quad AP = PA, \quad \sigma(AP) = \{0\}, \quad A + P \text{ is nonsingular.} \tag{1.2}$$

By [2, Theorem 7.2.1], for each matrix $A \in \mathbb{C}^{d\times d}$ there exists a *core-nilpotent* (*block*) *form*

$$\begin{gathered} A = Q^{-1}\mathrm{diag}\,(A_1, A_2)Q, \\ A_1 \in \mathbb{C}^{m\times m} \text{ nonsingular}, \quad A_2 \text{ nilpotent}, \quad 0 \le m \le d. \end{gathered} \tag{1.3}$$

Either block A_i may be empty. Using (1.2) it is easy to verify that $A^\pi = Q^{-1}\mathrm{diag}\,(0, I)Q$.

Relative to the form (1.3), the *core part* C_A and the *nilpotent part* N_A of A are defined by

$$C_A = Q^{-1}\mathrm{diag}\,(A_1, 0)Q, \quad N_A = Q^{-1}\mathrm{diag}\,(0, A_2)Q, \tag{1.4}$$

respectively. The *Drazin inverse* of A is defined by

$$A^{\mathrm{D}} = Q^{-1}\mathrm{diag}\,(A_1^{-1}, 0)Q. \tag{1.5}$$

To prove that C_A, N_A and A^{D} are well defined, verify the equations

$$\begin{aligned} C_A &= Q^{-1}\mathrm{diag}\,(A_1, 0)Q = A(I - A^\pi), \\ N_A &= Q^{-1}\mathrm{diag}\,(0, A_2)Q = AA^\pi, \\ A^{\mathrm{D}} &= Q^{-1}\mathrm{diag}\,(A_1^{-1}, 0)Q = (A + A^\pi)^{-1}(I - A^\pi) = (I - A^\pi)(A + A^\pi)^{-1}. \end{aligned} \tag{1.6}$$

By (1.1) and (1.6),

$$\mathcal{R}(A^{\mathrm{D}}) = \mathcal{R}(I - A^\pi) = \mathcal{R}(A^k), \quad \mathcal{N}(A^{\mathrm{D}}) = \mathcal{N}(I - A^\pi) = \mathcal{N}(A^k), \tag{1.7}$$

where $k = \mathrm{ind}\,(A)$.

Convention 1.2 Throughout this paper $A \in \mathbb{C}^{d\times d}$ and the notations Q, A_i and m have the meaning given them in (1.3). Further, for any $B \in \mathbb{C}^{d\times d}$, B_{ij} for $i, j \in \{1, 2\}$ denote the blocks such that $B = Q^{-1}(B_{ij})Q$ with $B_{11} \in \mathbb{C}^{m\times m}$; B_i will stand for B_{ii}.

The main purpose of our paper is to determine when $A^\pi = B^\pi$ and apply these conditions to perturbations of the Drazin inverse and related topics.

2 Characterization of matrices with the same eigenprojection

Here we characterize matrices $B \in \mathbb{C}^{d\times d}$ with $B^\pi = A^\pi$.

Theorem 2.1 *Let $A \in \mathbb{C}^{d\times d}$. The following conditions on $B \in \mathbb{C}^{d\times d}$ are equivalent:*

(i) $A^\pi = B^\pi$;

(ii) $A^\pi B = BA^\pi$, BA^π *is nilpotent and* $B + A^\pi$ *is nonsingular*;

(iii) B_1 *is invertible,* B_2 *is nilpotent and* $B_{ij} = 0$ *for* $i \neq j$;

(iv) $I + A^{\mathrm{D}}(B - A)$ *is nonsingular,* $A^\pi B = BA^\pi$ *and* BA^π *is nilpotent*;

(v) $B^{\mathrm{D}} = (I + A^{\mathrm{D}}(B - A))^{-1}A^{\mathrm{D}}$;

(vi) $B^{\mathrm{D}} - A^{\mathrm{D}} = A^{\mathrm{D}}(A - B)B^{\mathrm{D}}$.

