Invertibility of the difference of idempotents

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Abstract

We study conditions equivalent to the invertibility of $f-g$ when f and g are idempotents in a unital ring, and give applications to bounded linear operators in Banach and Hilbert spaces. In the setting of rings we are able to show that many conditions previously linked to finite dimensionality, rank equalities, norm topology of bounded linear operators or to properties of C^*-algebras can be in fact proved by simple algebraic arguments.

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1 Introduction

In this paper we find several equivalent conditions that ensure the invertibility of $f-g$, where f, g are general idempotents in a unital ring, and idempotent operators on a Banach or Hilbert space. The question of the invertibility of $P-Q$, where P, Q are idempotent hermitian matrices or, more generally, hermitian idempotent operators on a Hilbert space H, is of great interest in operator theory as it is connected with the question of when the space H is the direct sum $H = R(P) \oplus R(Q)$ of the ranges, and with the existence of an idempotent operator F satisfying the equations

$$PF = F, \quad FP = P, \quad Q(I-F) = I-F, \quad (I-F)Q = Q.$$

These problems were considered by several mathematicians, for instance Ljance [10], Pták [13] and Vidav [15], and were recently revisited by Buckholtz in [2, 3], Wimmer

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[16, 17], Rakočević in [14] in the setting of Hilbert spaces, and by Koliha [8] and the present authors [9] in the setting of C^*-algebras.

In most of the above papers the proofs rely on fairly involved spatial arguments in Hilbert spaces, on properties of the norm and involution, and on spectral theory in C^*-algebras. In [4], Gross and Trenkler consider the invertibility of $P - Q$ for general matrix idempotents. Their methods rely strongly on finite dimensionality of the underlying space, and on the complex theory of equalities for the ranks of matrices developed by Marsaglia and Styan [12].

In contrast to the works cited above, our paper initially uses only techniques based in algebra. Although the problem has its origin in matrix and operator theory, we have opted for a purely algebraic setting of rings to obtain greater clarity, to provide a unified approach to this topic and to separate the properties dependent on topology from the ones dependent only on algebra. Such setting was previously adopted, for example, by Harte [6] to investigate quasinilpotents in rings without the concept of spectrum which is unavailable there.

In Sections 5 and 6 we turn to the original setting of matrices and operators in Banach and Hilbert spaces to obtain new results and to recover known results with greatly simplified proofs.

2 Preliminaries

For basic facts about rings we refer the reader to McCoy's book [11]. By $\mathcal{R}$ we denote an associative ring with unit $1 \neq 0$; the set of all invertible elements in $\mathcal{R}$ is written as $\mathcal{R}^{-1}$. An *idempotent* in $\mathcal{R}$ is an element h satisfying $h^2 = h$. If $\mathcal{R}$ has an involution $x \mapsto x^*$, an element $p \in \mathcal{R}$ which is both idempotent and hermitian ($p^2 = p = p^*$) is called a *projection*. With each element a of a unital ring $\mathcal{R}$ we associate two image ideals

$$a\mathcal{R} = \{ax : x \in \mathcal{R}\}, \qquad \mathcal{R}a = \{xa : x \in \mathcal{R}\} \tag{2.1}$$

and two kernel ideals

$$a^0 = \{x \in \mathcal{R} : ax = 0\}, \qquad {}^0a = \{x \in \mathcal{R} : xa = 0\}. \tag{2.2}$$

If $f,\, h \in \mathcal{R}$ are idempotents, then

$$f\mathcal{R} = h\mathcal{R} \iff {}^0f = {}^0h \iff (hf = f \text{ and } fh = h), \tag{2.3}$$

$$\mathcal{R}f = \mathcal{R}h \iff f^0 = h^0 \iff (hf = h \text{ and } fh = f), \tag{2.4}$$

$$f^0 = (1-f)\mathcal{R}, \qquad {}^0f = \mathcal{R}(1-f). \tag{2.5}$$

The next result relates direct sums of image and kernel ideals of idempotent elements in $\mathcal{R}$.

Lemma 2.1. *If $f,\, g \in \mathcal{R}$ are idempotents, then the following are equivalent*:

(i) *There exists an idempotent $h \in \mathcal{R}$ such that*

$$h\mathcal{R} = f\mathcal{R}, \quad (1-h)\mathcal{R} = g\mathcal{R}, \ \textit{that is,} \tag{2.6}$$

$$hf = f, \quad fh = h, \quad (1-h)g = g, \quad g(1-h) = 1-h; \tag{2.7}$$

h is unique if it exists.

(ii) $\mathcal{R} = f\mathcal{R} \oplus g\mathcal{R}$.

(iii) $\mathcal{R} = \mathcal{R}(1-f) \oplus \mathcal{R}(1-g)$.

(iv) $\mathcal{R} = {}^0f \oplus {}^0g$.

(v) $\mathcal{R} = (1-f)^0 \oplus (1-g)^0$.

Proof. (i) $\Longrightarrow$ (ii). Follows from $\mathcal{R} = h\mathcal{R} \oplus (1-h)\mathcal{R}$.

