The nullity and rank of linear combinations of idempotent matrices

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Abstract

Baksalary and Baksalary (Linear Algebra Appl. 388 (2004) 25–29) proved that the nonsingularity of $P_1 + P_2$, where P_1 and P_2 are idempotent matrices, is equivalent to the nonsingularity of any linear combinations $c_1P_1 + c_2P_2$, where $c_1, c_2 \neq 0$ and $c_1 + c_2 \neq 0$. In the present note this result is strenghtened by showing that the nullity and rank of $c_1P_1 + c_2P_2$ are constant. Furthermore, a simple proof of the rank formula of Groß and Trenkler (SIAM J. Matrix Anal. Appl. 21 (1999) 390–395) is obtained.

Key words: oblique projector, linear combinations of projectors, rank, nullity
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1 Introduction

If A is a matrix (linear transformation) on $\mathbb{C}^n$, we write $\mathcal{N}(A)$ and $\mathcal{R}(A)$ for the nullspace and the range of A. The *rank* of A, $\mathrm{rk}\,(A)$, is the dimension of $\mathcal{R}(A)$, and the *nullity* of A, $\mathrm{nul}\,(A)$, is the dimension of $\mathcal{N}(A)$. Let $\mathcal{P}$ be the set of all $n \times n$ complex idempotent matrices P ($P^2 = P$). In this note we

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consider nontrivial linear combinations of given $P_1, P_2 \in \mathcal{P}$, that is, matrices of the form

$$c_1P_1 + c_2P_2, \qquad c_1, c_2 \neq 0, \quad c_1 + c_2 \neq 0.$$

Groß and Trenkler [2] proved that the nonsingularity of $P_1 - P_2$ implies the nonsingularity of $P_1 + P_2$. Their methods rely strongly on relations for the ranks of matrices developed by Marsaglia and Styan [5], while Koliha et al. [3] obtained new simple proofs, without reference to rank theory, and pointed out explicitly a condition, which combined with the nonsingularity of P_1+P_2 implies the nonsingularity of P_1-P_2. Baksalary and Baksalary [1] proved that the nonsingularity of $P_1 + P_2$ is equivalent to the nonsingularity of any linear combinations $c_1P_1+c_2P_2$, where $c_1, c_2 \neq 0$ and $c_1+c_2 \neq 0$. In the present note this result is strengthened by showing that the nullity and rank of $c_1P_1+c_2P_2$ are constant. Furthermore, we obtain a simple proof of the rank formula of Groß and Trenkler [2].

2 Results

First a useful auxiliary result whose proof is left to the reader.

Lemma 2.1 *If $P_1, P_2 \in \mathcal{P}$, we define A as the restriction of $(I - P_1)P_2$ to $\mathcal{N}(P_1)$, that is,*

$$A\colon \mathcal{N}(P_1) \to [(I - P_1)P_2]\mathcal{N}(P_1), \quad x \mapsto Ax = (I - P_1)P_2x.$$

Then

$$\mathcal{N}(A) = \mathcal{N}[(I - P_1)P_2] \cap \mathcal{N}(P_1), \qquad \mathcal{R}(A) = \mathcal{R}[(I - P_1)P_2(I - P_1)]. \quad (2.1)$$

We start our observations with the following result.

Theorem 2.2 *Let $P_1, P_2 \in \mathcal{P}$, $c_1, c_2 \in \mathbb{C}\setminus\{0\}$ and $c_1+c_2 \neq 0$. If A is defined as in Lemma 2.1, then $\mathcal{N}(c_1P_1 + c_2P_2)$ is isomorphic to $\mathcal{N}(A)$.*

PROOF. In the proof we use Lemma 2.1.

Let $\mathcal{N} = \mathcal{N}(c_1P_1 + c_2P_2)$ and $c \neq 0$. First we show that

$$\mathcal{N} \cong (I - P_1)\mathcal{N} \quad \text{and} \quad \mathcal{N}(A) \cong (cI - P_2)\mathcal{N}(A). \quad (2.2)$$

Let $x \in \mathcal{N}$ and $(I - P_1)x = 0$. Then $x = P_1x$, and $(c_1 + c_2)P_2x = P_2(c_1P_1 + c_2P_2)x = 0$. Therefore $P_2x = 0$, and so $x = c_1^{-1}(c_1P_1 + c_2P_2)x = 0$. Hence $I - P_1$ restricted to acting from $\mathcal{N}$ to $(I - P_1)\mathcal{N}$ is an isomorphism.

If $x \in \mathcal{N}(A)$ and $(cI - P_2)x = 0$, then $P_1 x = 0$, $P_2 x = P_1 P_2 x$ and $P_2 x = cx$. Thus, $P_2 x = P_1 cx = 0$, that is $x = 0$. Hence, $cI - P_2$ restricted to acting from $\mathcal{N}(A)$ to $(cI - P_2)\mathcal{N}(A)$ is an isomorphism.