Proof We observe that $A^\pi B = BA^\pi$ implies

$$B = Q^{-1}\mathrm{diag}\,(B_1, B_2)Q. \tag{2.1}$$

Indeed, from $A^\pi = Q^{-1}\mathrm{diag}\,(0, I)Q$,

$$\begin{bmatrix} 0 & 0 \\ B_{21} & B_2 \end{bmatrix} = A^\pi B = BA^\pi = \begin{bmatrix} 0 & B_{12} \\ 0 & B_2 \end{bmatrix},$$

that is, $B_{12} = 0 = B_{21}$.

In view of Lemma 1.1, (i) and (ii) are equivalent.

(ii) $\Longrightarrow$ (iii) B_1 is nonsingular because $B + A^\pi = Q^{-1}\mathrm{diag}\,(B_1, B_2 + I)Q$ is, and B_2 is nilpotent since $BA^\pi = Q^{-1}\mathrm{diag}\,(0, B_2)Q$ is.

(iii) $\Longrightarrow$ (iv) A calculation gives

$$I + A^{\mathrm{D}}(B - A) = Q^{-1}\mathrm{diag}\,(A_1^{-1}B_1, I)Q, \tag{2.2}$$

which is nonsingular. The remaining two equations are easily verified.

(iv) $\Longrightarrow$ (v) B_1 is nonsingular because the left hand side of (2.2) is assumed to be. Since $BA^\pi = Q^{-1}\mathrm{diag}\,(0, B_2)Q$ is nilpotent, so is B_2. Hence (2.1) is a core-nilpotent form for B, and

$$
\begin{aligned}
(I + A^{\mathrm{D}}(B - A))B^{\mathrm{D}} &= Q^{-1}\mathrm{diag}\,(A_1^{-1}B_1, I)\,\mathrm{diag}\,(B_1^{-1}, 0)Q \\
&= Q^{-1}\mathrm{diag}\,(A_1^{-1}, 0)Q = A^{\mathrm{D}}.
\end{aligned}
$$

(v) $\Longrightarrow$ (vi) is verified by a calculation.

(vi) $\Longrightarrow$ (i) From $A^{\mathrm{D}} = (I - A^{\mathrm{D}}(A - B))B^{\mathrm{D}}$ we have $\mathcal{N}(B^{\mathrm{D}}) \subset \mathcal{N}(A^{\mathrm{D}})$, and $B^{\mathrm{D}} = A^{\mathrm{D}}(I + (A - B)B^{\mathrm{D}})$ implies $\mathcal{R}(B^{\mathrm{D}}) \subset \mathcal{R}(A^{\mathrm{D}})$. Then

$$
d = \mathrm{nullity}\,(B^{\mathrm{D}}) + \mathrm{rank}\,(B^{\mathrm{D}}) \le \mathrm{nullity}\,(A^{\mathrm{D}}) + \mathrm{rank}\,(A^{\mathrm{D}}) = d.
$$

Hence $\mathcal{N}(A^{\mathrm{D}}) = \mathcal{N}(B^{\mathrm{D}})$, $\mathcal{R}(A^{\mathrm{D}}) = \mathcal{R}(B^{\mathrm{D}})$. By (1.7) and (1.1), $A^\pi = B^\pi$. $\square$

Remark 2.2 It is not hard to show that conditions (iv)–(vi) of the preceding theorem have the following equivalent versions:

(iv)$'$ $I + (B - A)A^{\mathrm{D}}$ is nonsingular, $A^\pi B = BA^\pi$ and BA^π is nilpotent;

(v)$'$ $B^{\mathrm{D}} = A^{\mathrm{D}}(I + (B - A)A^{\mathrm{D}})^{-1}$;

(vi)$'$ $B^{\mathrm{D}} - A^{\mathrm{D}} = B^{\mathrm{D}}(A - B)A^{\mathrm{D}}$.