(ii) $\Longrightarrow$ (i). Suppose that $\mathcal{R} = f\mathcal{R} \oplus g\mathcal{R}$. Towards the existence of h we note that the unit element 1 is decomposed as $1 = fu + gv$ for some $u,\, v \in \mathcal{R}$. Let $h = fu$. Then $h = h^2 + fugv = h^2 + gvfu$, and $h - h^2 \in f\mathcal{R} \cap g\mathcal{R} = \{0\}$. Hence h is idempotent. From $f - hf = f(1-uf) = gvf \in f\mathcal{R} \cap g\mathcal{R}$ we conclude that $f - hf = 0$ and $hf = f$. Further, $fh = f^2u = fu = h$. This gives $h\mathcal{R} = f\mathcal{R}$. A similar argument with $1-h$ in place of h and g in place of f yields $(1-h)g = g$ and $g(1-h) = 1-h$, that is, $(1-h)\mathcal{R} = g\mathcal{R}$.

To prove the uniqueness of h assume that (2.7) is satisfied with h_1, h_2 in place of h. Then $h_1f = f$ implies $h_1fh_2 = fh_2$ and $h_1h_2 = h_2$. On the other hand, $h_1g = 0$ implies $h_1gh_2 = 0$, $h_1(g + h_2 - 1) = 0$ and $h_1h_2 = h_1$. Hence $h_1 = h_2$.

(i) $\iff$ (iii). By an argument similar to the one used above we show that condition (iii) is equivalent to the existence of an idempotent $k \in \mathcal{R}$ satisfying $\mathcal{R}k = \mathcal{R}(1-f)$ and $\mathcal{R}(1-k) = \mathcal{R}(1-g)$, that is,

$$k(1-f) = k, \quad (1-f)k = 1-f,$$
$$(1-k)(1-g) = 1-k, \quad (1-g)(1-k) = 1-g.$$

This is equivalent to (2.7) when we set $k = 1-h$.

The equivalence of (iv) with (iii) and of (v) with (ii) follows from (2.5). □

Regularity of elements in $\mathcal{R}$ plays an important role in our investigation. We say that $a \in \mathcal{R}$ is *regular* (in the sense of von Neumann) if $a \in a\mathcal{R}a$. The next result links regularity and (left, right) invertibility.

Lemma 2.2. *Let $a \in \mathcal{R}$. Then*:

(i) *a is left invertible in $\mathcal{R}$ $\iff$ ($a \in a\mathcal{R}a$ and $a^0 = \{0\}$).*

(ii) *a is right invertible in $\mathcal{R}$ $\iff$ ($a \in a\mathcal{R}a$ and ${}^0a = \{0\}$).*

(iii) *$a \in \mathcal{R}^{-1}$ $\iff$ ($a \in a\mathcal{R}a$ and $a^0 = \{0\} = {}^0a$).*

Proof. Let $a = aba$ for some $b \in \mathcal{R}$. If $a^0 = \{0\}$, then $a(1-ba) = 0$ implies $ba = 1$. Similarly, if ${}^0a = \{0\}$, then from $(1-ab)a = 0$ follows $ab = 1$.

The converse implications are proved similarly. □

For our future use we state the following known result on invertibility.

Lemma 2.3. *Let a, $b \in \mathcal{R}$. Then*

$$1 - ab \in \mathcal{R}^{-1} \iff 1 - ba \in \mathcal{R}^{-1}.$$

Proof. Let $1-ab$ be invertible in $\mathcal{R}$ with $u = (1-ab)^{-1}$. Then $abu = u-1$, and

$$\begin{aligned}(1-ba)(1+bua) &= 1 + bua - ba - b(abu)a \\ &= 1 + bua - ba - b(u-1)a \\ &= 1.\end{aligned}$$

Similarly we show that $(1+bua)(1-ba) = 1$. □

Our last preliminary result is often called the Atkinson Lemma, although to the best of our knowledge it was first obtained by Brown and McCoy [1, Lemma 1]. We give a proof to demonstrate the simplicity of the argument.

Lemma 2.4. *Let a, $b \in \mathcal{R}$. If $a - aba$ is regular, then so is a.*

Proof. If $a - aba$ is regular, then there exists $z \in \mathcal{R}$ such that

$$(a - aba)z(a - aba) = a - aba.$$

Hence

$$a = a - aba + aba = (a - aba)z(a - aba) + aba = axa$$

with $x = (1 - ba)z(1 - ab) + b$. □

3 Main results

In [9, Theorem 3.1], the present authors derived necessary and sufficient conditions for the invertibility of $p - q$, where p, q are nontrivial projections in a C^*-algebra. Stripping away the conditions of that theorem that explicitly involve norms of elements in a C^*-algebra, we can state the results for rings with involution, and ask whether they also holds in this setting. The answer is affirmative, but since the methods of proof in [9] are based on spectral theory of C^*-algebras and topological arguments, we need to devise new algebraic techniques to prove it for rings with involution. However, it is convenient to postpone the proof of such a theorem after we have dealt with idempotents in rings without involution.

The first step is the following observation.

