620123 - LECTURE SUMMARY 5
APPLICATIONS OF FIRST ORDER ODE’S
Marty Ross
August 22, 2006

This is a summary of my lectures 11-12 (roughly equal to lectures 10-11 in the subject guide). See the indicated sections of Thomas [T] for further discussion and worked examples. See Olc’s notes [9 am stream] for details on chemical reactions and financial models.

Though solving ODE’s is lots of fun [?], the point of ODE’s is that they can be used to model physical and biological and chemical and financial - and mathematical - phenomena. Then, solving the ODE’s can give us deep insight into the phenomena in question. In this procedure, it is important to keep in mind three steps of mathematical modelling (and especially which step is being asked of you!):

PART 1: Setting up the ODE.

PART 2: Solving the ODE.

PART 3: Interpreting the solution.

Note that for PART 1, we definitely need to know (or be told) pretty clearly what’s going on with the phenomenon to be modelled. By comparison, in PART 2 we can be quite clueless, and treat the ODE as ‘just another ODE’. (Of course, in PART 2, we’re not permitted to be clueless about solving ODE’s!). In PART 3, again some sense is needed of what’s really going on.

One aspect of modelling should always be kept in mind: a model is a model is a model.

A model is not reality: if the conclusion of the model is too weird or surprising, you should be questioning the validity of the model as much as adjusting your view of reality.

1 Population Models ([T; 9.5, 674-679])

1.1 The Malthus/Exponential Model

In any community of beings, the Population $P$ can be thought of as a function of time. We can then ask: what determines how the population changes with time? The simplest and most intuitive approach is the Malthus (or Exponential) Model (we prefer to refer to it as Malthus, to emphasise that we are not assuming that the population will in some sense be exponential, even if we eventually conclude that). The Malthus assumption is that

Malthus: In a given time period, the increase in Population is proportional to the population.

In differential terms, we write:

$$\Delta P = kP\Delta t.$$ Dividing by $\Delta t$ and taking the limit, we arrive at the Malthus Equation:

$$\frac{dP}{dt} = kP.$$ We know how to solve the Malthus equation (Handout 2, §§2.3, 3.2):

$$P = P_0 e^{kt}.$$ Here, there are two unknowns, the initial population $P_0$, and the parameter $k$. This raises an obvious but important point: to obtain a specific solution for any (first order) model, we need an initial condition, together with an extra condition for each parameter in the model.

1.1.1 Example: World Population à la Malthus

If we are given an initial condition, we can evaluate $P$. For example the population of the earth in 2000 was (approximately) 6 billion. So, $P(0) = P_0 = 6$, and we can write

$$P = 6e^{kt}.$$ Note that here, we are taking the year 2000 to be $t = 0$ (we can start our clock at any convenient time), and that $P$ is measured in billions of people (to avoid carrying around $10^9$ in our expressions).

Now, to evaluate $k$, we can need an extra condition. For example, the World population in 1900 was (roughly) 2.5 billion. So, measuring time in centuries, we have

$$P(-\frac{1}{4}) = 2.5.$$ Plugging that into (M3), we find

$$2.5 = 6e^{-\frac{k}{4}}$$

$$\Rightarrow e^{\frac{k}{4}} = \frac{2.5}{6}$$

$$\Rightarrow k = 2\log(2.4) = 1.75.$$
Thus, the Malthus model (with our input data) predicts the World population to take the form
\[(M4) \quad P = 60^{1.75t} \quad (P \text{ measured in billions, } t \text{ in centuries}).\]

As is well known, the Malthus model predicts exponential population growth. This is reflected by the notion of doubling time, the time \(T\) taken for a population to grow from some \(P_0\) to \(2P_0\). From (M2), we can calculate:
\[
2P_0 = P_0 e^{kT} \\
\Rightarrow T = \frac{\log 2}{k} = \frac{0.396}{0.002} = 198 \text{ years.}
\]

(Note that \(k\) has the units of \(1/\text{century}\), so \(T\) correctly has units of time). The important aspect is that the doubling time \(T\) does not depend upon \(P_0\). Such growth is thus often called geometric: the population increases by a fixed factor for each period of time.

For our particular world model (M4), we calculate a doubling time of
\[
T = \frac{\log 2}{1.75} = 0.396 \approx 40 \text{ years.}
\]

This quickly leads to unsettling (and implausible) conclusions:

- In 400 years, the Earth’s population will double 10 times; so, since \(2^{10} = 1024 \approx 1000\), the prediction is that the population in the year 2400, will be about 6 trillion.
- In 800 years, the population will supposedly be \(6 \times 10^{15}\). This will give each person roughly 0.1m² to stand on (and some people will get the wet bits).
- In about 2200 years (not a huge amount of time in historical terms), the population will be about \(2 \times 10^{20}\), so large that the ball of people will reach to the Moon.

Clearly, something has got to give, and it was Thomas Malthus in 1798 who first pointed this out, in the very influential *An Essay On the Principle of Population*. Malthus didn’t quite word things as we have. What Malthus said was that population would grow geometrically (as we have predicted), but that food supply could only increase arithmetically (i.e. increasing by a fixed added amount in each time period); thus, eventually we’ll all starve, and/or there will be huge wars for scarce resources. But even if Malthus’s take on the problem was slightly different, he clearly saw the essence of the problem: geometric population growth is unsustainable.