Now we specialize to the case where A and B have the same nilpotent part. This is an important special case in applications. The rank condition (vii) of the following corollary can often be verified directly.

Corollary 2.3 *Let $A \in \mathbb{C}^{d\times d}$. The following conditions on $B \in \mathbb{C}^{d\times d}$ satisfying $A^\pi B = A^\pi A$ are equivalent*:

(i) $A^\pi = B^\pi$;

(ii) $A^\pi B = BA^\pi$ *and* $B + A^\pi$ *is nonsingular*;

(iii) B_1 *is invertible,* $B_2 = A_2$ *and* $B_{ij} = 0$ *for* $i \ne j$;

(iv) $A^\pi B = BA^\pi$ *and* $I + A^{\mathrm{D}}(B - A)$ *is nonsingular*;

(v) $B^{\mathrm{D}} = (I + A^{\mathrm{D}}(B - A))^{-1}A^{\mathrm{D}}$;

(vi) $B^{\mathrm{D}} - A^{\mathrm{D}} = A^{\mathrm{D}}(A - B)B^{\mathrm{D}}$;

(vii) $A^\pi B = BA^\pi$ *and* $\mathrm{rank}\,(A) = \mathrm{rank}\,(B)$.

If $A^\pi B = A^\pi A$ and any one of (i)–(vii) *holds, then $N_A = N_B$ and* $\mathrm{ind}\,(A) = \mathrm{ind}\,(B)$.

Proof By the equivalence of (i), (iii), (v) and (vi) in Theorem 2.1, $A^\pi B = BA^\pi$ may be viewed, at least implicitly, as part of each (i)–(vii). Then $BA^\pi =$

$A^\pi B = A^\pi A$, which is nilpotent, so the condition that BA^π is nilpotent is at least implicitly part of each (i)–(vii). Hence the conditions (i)–(vi) are equivalent by Theorem 2.1. In all cases

$$B = Q^{-1}\mathrm{diag}\,(B_1, A_2)Q. \tag{2.3}$$

From (i) we know that B_1 nonsingular, so (vii) holds. The rank condition in (vii) implies B_1 nonsingular, so (i) holds. To prove the last sentence: $\mathrm{ind}\,(A) = \mathrm{ind}\,(A_2) = \mathrm{ind}\,(B)$. □

3 Perturbation of the Drazin inverse

The continuity of the Drazin inverse without explicit error bounds was studied in [1–5,8–11].

It is known that if A and A_n are matrices in $\mathbb{C}^{d\times d}$ such that $A_n \to A$, then

$$A_n^{\mathrm{D}} \to A^{\mathrm{D}} \iff A_n^\pi \to A^\pi. \tag{3.1}$$

Numerical estimates for $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|$ have been obtained in the case that a matrix B has a core-nilpotent form

$$B = Q^{-1}\mathrm{diag}\,(A_1 + E_1, A_2 + E_2)Q, \tag{3.2}$$

where either $E_2 = 0$ (see [10,11]) or $A_2E_2 = E_2A_2$ and E_2 is nilpotent (see [4]). In this section we derive error estimates for $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|$ whenever $A^\pi = B^\pi$, that is, whenever $A_1 + E_1$ is invertible and $A_2 + E_2$ is nilpotent.

From this point on we assume that $\|\cdot\|$ is a matrix norm in $\mathbb{C}^{d\times d}$ (which need not satisfy $\|I\| = 1$). Thus $\|A + B\| \le \|A\| + \|B\|$, $\|-A\| = \|A\|$ and $\|AB\| \le \|A\|\|B\|$ for any A, $B \in \mathbb{C}^{d\times d}$.