Lemma 3.1. *Let f, g be idempotents in a unital ring $\mathcal{R}$. Then*

$$f - g \in \mathcal{R}^{-1} \implies 1 - fgf \in \mathcal{R}^{-1} \iff 1 - fg \in \mathcal{R}^{-1}. \tag{3.1}$$

Proof. Suppose that $f - g \in \mathcal{R}^{-1}$. Then the element $(f - g)^2$ is invertible in $\mathcal{R}$ and commutes with f:

$$f(f - g)^2 = f - fgf = (f - g)^2 f.$$

(The commutativity of $(f-g)^2$ with f and g is well known—see Kato [7, Chapter 1, § 4.6].) Writing $1-fgf = (1-fgf)f + (1-fgf)(1-f)$, we obtain

$$1 - fgf = (f-g)^2 f + 1 - f.$$

From this equation we deduce that $(f-g)^{-2}f + 1 - f$ is the inverse of $1-fgf$ in $\mathcal{R}$. (See also [5, Theorem 7.5.1].)

Applying Lemma 2.3 first with $a = fg$, $b = f$, and then with $a = f$, $b = gf$, we conclude that

$$1 - fgf \in \mathcal{R}^{-1} \iff 1 - fg \in \mathcal{R}^{-1} \iff 1 - gf \in \mathcal{R}^{-1}. \qquad \square$$

We can now state our main theorem.

Theorem 3.2. *Let f, g be idempotents in a unital ring $\mathcal{R}$. Then the following conditions are equivalent*:

(i) $f - g \in \mathcal{R}^{-1}$.

(ii) $\mathcal{R} = f\mathcal{R} \oplus g\mathcal{R}$ *and* $\mathcal{R} = \mathcal{R}f \oplus \mathcal{R}g$.

(iii) *There exist idempotents h, $k \in \mathcal{R}$ such that $h\mathcal{R} = f\mathcal{R}$, $(1-h)\mathcal{R} = g\mathcal{R}$, and $\mathcal{R}k = \mathcal{R}f$, $\mathcal{R}(1-k) = \mathcal{R}g$; h and k are unique if they exist.*

(iv) $1 - fg \in \mathcal{R}^{-1}$, $\mathcal{R} = f\mathcal{R} + g\mathcal{R}$ *and* $f^0 \cap g^0 = \{0\}$.

(v) $f + g - fg \in \mathcal{R}^{-1}$, $f\mathcal{R} \cap g\mathcal{R} = \{0\}$ *and* $\mathcal{R} = f^0 + g^0$.

(vi) $1 - fg \in \mathcal{R}^{-1}$ *and* $f + g - fg \in \mathcal{R}^{-1}$.

Proof. (i) $\Longrightarrow$ (vi). Suppose that $f - g \in \mathcal{R}^{-1}$. By Lemma 3.1, $1 - fg \in \mathcal{R}^{-1}$. Since $(1-f) - (1-g) = g - f \in \mathcal{R}^{-1}$, we also have $1 - (1-f)(1-g) = f + g - fg \in \mathcal{R}^{-1}$.

(vi) $\Longrightarrow$ (iv). Let $w \in \mathcal{R}$ be the inverse of $f + g - fg$. From $1 = (f+g-fg)w = f(1-g)w + gw$ we obtain $\mathcal{R} = f\mathcal{R} + g\mathcal{R}$, and from $1 = w(f+g-fg) = wf + w(1-f)g$ we conclude that $f^0 \cap g^0 = \{0\}$.

(iv) $\Longrightarrow$ (ii). Let $x \in f\mathcal{R} \cap g\mathcal{R}$. Then $x = fx = gx = fgx$ and $(1-fg)x = 0$; hence $x = 0$, and $f\mathcal{R} \cap g\mathcal{R} = \{0\}$. Consequently, $\mathcal{R} = f\mathcal{R} \oplus g\mathcal{R}$. Write $\widetilde{f} = 1 - f$

and $\widetilde{g} = 1 - g$. Then $\widetilde{f} + \widetilde{g} - \widetilde{f}\widetilde{g} = 1 - fg \in \mathcal{R}^{-1}$. As in the proof of (vi) $\Longrightarrow$ (iv) we show that $\mathcal{R} = \widetilde{f}\mathcal{R} + \widetilde{g}\mathcal{R} = f^0 + g^0$. Hence $\mathcal{R} = f^0 \oplus g^0 = \mathcal{R}f \oplus \mathcal{R}g$.

(ii) $\Longrightarrow$ (iii). This follows from Lemma 2.1.

(iii) $\Longrightarrow$ (i). From the equality of image ideals we get

$$hf = f, \quad fh = h, \quad (1-h)g = g, \quad g(1-h) = 1-h, \tag{3.2}$$

$$kf = k, \quad fk = f, \quad (1-k)g = 1-k, \quad g(1-k) = g. \tag{3.3}$$

Using these equations, we show that $h + k - 1$ is the inverse of $f - g$:

$$(h + k - 1)(f - g) = hf + kf - f - hg - kg + g = 1,$$
$$(f - g)(h + k - 1) = fh + fk - f - gh - gk + g = 1.$$

We note that (v) is obtained from (iv) by substituting $\widetilde{f} = 1 - f$ and $\widetilde{g} = 1 - g$ for f and g. Since $\widetilde{f} - \widetilde{g} = g - f$, $\widetilde{f} - \widetilde{g} \in \mathcal{R}^{-1}$ if and only if $f - g \in \mathcal{R}^{-1}$. Hence (v) is equivalent to (i). □