### 1.2 The Verhulst/Logistic Model

As a population in a limited environment grows, the effects of crowding (war, famine, disease) will become significant. So, we would like to adjust the Malthus model to take account of these affects. This was first attempted in 1838 by Pierre Verhulst, in direct reaction to Malthus’s Essay. Verhulst’s idea was that crowding, and thus its bad effects, could be measured by the number of interactions between members of the populations. So, if the population is size \(P\), then the number of possible interactions is \(P \times (P - 1) \approx P^2\). Verhulst thus hypothesised:

**Verhulst: The negative effect of crowding is proportional to the square of the population.**

In differential terms, we then obtain the adjusted version of the Malthus model:
\[
\frac{\Delta P}{\Delta t} = \alpha P - \beta P^2.
\]

Dividing by \(\Delta t\) and taking the limit, we arrive at the Verhulst Equation:
\[(V1) \quad \frac{dP}{dt} = \alpha P - \beta P^2 .
\]

We can (and will) solve this ODE. However, before doing so, notice that we can get a very good idea of the solutions directly from (V1). To begin, notice that the constant functions \(P = 0\) and \(P = \frac{\alpha}{\beta}\) are solutions of (V1). Next, suppose we start with \(P_0 > \frac{\alpha}{\beta}\); then, by (V1), \(\frac{dP}{dt} < 0\) unless we cross \(P_0 = \frac{\alpha}{\beta}\). But in fact, the Existence and Uniqueness Theorem promises that we can never cross (at any such crossing point, we would have two solutions to (V1) with the same initial condition, which the Theorem forbids). So, this suggests that if we start with \(P_0 > \frac{\alpha}{\beta}\), then \(P_0 \to \frac{\alpha}{\beta}\) as \(t \to \infty\). And, if \(0 < P_0 < \frac{\alpha}{\beta}\), we can argue similarly to reach the same conclusion. Thus, with any (non-zero) initial condition, we expect \(P \to \frac{\alpha}{\beta}\) to be the long-term stable population.

Now, we solve (V1) explicitly. This is a separable ODE, and so to solve we divide by the RHS, giving
\[
\int \frac{1}{\alpha P - \beta P^2} \, dP = \int \frac{1}{dt}.
\]
To integrate the LHS, we apply partial fractions:
\[
\frac{1}{aP - bP^2} = \frac{A}{P} + \frac{B}{a-bP} = \frac{1}{aP} + \frac{b}{a(a-bP)}
\]  
(setting \( P = 0 \) and \( P = \frac{1}{a} \) to evaluate \( A \) and \( B \)).

So, integrating, we find:
\[
\int \frac{1}{a} \log |P| - \frac{1}{a} \log |a-bP| = t + c
\]

\[\Rightarrow \log \left| \frac{P}{a-bP} \right| = at + d \quad (d = ac)
\]

\[\Rightarrow \frac{P}{a-bP} = Le^{at} \quad (L = \pm e^d).
\]

We can now solve for \( P \). But, before doing so, we’ll incorporate a generic initial condition:
\[P(0) = P_0 \quad \Rightarrow \quad \frac{P_0}{a-bP_0} = L.
\]

Substituting for \( L \) and getting rid of denominators, we have
\[P(a-bP_0) = P_0 e^{at} (a-bP).
\]

This is a linear equation for \( P \), which we can solve to give
\[P = \frac{aP_0 e^{at}}{(a-bP_0) + bP e^{at}}.
\]

If we divide numerator and denominator, we obtain the alternative expression
\[P = \frac{aP_0}{(a-bP_0) e^{-at} + bP}.
\]

An equation of this form is called a **logistic**, which is why the Verhulst model is commonly known by this name. Notice that in this form, it is clear (as we suspected) that \( P \to \frac{a}{b} \) as \( t \to \infty \) (as long as \( P_0 > 0 \)).

### 1.2.1 Example: World Population à la Verhulst

Given the population of 6 billion in the year 2000, we can plug in \( P_0 = 6 \) into (V2). We then need two further conditions to evaluate the parameters \( a \) and \( b \). As above, the population in 1950 gives us the condition
\[P(-\frac{1}{2}) = 2.5.
\]

Then, since the population in 1980 was (roughly) 4.44 billion, we have the further condition
\[P(-2) = 4.44.
\]

Plugging into (V2), we then have two equations for \( a \) and \( b \). However, these equations cannot be explicitly solved for \( a \) and \( b \).

So, if we can’t solve for \( a \) and \( b \), the question is, what do we do? There are various answers, with varying levels of integrity:

**The 123 answer:** In practice, for this subject, we shall be given \( a \) and \( b \). (in particular, though the discussion to follow is hopefully of interest, it can also be safely ignored).