Theorem 3.1 *Let A, $B \in \mathbb{C}^{d\times d}$ with $A^\pi = B^\pi$. Then*

$$\frac{\|A^{\mathrm{D}}\|}{1 + \|A^{\mathrm{D}}(B - A)\|} \le \|B^{\mathrm{D}}\|. \tag{3.3}$$

If $\|A^{\mathrm{D}}(B - A)\| < 1$, then

$$\|B^{\mathrm{D}}\| \le \frac{\|A^{\mathrm{D}}\|}{1 - \|A^{\mathrm{D}}(B - A)\|} \tag{3.4}$$

and

$$\frac{\|B^{\mathrm{D}} - A^{\mathrm{D}}\|}{\|A^{\mathrm{D}}\|} \le \frac{\|A^{\mathrm{D}}(B - A)\|}{1 - \|A^{\mathrm{D}}(B - A)\|}. \tag{3.5}$$

Proof By Theorem 2.1 (vi), $B^{\mathrm{D}} - A^{\mathrm{D}} = A^{\mathrm{D}}(A-B)B^{\mathrm{D}}$. Apply the norm to $A^{\mathrm{D}} = B^{\mathrm{D}} + A^{\mathrm{D}}(B-A)B^{\mathrm{D}}$ to obtain (3.3). Let $\|A^{\mathrm{D}}(A-B)\| < 1$. Apply the norm to $B^{\mathrm{D}} = A^{\mathrm{D}} + A^{\mathrm{D}}(A-B)B^{\mathrm{D}}$ to obtain $\|B^{\mathrm{D}}\| \le \|A^{\mathrm{D}}\| + \|A^{\mathrm{D}}(A-B)\|\|B^{\mathrm{D}}\|$, and consequently (3.4). Finally apply the norm to $B^{\mathrm{D}} - A^{\mathrm{D}} = A^{\mathrm{D}}(A-B)(A^{\mathrm{D}} + (B^{\mathrm{D}} - A^{\mathrm{D}}))$ to first obtain

$$\|B^{\mathrm{D}} - A^{\mathrm{D}}\| \le \|A^{\mathrm{D}}(B-A)\|(\|A^{\mathrm{D}}\| + \|B^{\mathrm{D}} - A^{\mathrm{D}}\|)$$

and then (3.5). □

Remark 3.2 Of course $A^{\pi} = B^{\pi}$ could be replaced with any of the equivalent formulations discussed above.

Remark 3.3 If $\|A^{\mathrm{D}}\|\|B-A\| < 1$ in the preceding theorem, then (3.5) becomes

$$\frac{\|B^{\mathrm{D}} - A^{\mathrm{D}}\|}{\|A^{\mathrm{D}}\|} \le \frac{\|A^{\mathrm{D}}(B-A)\|}{1 - \|A^{\mathrm{D}}(B-A)\|} \le \frac{\kappa_{\mathrm{D}}(A)\Delta}{1 - \kappa_{\mathrm{D}}(A)\Delta},$$

where $\kappa_{\mathrm{D}}(A) = \|A^{\mathrm{D}}\|\|A\|$ is the Drazin condition number of A and where $\Delta = \|B-A\|/\|A\|$.

Remark 3.4 In [4, Theorem 2.3], (3.5) was proved assuming $E = B - A$ satisfies

$$\|A^{\mathrm{D}}E\| < 1, \tag{3.6}$$
$$AA^{\mathrm{D}}E = EAA^{\mathrm{D}}, \tag{3.7}$$
$$E^k = AA^{\mathrm{D}}E^k \text{ for some } k \in \mathbb{N}, \tag{3.8}$$
$$AE(I - AA^{\mathrm{D}}) = EA(I - AA^{\mathrm{D}}). \tag{3.9}$$

Conditions (3.7)–(3.9) say respectively $E_{ij} = 0$ for $i \ne j$, E_2 is nilpotent of order k, and $A_2E_2 = E_2A_2$. Then B has the form (3.2) with $A_2 + E_2$ nilpotent. (3.6) implies $I + A^{\mathrm{D}}E$ invertible. Then (2.2) shows $B_1 = A_1 + E_1$ is invertible. Hence $A^{\pi} = B^{\pi}$ and Theorem 3.1 applies giving [4, Theorem 2.3].