Corollary 3.3. *Let f, g be idempotents in a unital ring $\mathcal{R}$. If $f - g \in \mathcal{R}^{-1}$, then*

$$(f - g)^{-1} = h + k - 1, \tag{3.4}$$

$$h = (f - g)^{-1}(1 - g), \qquad k = (1 - g)(f - g)^{-1}. \tag{3.5}$$

where h and k are the idempotents defined in part (iii) *of the preceding theorem.*

Proof. The formula for $(f - g)^{-1}$ was derived in the implication (iii) $\Longrightarrow$ (i) of the preceding proof. To prove (3.5) we use (3.2) and (3.3):

$$(f - g)^{-1}(1 - g) = (h + k - 1)(1 - g) = h,$$
$$(1 - g)(f - g)^{-1} = (1 - g)(h + k - 1) = k.$$

□

Remark 3.4. Condition (i) is invariant under the reversal of the ring multiplication, and under the substitution of the complementary idempotents $1 - f$, $1 - g$ for f, g. By applying these two transformations (reversal and substitution) to the rest of Theorem 3.2, we may obtain new equivalent conditions.

• Conditions (i), (ii), (iii) and (vi) are invariant under the reversal of the ring multiplication. Precisely speaking, we apply each of these conditions to the ring $(\mathcal{R}, \circ)$, where $x \circ y = yx$, and then reinterpret the results in the original ring. Condition (i) is unchanged, in (ii) (and in (vi)) the two conditions get interchanged. In (iii), the roles of h and k get interchanged. Conditions (iv) and (v) yield two new equivalent conditions:

(iv)′ $1 - fg \in \mathcal{R}^{-1}$, $\mathcal{R} = \mathcal{R}f + \mathcal{R}g$ and ${}^0 f \cap {}^0 g = \{0\}$.

(v)′ $f + g - fg \in \mathcal{R}^{-1}$, $\mathcal{R}f \cap \mathcal{R}g = \{0\}$ and $\mathcal{R} = {}^0 f + {}^0 g$.

• Next we substitute the complementary idempotents $\widetilde{f} = 1 - f$, $\widetilde{g} = 1 - g$ for f, g. Taking into account Lemma 2.1, we observe that conditions (i), (ii), (iii) and (vi) are self-complementary, while (iv) and (v) are complementary to each other. No new conditions are obtained.

• It is often convenient to interpret various direct sums in light of Lemma 2.1; for instance, condition (ii) may take the form $\mathcal{R} = f^0 \oplus g^0$ and $\mathcal{R} = {}^0 f \oplus {}^0 g$. Let us also remark that the element $1 - fg$ in condition (iv) can be replaced by $1 - fgf$ or by $1 - gf$ in view of Lemma 3.1.

The direct decompositions in condition (ii) of Theorem 3.2 can be replaced by weaker assumptions, but then we have to assume the regularity of $f - g$.

Theorem 3.5. *Let f, g be idempotents in a unital ring $\mathcal{R}$. Then the following conditions are equivalent*:

(i) $f - g \in \mathcal{R}^{-1}$.

(ii) $f - g$ *is regular and*

$$f\mathcal{R} \cap g\mathcal{R} = \{0\}, \; f^0 \cap g^0 = \{0\}, \; \mathcal{R}f \cap \mathcal{R}g = \{0\}, \; {}^0 f \cap {}^0 g = \{0\}. \tag{3.6}$$

Proof. (i) $\Longrightarrow$ (ii). If $f - g \in \mathcal{R}^{-1}$, then $f - g$ is regular, and condition (ii) of Theorem 3.2 implies the range relations in (3.6). The kernel relations of (3.6) follow from the last part of Remark 3.4.

(ii) $\Longrightarrow$ (i). Suppose that (ii) holds. First we show that $(f - g)^0 = \{0\}$. Let $(f - g)x = 0$. Then $fx = gx \in f\mathcal{R} \cap g\mathcal{R} = \{0\}$, and $x \in f^0 \cap g^0 = \{0\}$. Hence

$x = 0$. From the regularity of $f - g$ and $(f-g)^0 = \{0\}$ it follows by Lemma 2.2 that $f - g$ is left invertible. By a symmetrical argument we conclude that $f - g$ is right invertible. □

We apply Theorem 3.2 to characterize the invertibility of the commutator of idempotents.

Corollary 3.6. *Let f, g be idempotents in a unital ring $\mathcal{R}$. Then the following conditions are equivalent*:

(i) $fg - gf \in \mathcal{R}^{-1}$.

(ii) $f - g \in \mathcal{R}^{-1}$ *and* $1 - f - g \in \mathcal{R}^{-1}$.

(iii) $\mathcal{R} = f\mathcal{R} \oplus g\mathcal{R} = \mathcal{R}f \oplus \mathcal{R}g = f^0 \oplus g\mathcal{R} = {}^0 f \oplus \mathcal{R}g$.

Proof. The equivalence of (i) and (ii) follows from

$$fg - gf = (1 - f - g)(f - g) = (f - g)(f + g - 1),$$

and the equivalence of (ii) and (iii) from Theorem 3.2 first applied to the pair f, g and then to the pair $1 - f$, g. □

4 Rings with involution

If the ring has an involution, there is a duality for the image and kernel ideals expressed in the following lemma, in which $S^* = \{x^* : x \in S\}$ for any $S \subset \mathcal{R}$.