Next we prove that

$$(I - P_1)\mathcal{N} \subset \mathcal{N}(A) \quad \text{and} \quad (cI - P_2)\mathcal{N}(A) \subset \mathcal{N} \text{ for some } c \neq 0. \tag{2.3}$$

Suppose that $x \in \mathcal{N}$. Then $P_1 x = -(c_2/c_1)P_2 x$, and

$$\begin{aligned} A(I - P_1)x &= (I - P_1)P_2(I - P_1)x = (I - P_1)(P_2 x - P_2 P_1 x) \\ &= (I - P_1)(P_2 x + (c_2/c_1)P_2 x) \\ &= \frac{c_1 + c_2}{c_1 c_2}(I - P_1)(c_1 P_1 x + c_2 P_2 x) = 0, \end{aligned}$$

that is, $(I - P_1)x \in \mathcal{N}(A)$. This proves the first inclusion in (2.3).

Suppose that $x \in \mathcal{N}(A)$ and set $c = 1 + c_1/c_2$. Then $P_1 x = 0$ and $P_1 P_2 x = P_2 x$. Thus

$$\begin{aligned} (c_1 P_1 + c_2 P_2)\Big(cI - P_2\Big)x &= c_1 c P_1 x - c_1 P_1 P_2 x - c_2 P_2 x + c_2 c P_2 x \\ &= -(c_1 + c_2)P_2 x + (c_1 + c_2)P_2 x = 0, \end{aligned}$$

that is, $(cI - P_2)x \in \mathcal{N}$. This proves the second inclusion in (2.3).

The proof is completed by combining (2.2) with (2.3). □

The following theorem subsumes a recent result of Baksalary and Baksalary [1, Theorem 1].

Theorem 2.3 *Let $P_1, P_2 \in \mathcal{P}$, let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, $c_1 + c_2 \neq 0$, and let A be defined as in Lemma* 2.1*. Then the nullity of $c_1 P_1 + c_2 P_2$ is constant, equal to*

$$\operatorname{nul}(c_1 P_1 + c_2 P_2) = \operatorname{nul}(P_1 + P_2) = \operatorname{nul}(A) = \dim[\mathcal{N}[(I - P_1)P_2] \cap \mathcal{N}(P_1)].$$

In particular, $c_1 P_1 + c_2 P_2$ is nonsingular if and only if $P_1 + P_2$ is.

PROOF. The result follows from Theorem 2.2 and from the fact that $A \in \mathbb{C}^{n \times n}$ is nonsingular if and only if $\operatorname{nul}(A) = 0$.

Furthermore, as a corollary of Theorem 2.2 we obtain the following theorem on the constancy of the rank of $c_1 P_1 + c_2 P_2$, which in the case of $c_1 = c_2 = 1$ yields a simple proof of Groß-Trenkler's [2] rank formula for the sum of oblique projections (see also [3]).

Theorem 2.4 *Let $P_1, P_2 \in \mathcal{P}$, $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, $c_1 + c_2 \neq 0$, and let A be defined as in Lemma* 2.1. *Then the rank of $c_1P_1 + c_2P_2$ is constant, equal to*

$$\begin{aligned} \mathrm{rk}\,(c_1P_1 + c_2P_2) = \mathrm{rk}\,(P_1 + P_2) &= \mathrm{rk}\,(P_1) + \mathrm{rk}\,(A) \\ &= \mathrm{rk}\,(P_1) + \mathrm{rk}[(I - P_1)P_2(I - P_1)] \\ &= n - \dim[\mathcal{N}[(I - P_1)P_2] \cap \mathcal{N}(P_1)]. \end{aligned} \tag{2.4}$$

PROOF. Since $\mathrm{rk}\,(c_1P_1 + c_2P_2) = n - \mathrm{nul}\,(c_1P_1 + c_2P_2)$, the rank of $c_1P_1 + c_2P_2$ is constant, equal to the rank of $P_1 + P_2$ by using Theorem 2.3. According to Lemma 2.1 and Theorem 2.2,

$$\mathrm{rk}\,(A) = \mathrm{nul}\,(P_1) - \mathrm{nul}\,(A) = n - \mathrm{rk}\,(P_1) - \mathrm{nul}\,(A),$$

which implies $\mathrm{rk}\,(P_1 + P_2) = n - \mathrm{nul}\,(P_1 + P_2) = n - \mathrm{nul}\,(A) = \mathrm{rk}\,(P_1) + \mathrm{rk}\,(A)$. Then (2.4) follows from Lemma 2.1. □

Open problem. In [4] we studied Fredholm properties of the difference of orthogonal projections in a Hilbert space. Suppose that P_1, P_2 are orthogonal (or oblique) projections in a Hilbert space. Is it true that $P_1 + P_2$ is Fredholm if and only if any linear combinations $c_1P_1 + c_2P_2$ is Fredholm, where $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ and $c_1 + c_2 \neq 0$?

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