Remark 3.5 The first part of

$$AA^{\mathrm{D}}EAA^{\mathrm{D}} = E \quad \text{and} \quad \|A^{\mathrm{D}}E\| < 1 \tag{$\mathcal{W}$}$$

says that E_{12}, E_{21} and E_2 are 0. Hence, as in Remark 3.4, $B_1 = A_1 + E_1$ is invertible, and Theorem 3.1 applies. So Theorems 2.1 and 3.1 of [11] follow from our Theorem 3.1.

4 Bounds for $\|B^\sharp\|$, $\|B^\sharp B\|$, $\|B^\sharp - A^{\mathrm{D}}\|$ and $\|B^\sharp B - A^{\mathrm{D}}A\|$

Here we show that the norm bounds proved in [10] by assuming that $\operatorname{ind}(A) = 1$ are also valid when $\operatorname{ind}(A)$ is not 1. We recall that A^{D} is called the *group inverse* if $\operatorname{ind}(A) \leq 1$; it is usually denoted by $A^\sharp$. First we generalize [10, Lemma 2.3].

Lemma 4.1 *If $A \in \mathbb{C}^{d\times d}$ is singular, then*

$$\|A^{\mathrm{D}}A\| = \|I - A^{\mathrm{D}}A\|,$$

where $\|A\| = r(A^{\mathrm{H}}A)^{1/2}$ is the spectral norm of A.

Proof Let $k = \operatorname{ind}(A)$ and $B = A^k$. Then $\operatorname{ind}(B) = 1$ since A is singular, so by [10, Lemma 2.3] $\|B^\sharp B\| = \|I - B^\sharp B\|$. Since $A^{\mathrm{D}}A = (A^{\mathrm{D}}A)^k = B^\sharp B$, we are done. □

The next result is a generalization of [10, Theorem 4.2].

Theorem 4.2 *Let $\|\cdot\|$ be the spectral norm and let $A \in \mathbb{C}^{d\times d}$ be singular with $\operatorname{ind}(A) = k$. Let B satisfy $\operatorname{ind}(B) \leq 1$, $\operatorname{rank}(B) = \operatorname{rank}(A^k)$, and let $\|A^{\mathrm{D}}\|\|B - A\| < (1 + \|A^{\mathrm{D}}A\|)^{-1}$. Then, writing $E = B - A$,*

$$\|B^\sharp\| \leq \|A^{\mathrm{D}}\| \frac{1 - \|A^{\mathrm{D}}\|\|E\|}{(1 - \|A^{\mathrm{D}}\|\|E\| - \|A^{\mathrm{D}}\|\|E\|\|AA^{\mathrm{D}}\|)^2}, \tag{4.1}$$

and

$$\|B^\sharp B\| \leq \|A^{\mathrm{D}}A\| \frac{1 - \|A^{\mathrm{D}}\|\|E\| + \|A^{\mathrm{D}}\|\|E\|\|AA^{\mathrm{D}}\|}{1 - \|A^{\mathrm{D}}\|\|E\| - \|A^{\mathrm{D}}\|\|E\|\|AA^{\mathrm{D}}\|}. \tag{4.2}$$

Proof First we need to verify the formulae for $B^\sharp$ and $BB^\sharp$ given in [10, Theorem 4.1] under the assumptions of our theorem when $A^\sharp$ is replaced everywhere by A^{D}. For this we modify the matrix C defined in the proof of [10, Theorem 4.1] by writing

$$C = P^{-1}AP = \operatorname{diag}(D, N),$$

where D is invertible and N is nilpotent. Then all the calculations in the proof of [10, Theorem 4.1] carry through with modifications to account for the change in C. In order that the formula for $(C + F)^{\mathrm{D}}$ can be expressed in the form required in [10, Lemma 2.2], we need the assumption $\operatorname{ind}(C + F) = \operatorname{ind}(B) = 1$, where $F = P^{-1}EP$. Then the formulae of [10, Theorem 4.1] hold with the modifications described above.