Lemma 4.1. *Let $\mathcal{R}$ be a ring with involution. If a, $b \in \mathcal{R}$, then*

$$({}^0 a)^* = (a^*)^0, \qquad (a\mathcal{R})^* = \mathcal{R}a^*, \tag{4.1}$$

$$\mathcal{R} = a^*\mathcal{R} \oplus b^*\mathcal{R} \iff \mathcal{R} = \mathcal{R}a \oplus \mathcal{R}b. \tag{4.2}$$

Suppose now that $\mathcal{R}$ is a unital ring with an involution $x \mapsto x^*$. Theorem 3.2 specializes to the following.

Theorem 4.2. *Let f, g be idempotents in a unital ring $\mathcal{R}$ with involution. Then the following conditions are equivalent*:

(i) $f - g \in \mathcal{R}^{-1}$.

(ii) $\mathcal{R} = f\mathcal{R} \oplus g\mathcal{R}$ *and* $\mathcal{R} = f^*\mathcal{R} \oplus g^*\mathcal{R}$.

(iii) *There exist idempotents* $h, k \in \mathcal{R}$ *such that* $h\mathcal{R} = f\mathcal{R}$, $(1-h)\mathcal{R} = g\mathcal{R}$, *and* $k\mathcal{R} = f^*\mathcal{R}$, $(1-k)\mathcal{R} = g^*\mathcal{R}$; h *and* k *are unique if they exist.*

(iv) $1 - fg \in \mathcal{R}^{-1}$, $\mathcal{R} = f\mathcal{R} + g\mathcal{R}$ *and* $\mathcal{R} = f^*\mathcal{R} + g^*\mathcal{R}$.

(v) $f + g - fg \in \mathcal{R}^{-1}$, $\mathcal{R} = f^0 + g^0$ *and* $\mathcal{R} = (f^*)^0 + (g^*)^0$.

(vi) $1 - fg \in \mathcal{R}^{-1}$ *and* $f + g - fg \in \mathcal{R}^{-1}$.

If $f - g \in \mathcal{R}^{-1}$, *then*

$$(f-g)^{-1} = h + k^* - 1, \tag{4.3}$$

$$h = (f-g)^{-1}(1-g), \quad k = (f^* - g^*)^{-1}(1-g^*). \tag{4.4}$$

Proof. In view of Theorem 3.2 we only need to prove

(a) $\mathcal{R} = f^*\mathcal{R} + g^*\mathcal{R}$ implies $f^0 \cap g^0 = \{0\}$,

(b) condition (v) is complementary to (iv) in the sense of substituting the complementary idempotents $\widetilde{f} = 1 - f$, $\widetilde{g} = 1 - g$ for f, g,

(c) the formulae involving $(f-g)^{-1}$.

Towards (a) we note that $f^*u + g^*v = 1 = u^*f + v^*g$ for some $u, v \in \mathcal{R}$; if $x \in f^0 \cap g^0$, then $x = u^*fx + v^*gx = 0$. The complementarity of (iv) and (v) is easily verified, and (b) follows. To prove (c) we check that

$$k\mathcal{R} = f^*\mathcal{R} \text{ and } (1-k)\mathcal{R} = g^*\mathcal{R} \iff \mathcal{R}k^* = \mathcal{R}f \text{ and } \mathcal{R}(1-k^*) = \mathcal{R}g. \qquad \square$$

Remark 4.3. As in Remark 3.4 we may consider the effect of the reverse of multiplication, and the substitution of $1-f$, $1-g$ for the pair f, g on the preceding theorem. There si no change when we replace f, g by f^*, g^*.

• Under the reversal of multiplication in $\mathcal{R}$, (i) and (vi) are unchanged, in the remaining conditions the left image ideals are replaced by the right image idelas, and the right kernel ideals are replaced by the left kernel ideals.

• Under the substitution of the complementary idempotents, (iv) and (v) get interchanged, the rest is essentially invariant (taking into account Lemma 2.1).

Theorem 4.2 and Lemma 4.1 yield the following result for projections, that is, hermitian idempotents.

Theorem 4.4. *Let $\mathcal{R}$ be a ring with involution and let $p, q \in \mathcal{R}$ be projections. Then the following conditions are equivalent*:

(i) $p - q \in \mathcal{R}^{-1}$.

(ii) $\mathcal{R} = p\mathcal{R} \oplus q\mathcal{R}$ (*or* $\mathcal{R} = p^0 \oplus q^0$).

(iii) *There exists an idempotent* $h \in \mathcal{R}$ *such that* $h\mathcal{R} = p\mathcal{R}$ *and* $(1-h)\mathcal{R} = q\mathcal{R}$ (*or an idempotent* k *such that* $k\mathcal{R} = p^0$ *and* $(1-k)\mathcal{R} = q^0$); h *and* k *are unique if they exist.*

(iv) $1 - pq \in \mathcal{R}^{-1}$ *and* $\mathcal{R} = p\mathcal{R} + q\mathcal{R}$.

(v) $p + q - pq \in \mathcal{R}^{-1}$ *and* $\mathcal{R} = p^0 + q^0$.

(vi) $1 - pq \in \mathcal{R}^{-1}$ *and* $p + q - pq \in \mathcal{R}^{-1}$.

If $p - q \in \mathcal{R}^{-1}$, *then*

$$(p-q)^{-1} = h + h^* - 1, \qquad h = (p-q)^{-1}(1-q), \tag{4.5}$$

where h *is the idempotent defined in part* (iii).

We note that the idempotents h and k in the preceding theorem satisfy $k = 1 - h^*$. The equivalence of (i)–(iv) in Theorem 4.4 was proved for C^*-algebras as [9, Theorem 3.1]. The proof in [9] uses spectral arguments and additional norm conditions in a C^*-algebra.

For hermitian idempotents Theorem 3.5 simplifies to the following.

Theorem 4.5. *Let p, q be projections in a unital ring $\mathcal{R}$ with involution. Then the following conditions are equivalent*:

(i) $p - q \in \mathcal{R}^{-1}$.

(ii) *$p-q$ is regular, $p\mathcal{R}\cap q\mathcal{R}=\{0\}$ and $p^0\cap q^0=\{0\}$.*

Proof. Applying involution to conditions $p\mathcal{R}\cap q\mathcal{R}=\{0\}$ and $p^0\cap q^0=\{0\}$, we obtain $\mathcal{R}p\cap\mathcal{R}q=\{0\}$ and ${}^0p\cap{}^0q=\{0\}$. The result follows from Theorem 3.5. □

5 Difference of idempotent operators

In this section we turn our attention to the case when $\mathcal{R}$ is the Banach algebra $B(X)$ of all bounded linear operators on a Banach or Hilbert space X. We write $R(A)$ and $N(A)$ for the range and nullspace of any operator $A\in B(X)$. First we need to link the ring theoretical image and kernel ideals defined in (2.1) and (2.2) with the spatial ranges and nullspaces of operators. An idempotent operator is often called an *oblique projection*. We observe that the range of an idempotent operator $F\in B(X)$ is a closed subspace of X.

It will be convenient to introduce the following notation. If a Banach space X is the direct sum $X=M\oplus N$ of closed subspaces M, N, then there exists a unique oblique projection $H\in B(X)$ with $R(H)=M$ and $N(H)=N$—the so-called projection of X onto M along N. We will write

$$H=P_{M,N}.$$

Lemma 5.1. *Suppose that F,G are oblique projections on a Banach space X. Then*

(i) $FB(X)\cap GB(X)=\{0\}\iff R(F)\cap R(G)=\{0\}$.

(ii) $F^0\cap G^0=\{0\}\iff N(F)\cap N(G)=\{0\}$.

(iii) $B(X)=FB(X)+GB(X)\implies X=R(F)+R(G)$.

(iv) $B(X)=(I-F)B(X)+(I-G)B(X)\implies X=N(F)+N(G)$.

(v) $B(X)=FB(X)\oplus GB(X)\iff X=R(F)\oplus R(G)$.

(vi) $B(X)=B(X)F\oplus B(X)G\iff X=N(F)\oplus N(G)$.

Proof. (i) Let $FB(X) \cap GB(X) = \{0\}$ and let $x \in R(F) \cap R(G)$. Then $x = Fx = Gx$. By the Hahn-Banach theorem there exists $x^* \in X^*$ such that $\|x^*\| = 1$ and $x^*(x) = \|x\|$. Define an operator $T_x : X \to X$ by $T_x u = x^*(u)x$ for all $u \in X$. Then $T_x \in B(X)$, and $(F - G)T_x u = x^*(u)(Fx - Gx) = 0$ for all $u \in X$, that is, $FT_x = GT_x$. By hypothesis, $FT_x = 0$. Then $FT_x x = x^*(x)Fx = \|x\|x = 0$, which implies $x = 0$. This proves $R(F) \cap R(G) = \{0\}$.

Assume that $R(F) \cap R(G) = \{0\}$, and that $FU = GV$ for some $U, V \in B(X)$. For each $x \in X$, $FUx = GVx$, so that $FUx = 0 = GVx$ by hypothesis. This implies $FU = 0 = GV$, and (i) is proved.

(ii) follows from (i) when we substitute $I - F$ and $I - G$ for F and G.

(iii) follows from the decomposition $I = FA + GB$ for some $A, B \in B(X)$; (iv) is obtained when we substitute $I - F$ and $I - G$ for F and G in (iii).

(v) Suppose that $B(X) = FB(X) \oplus GB(X)$. The decomposition $X = R(F) \oplus R(G)$ follows from (i) and (iii).

Conversely suppose that $X = R(F) \oplus R(G)$. The oblique projection $H = P_{R(F),R(G)}$ satisfies $FH = H$ and $G(I - H) = I - H$. For any $A \in B(X)$,

$$A = HA + (I - H)A = FHA + G(I - H)A \in FB(X) + GB(X),$$

which proves $B(X) = FB(X)+GB(X)$. The remaining relation $FB(X) \cap GB(X) = \{0\}$ follows form part (i).