We then proceed as in the proof of [10, Theorem 4.2] replacing $A^\sharp$ everywhere by A^{D}, and employing our Lemma 4.1 instead of [10, Lemma 2.3]. The result then follows. □

Example 4.3 We display a class of matrices A and B to which Theorem 4.2 applies but [10, Theorem 4.2] does not because $\mathrm{ind}\,(A) > 1$. Let $0 < m < d-1$, let $M \in \mathbb{C}^{m\times m}$ be nonsingular and of (spectral) norm 1 with the smallest singular value $4p > 0$, and let $N \in \mathbb{C}^{(d-m)\times(d-m)}$ be nilpotent of index $k > 1$ and norm 1. Set $A = \mathrm{diag}\,(M, pN)$ and $B = \mathrm{diag}\,((1+p)M, 0)$. These are core-nilpotent forms of A and B. Then A is singular, $\mathrm{ind}\,(B) = 1$, $\mathrm{rank}\,(B) = \mathrm{rank}\,(\mathrm{diag}\,(M^k, 0)) = \mathrm{rank}\,(A^k)$, $\|A^{\mathrm{D}}\| = \|\mathrm{diag}\,(M^{-1}, 0)\| = 1/(4p)$ (the spectral norm is the largest singular value), $\|B - A\| = \|\mathrm{diag}\,(pM, -pN)\| = p$. Hence $\|A^{\mathrm{D}}\|\|B - A\| = 1/4 < 1/2 = (1 + \|A^{\mathrm{D}}A\|)^{-1}$, satisfying the last of the hypotheses of Theorem 4.2.

The following corollary generalizes [10, Theorem 4.5], and is obtained by a similar argument as the preceding theorem.

Corollary 4.4 *Under the assumptions of the preceding theorem,*

$$\begin{aligned}\frac{\|B^\sharp - A^{\mathrm{D}}\|}{\|A^{\mathrm{D}}\|} \le\ & \frac{1 - \|A^{\mathrm{D}}\|\|E\| + \|A^{\mathrm{D}}\|\|E\|\|AA^{\mathrm{D}}\|}{(1 - \|A^{\mathrm{D}}\|\|E\| - \|A^{\mathrm{D}}\|\|E\|\|AA^{\mathrm{D}}\|)^4} \\ & \times \Big\{(1 - \|A^{\mathrm{D}}\|\|E\| - \|A^{\mathrm{D}}\|\|E\|\|AA^{\mathrm{D}}\|)^2 \\ & + \|AA^{\mathrm{D}}\|(1 + (1 - \|A^{\mathrm{D}}\|\|E\| - \|A^{\mathrm{D}}\|\|E\|\|AA^{\mathrm{D}}\|)^3)\Big\} \end{aligned} \tag{4.3}$$

and

$$\frac{\|B^\sharp B - A^{\mathrm{D}}A\|}{\|A^{\mathrm{D}}A\|} \le \frac{\|A^{\mathrm{D}}\|\|E\|(1 - \|A^{\mathrm{D}}\|\|E\|)(2 - \|A^{\mathrm{D}}\|\|E\|)}{(1 - \|A^{\mathrm{D}}\|\|E\| - \|A^{\mathrm{D}}\|\|E\|\|A^{\mathrm{D}}A\|)^2}. \tag{4.4}$$

The upper bounds obtained in this section generalize for unrestricted $\mathrm{ind}\,(A)$ a result of Campbell and Meyer [3] which Wei [10] extended assuming that $\mathrm{ind}\,(A) = 1$. The formulae for $B^\sharp$ and $B^\sharp B$ that we obtained as modifications of the corresponding formulae in [10, Theorem 4.1] further generalize Meyer's results [7].