(vi) Follows from part (v) of this theorem combined with the equivalence

$$B(X) = B(X)F \oplus B(X)G \iff B(X) = (I - F)B(X) \oplus (I - G)B(X)$$

obtained (for arbitrary rings) in Lemma 2.1, and equations $R(I - F) = N(F)$, $R(I - G) = N(G)$. □

Theorem 5.2. *Let X be a Banach space and let $F, G \in B(X)$ be oblique projections. Then the following conditions are equivalent*:

(i) *$F - G$ is invertible.*

(ii) *$X = R(F) \oplus R(G)$ and $X = N(F) \oplus N(G)$.*

(iii) *There exist oblique projections* $H = P_{R(F),R(G)}$ *and* $K = P_{N(F),N(G)}$.

(iv) $I - FG$ *is invertible,* $X = R(F) + R(G)$ *and* $N(F) \cap N(G) = \{0\}$.

(v) $F + G - FG$ *is invertible,* $R(F) \cap R(G) = \{0\}$ *and* $X = N(F) + N(G)$.

(vi) $I - FG$ *and* $F + G - FG$ *are invertible.*

If $F - G$ *is invertible, then*

$$(F - G)^{-1} = H - K = P_{R(F),R(G)} - P_{N(F),N(G)},$$
$$H = P_{R(F),R(G)} = (F - G)^{-1}(I - G),$$
$$K = P_{N(F),N(G)} = (F - I)(F - G)^{-1}.$$

Proof. (vi) $\Longrightarrow$ (iv). As in the proof of Theorem 3.2, the invertibility of $F + G - FG$ implies that $B(X) = FB(X) + GB(X)$, which implies $X = R(F) + R(G)$ by Lemma 5.1 (iii). The invertibility of $I - FG$ implies $F^0 \cap G^0 = \{0\}$, which is equivalent to $N(F) \cap N(G) = \{0\}$ by Lemma 5.1 (ii).

(iv) $\Longrightarrow$ (ii). Let $x \in R(F) \cap R(G)$. Then $x = Fx = Gx = FGx$, and $(I - FG)x = 0$, that is, $x = 0$. Therefore $R(F) \cap R(G) = \{0\}$ and $X = R(F) \oplus R(G)$. Write $\widetilde{F} = I - F$ and $\widetilde{G} = I - G$. As in the proof of Theorem 3.2 we show that $B(X) = \widetilde{F}B(X) + \widetilde{G}B(X)$, from which $X = R(\widetilde{F}) + R(\widetilde{G}) = N(F) + N(G)$ follows by Lemma 5.1 (iii). This proves $X = N(F) \oplus N(G)$.

We observe that condition (v) is complementary to (iv) in the sense of substituting the complementary oblique projections $\widetilde{F} = I - F$ and $\widetilde{G} = I - G$ for F and G.

The rest follows from Theorem 3.2 and Lemma 5.1. □

6 Applications to Hilbert space operators

First we consider oblique projections.

Theorem 6.1. *Let* X *be a Hilbert space and let* $F, G \in B(X)$ *be oblique projections. Then the following conditions are equivalent:*

(i) *$F - G$ is invertible.*

(ii) *$X = R(F) \oplus R(G)$ and $X = R(F^*) \oplus R(G^*)$ (or $X = N(F) \oplus N(G)$ and $X = N(F^*) \oplus N(G^*)$).*

(iii) *There exist oblique projections $H = P_{R(F),R(G)}$ and $K = P_{R(F^*),R(G^*)}$ (or oblique projections $U = P_{N(F),N(G)}$ and $V = P_{N(F^*),N(G^*)}$).*

(iv) *$I - FG$ is invertible, $X = R(F) + R(G)$ and $R(F^*) + R(G^*)$.*

(v) *$F + G - FG$ is invertible, $X = N(F) + N(G)$ and $X = N(F^*) + N(G^*)$.*

(vi) *$I - FG$ and $F + G - FG$ are invertible.*

If $F - G$ is invertible, then

$$\begin{aligned}(F-G)^{-1} = H + K^* - I = P_{R(F),R(G)} - P_{N(F),N(G)},\\ H = P_{R(F),R(G)} = (F-G)^{-1}(I-G),\\ K = P_{R(F^*),R(G^*)} = (F^*-G^*)^{-1}(I-G^*).\end{aligned}$$

Proof. (vi) $\Longrightarrow$ (iv). As in the proof of Theorem 5.2, the invertibility of $F+G-FG$ implies $X = R(F) + R(G)$. Similarly, since $F^* + G^* - G^*F^*$ is invertible, we get $X = R(F^*) + R(G^*)$.

(iv) $\Longrightarrow$ (ii) Following the proof of Theorem 5.2, from the invertibility of $I - FG$ we obtain $X = R(F) \oplus R(G)$. Since $I - G^*F^*$ is also invertible, we have $X = R(F^*) \oplus R(G^*)$.

(v) is complementary to (iv) in the sense of substituting $I - F$ and $I - G$ for F and G.

The rest follows from Theorem 5.2. □

We specialize the preceding results to Hilbert space projections and add conditions that involve the operator norm in $B(X)$.

Theorem 6.2. *Let P, Q be projections on a Hilbert space X. The following are equivalent*:

(i) *$P - Q$ is invertible.*

(ii) $X = R(P) \oplus R(Q)$ (*or* $X = N(P) \oplus N(Q)$).

(iii) *There exists an oblique projection* $H = P_{R(P),R(Q)}$ (*or an oblique projection* $K = P_{N(P),N(Q)}$).

(iv) $I - PQ$ *is invertible and* $X = R(P) + R(Q)$.