5 EP matrices

A matrix $A \in \mathbb{C}^{d\times d}$ is called an EP matrix if $\mathcal{R}(A) = \mathcal{R}(A^{\mathrm{H}})$. According to [2, Theorem 4.3.1], A is EP if and only if in the core-nilpotent form (1.3), Q can be unitary and $A_2 = 0$. From this we can deduce that A is EP if and only if

$$AA^{\pi} = 0 \quad \text{and} \quad (A^{\pi})^{\mathrm{H}} = A^{\pi}. \tag{5.1}$$

We note that $AA^{\pi} = 0$ is equivalent to $\mathrm{ind}\,(A) \leq 1$.

EP matrices are of interest since they commute with their Moore–Penrose inverses [2, Definition 4.3.1 and Theorem 4.3.1]. This property is the key for the following result which gives a condition ensuring that a perturbation of an EP matrix is again EP, and establishes error bounds for the Moore–Penrose inverse of the perturbation.

Theorem 5.1 *Let $A \in \mathbb{C}^{d\times d}$ be an* EP *matrix and let $B = A + E$, where E satisfies condition $(\mathcal{W})$. Then B is also* EP, *and*

$$\frac{\|B^{\dagger} - A^{\dagger}\|}{\|A^{\dagger}\|} \leq \frac{\|A^{\dagger}E\|}{1 - \|A^{\dagger}E\|}.$$

Proof By Remark 3.5 the inequality is true for Drazin inverses and B satisfies (3.2) with $E_2 = 0$ and $A_1 + E_1$ invertible. Since A is EP, Q is unitary and $B_2 = A_2 = 0$. So B is EP also and since Drazin and Moore–Penrose inverses coincide for EP matrices by [2, Theorem 7.3.4], we are done. □

Here is a new characterization of EP matrices:

Theorem 5.2 *Let $A \in \mathbb{C}^{d\times d}$. Then the following conditions are equivalent*:

(i) *A is* EP;

(ii) $(A^{\mathrm{H}})^{\mathrm{D}}A^{\mathrm{H}} = A^{\mathrm{D}}A$ *and* $A^{\mathrm{D}}AA^{\mathrm{H}} = A^{\mathrm{H}}$;

(iii) $A^{\mathrm{D}}AA^{\mathrm{H}} = A^{\mathrm{H}} = A^{\mathrm{H}}A^{\mathrm{D}}A$ *and* $I + A^{\mathrm{H}} - A^{\mathrm{D}}A$ *is nonsingular*;

(iv) $A^{\mathrm{D}}AA^{\mathrm{H}} = A^{\mathrm{H}} = A^{\mathrm{H}}A^{\mathrm{D}}A$ *and* $I + A^{\mathrm{D}}A^{\mathrm{H}} - A^{\mathrm{D}}A$ *is nonsingular.*

Proof We show that conditions (i)–(iv) of the theorem can be rewritten in the following equivalent forms:

(i)′ $AA^{\pi} = 0$ and $(A^{\pi})^{\mathrm{H}} = A^{\pi}$;

(ii)′ $(A^{\mathrm{H}})^{\pi} = A^{\pi}$ and $A^{\pi}A^{\mathrm{H}} = 0 = A^{\mathrm{H}}A^{\pi}$;

(iii)′ $A^{\pi}A^{\mathrm{H}} = 0 = A^{\mathrm{H}}A^{\pi}$ and $A^{\mathrm{H}} + A^{\pi}$ is nonsingular;

(iv)′ $I + A^{\mathrm{D}}(A^{\mathrm{H}} - A)$ is nonsingular and $A^{\pi}A^{\mathrm{H}} = 0 = A^{\mathrm{H}}A^{\pi}$.

Equivalence of (i) and (i)′ is (5.1).

Note that we have $(A^\pi)^{\mathrm{H}} = (A^{\mathrm{H}})^\pi$, $(A^{\mathrm{D}})^{\mathrm{H}} = (A^{\mathrm{H}})^{\mathrm{D}}$, $A^\pi = I - A^{\mathrm{D}}A$. From (ii) we get

$$\begin{aligned}(A^{\mathrm{H}})^\pi = (A^\pi)^{\mathrm{H}} = (I - AA^{\mathrm{D}})^{\mathrm{H}} = I - (A^{\mathrm{H}})^{\mathrm{D}}A^{\mathrm{H}} = I - A^{\mathrm{D}}A = A^\pi,\\ A^\pi A^{\mathrm{H}} = (I - A^{\mathrm{D}}A)A^{\mathrm{H}} = A^{\mathrm{H}} - A^{\mathrm{D}}AA^{\mathrm{H}} = 0\end{aligned}$$

and

$$\begin{aligned}A^{\mathrm{H}}A^\pi &= A^{\mathrm{H}} - A^{\mathrm{H}}A^{\mathrm{D}}A = A^{\mathrm{H}} - A^{\mathrm{H}}(A^{\mathrm{H}})^{\mathrm{D}}A^{\mathrm{H}}\\ &= A^{\mathrm{H}} - (A^{\mathrm{H}})^{\mathrm{D}}A^{\mathrm{H}}A^{\mathrm{H}} = A^{\mathrm{H}} - A^{\mathrm{D}}AA^{\mathrm{H}} = A^\pi A^{\mathrm{H}} = 0.\end{aligned}$$

Thus (ii) implies (ii)′. The reverse implication and the equivalence of (iii) with (iii)′ and (iv) with (iv)′ follow easily from $B^\pi = I - B^{\mathrm{D}}B$. Then we apply Theorem 2.1 (i)–(iv) with $B = A^{\mathrm{H}}$ to conclude that conditions (i)–(iv) of the present theorem are equivalent. □

Conditions (v) and (vi) of Theorem 2.1 can be also used to produce new characterizations of EP matrices; we do not include them as they appear more technical.

Acknowledgements

The authors are indebted to the referee whose meticulous reading of the paper and detailed comments immeasurably improved the presentation. Condition (vi) of Theorem 2.1 is due to the referee, as is the current economical modification of Example 4.3.

References

[1] S. L. Campbell, Continuity of the Drazin inverse, Linear and Multilinear Algebra 8 (1980), 265–268.

[2] S. L. Campbell and C. D. Meyer, Generalized Inverses of Linear Transformations, Pitman, London, 1979.

[3] S. L. Campbell and C. D. Meyer, Continuity properties of the Drazin pseudoinverse, Linear Algebra Appl. 10 (1975), 77–83.

[4] N. Castro González, J. J. Koliha and I. Straškraba, Perturbation of the Drazin inverse, preprint, 1999.

[5] J. J. Koliha and V. Rakočević, Continuity of the Drazin inverse II, Studia Math. 131 (1998), 167–177.

[6] J. J. Koliha and I. Straškraba, Power bounded and exponentially bounded matrices, Appl. Math. 44 (1999), 289–308.

[7] C. D. Meyer, The condition number of a finite Markov chain and perturbation bounds for the limiting probabilities, SIAM J. Algebraic Discrete Methods 1 (1980), 273–283.

[8] V. Rakočević, Continuity of the Drazin inverse, J. Operator Theory 41 (1999), 55–68.

[9] G. Rong, The error bounds for the perturbation of the Drazin inverse, Linear Algebra Appl. 47 (1982), 159–168.

[10] Y.Wei, On the perturbation of the group inverse and oblique projections, Appl. Math. Comput. 98 (1999), 29–42.

[11] Y. Wei and G. Wang, The perturbation theory for the Drazin inverse and its applications, Linear Algebra Appl. 258 (1997), 179–186.