(v) $P + Q - PQ$ *is invertible and* $X = N(P) + N(Q)$.

(vi) $I - PQ$ *and* $P + Q - PQ$ *are invertible.*

If the idempotents P, Q *are nontrivial, then we have additional equivalent conditions*:

(vii) $\|PQ\| < 1$ *and* $X = R(P) + R(Q)$.

(viii) $\|(I - P)(I - Q)\| < 1$ *and* $X = N(P) + N(Q)$.

(ix) $\|P + Q - I\| < 1$.

If $P - Q$ *is invertible, then*

$$
\begin{aligned}
(P - Q)^{-1} &= H + H^* - I = P_{R(P),R(Q)} - P_{N(P),N(Q)}, \\
H &= P_{R(P),R(Q)} = (P - Q)^{-1}(I - Q), \\
I - H^* &= P_{N(P),N(Q)} = (Q - I)(P - Q)^{-1}.
\end{aligned}
$$

Proof. The equivalence of conditions (i)–(vi) follows from Theorem 5.2. If P, Q are nontrivial projections on a Hilbert space X, then according to [9, Lemma 2.4] we have

$$
\begin{aligned}
I - PQ \text{ is invertible } &\iff \|PQ\| < 1, \\
P + Q - PQ \text{ is invertible } &\iff \|(I - P)(I - Q)\| < 1.
\end{aligned}
$$

Thus (i)–(viii) are equivalent.

(iii) $\Longrightarrow$ (ix). By Ljance's formula [10] (see also [9, Corollary 3.2] for a C^*-algebra version), $\|PQ\| = (\|H\|^2 - 1)^{1/2}\|H\|^{-1}$, where $H = P_{R(P),R(Q)}$. Since H^* is also idempotent with $R(H^*) = R(I - P)$ and $R(I - H^*) = R(Q)$, Ljance's formula also gives $\|(I - P)(I - Q)\| = (\|H^*\|^2 - 1)^{1/2}\|H^*\|^{-1} = (\|H\|^2 - 1)^{1/2}\|H\|^{-1} = \|PQ\|$.

By the well known Akhiezer–Glazman equality,

$$\|P + Q - I\| = \|Q - (I - P)\| = \max\left(\|PQ\|, \|(I - P)(I - Q)\|\right) = \|PQ\|.$$

Since (iii) is equivalent to (vi), we have $\|Q + P - I\| < 1$.

(ix) $\Longrightarrow$ (i). Follows from Kato's decomposition [7] for a pair of projections:

$$(P - Q)^2 + (P + Q - I)^2 = I. \qquad \square$$

Corollary 6.3. *Let P, Q be nontrivial projections on a Hilbert space X. Then the invertibility of $P - Q$ implies the invertibility of $P + Q$.*

Proof. The inequality $\|I - P - Q\| < 1$ implies that $I - (I - P - Q) = P + Q$ is invertible. $\square$

Remark 6.4. For nontrivial projections P, Q we also have

$$\|PQ\| < 1 \iff \|PQP\| < 1 \iff I - PQP \text{ is invertible.}$$

Remark 6.5. Buckholtz [2, Main Theorem] proved the equivalence of (i), (ii) and (iii) of the preceding theorem by a different method. Vidav [15, Theorem 1] proved the equivalence of (ii), (iii) and (vii). In [9, Theorem 4.1], the preceding theorem was proved in the setting of C^*-algebra without condition (v).

Theorem 4.5 yields the following result for Hilbert space projections when we recall that an operator $A \in L(X)$ is regular if and only if it has closed range.

Theorem 6.6. *Let P, Q be projections in a Hilbert space X. Then the following conditions are equivalent*:

(i) $P - Q$ *is invertible.*

(ii) $P - Q$ *has closed range,* $R(P) \cap R(Q) = \{0\}$ *and* $N(P) \cap N(Q) = \{0\}$.

In a finite dimensional Hilbert space, condition (ii) of the preceding theorem (without the closed range hypothesis which is automatic) is enough to ensure the invertibility of the difference of arbitrary idempotents.

Theorem 6.7. *Let X be a finite dimensional Hilbert space and F, G idempotent operators in $B(X)$. Then the following statements are equivalent*:

(i) *The operator $F - G$ is invertible.*

(ii) $X = R(F) \oplus R(G)$ *and* $X = R(F^*) \oplus R(G^*)$.

(iii) $X = R(F) \oplus R(G)$ *and* $X = N(F) \oplus N(G)$.

(iv) $R(F) \cap R(G) = \{0\}$ *and* $N(F) \cap N(G) = \{0\}$.

Proof. In view of Theorems 5.2 and 6.1 we only need to prove the implications (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i).

(iii) $\Longrightarrow$ (iv) is clear.

(iv) $\Longrightarrow$ (i) It is enough to show that $N(F - G) = \{0\}$. Let $(F - G)x = 0$ for some $x \in X$. Then $Fx = Gx \in R(F) \cap R(G) = \{0\}$. This means that $x \in N(F) \cap N(G) = \{0\}$, that is, $x = 0$. □

In [4, Corollary 1], Gross and Trenkler proved the equivalence of (i), (iii) and (iv) of the preceding theorem for real matrices using involved relations for ranks of matrices obtained in [12].

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