**A naive numerical approach:** We can use (1) and (1) to obtain approximate solutions to \( a \) and \( b \) (it’s too much off-topic to go into the methods here). If we do this, we find that
\[a \approx 2.7 \quad b \approx .22.
\]

Notice that \( \frac{a}{b} \approx 10 \), which is suggesting a stable Earth population of about 10 billion.

**A genuine numerical approach:** Whether or not we can solve (1) and (1) for \( a \) and \( b \), it is actually more natural to use as much population data as we can get: one then fits a curve of the form (V2) to all the data as best we can (Thomas has a good discussion of this).

**Just Forget the Whole Thing:** In fact, this is probably the best strategy. The first serious attempt to use the Verhulst equation to model the Earth’s population was by Ray Pearle, in 1924. He estimated that the Earth’s asymptotic population would be 2 billion, a figure passed in 1930. In 1936, Pearle tried again, obtaining the estimate 2.6 billion; this figure was passed in 1952.

In reality, the logistic model is a very poor model of the Earth’s population: conditions on the Earth are just not sufficiently uniform to be modelled in such a simple manner. Nonetheless, for contained populations in uniform conditions, the Verhulst model is considered a legitimate modeling tool.

### 1.2.2 Adjustments to the Verhulst/Logistic Model

There are many ways to adapt the Verhulst model, to try to make it more reflective of some particular situation. We’ll just quickly consider one quick adjustment, a population equation of the form
\[
\frac{dP}{dt} = aP - bP^2 + c.
\]

Here, we can interpret the constant term as a constant rate of harvesting (\( c < 0 \)) or immigration (either \( c > 0 \) or \( c < 0 \)), or course one could also consider an immigration/harvesting
rate $c(t)$ which varies with time (and even with $P$ as well), but we shall not consider that here.

The equation (V3) is still separable, and we can solve it as above. Let’s just consider what we expect to happen. Let’s consider the case where $c < 0$ (harvesting). Then the quadratic on the RHS of (V3) either has two positive real roots (when $|c|$ is not too large), or a repeated positive root, or no roots at all (if $|c|$ large). Assuming the quadratic on the right has real roots $\alpha$ and $\beta$, we can then write (V3) as

\[
\frac{dP}{dt} = -b(P - \alpha)(P - \beta).
\]

Then, $P = \alpha$ and $P = \beta$ are particular constant solutions to (V3). Further, by considering the value of the initial population $P_0$, we know whether $\frac{dP}{dt}$ begins positive or negative, and we can predict the general behaviour of the solution. Note in particular that if $P_0 < \alpha$, then we expect the population to become extinct in some finite time.

Other cases, depending upon the relative values of $a, b$ and $c$, can be similarly analysed. Note in particular that if $c$ is negative and too large in magnitude, then we will expect the population to become extinct, no matter the size of the initial population $P_0$. Just what we would expect from over-harvesting.

2 Newton’s Law of Cooling ([T: 7.5, 507-508])

Newton’s Law of Cooling is a very simple model. Consider a room at temperature $R$ which contains some body (say a cup of coffee) at temperature $T$. The body will then cool (or heat up) according to the difference $T - R$ between $T$ and $R$; we further assume that the body is not large enough to also heat up the room; so $T$ will change with time, but $R$ is assumed constant. Newton’s Law of Cooling then takes the form

\[
\frac{dT}{dt} = -k(T - R).
\]

Note the minus sign: so, if $T > R$ then $k > 0$ would give $\frac{dT}{dt} < 0$, as we would expect.

It is easy enough to solve the (separable) equation (N1), but is also natural to introduce a new variable

\[
\theta = T - R.
\]

Then (N1) becomes

\[
\frac{d\theta}{dt} = -k\theta.
\]

We then have the general solution

\[
\theta = \theta_0 e^{-kt} \Rightarrow T = R + (T_0 - R)e^{-kt}
\]

To get a specific solution, we need the room temperature $R$, the initial temperature $T_0$ of the body, and one further condition to evaluate the parameter $k$. Note that independent of these specific conditions, we can conclude

\[
T \rightarrow R \text{ as } t \rightarrow \infty.
\]

3 Mixing Problems ([T: 9.2, 655-657])

Mixing problems are very standard, but are also problems where the modeling works very nicely. In such problems we have a tank (or room, lake, body organ, etc) containing water (or air, blood, etc). The water contains a certain concentration of salt (pollutant, drug, etc). There is also water, also perhaps containing salt, flowing into the tank. Finally the mixed water in the tank is flowing out at a certain rate.

In mixing problems, there are three quantities in which we are interested: the Volume $V$ of water in the tank, the Quantity $Q$ of salt in the tank, and the Concentration $C$ of salt in the tank. All three quantities can change with time, but at any time are related by the equation

\[
C = \frac{Q}{V}.
\]

Note that this equation is consistent with the units used: if $V$ is measured in litres, and $Q$ in kg, then $C$ is measured in kg/L.

An example (below) illustrates the general method of solving mixing problems. But we emphasise a couple of general points. First of all, $V(t)$ will usually be immediately obvious from the information given; so, if we know either $Q(t)$ or $C(t)$ then (M1) (together with $V$) gives us the other quantity as well. More importantly, it is usually more natural to obtain an ODE for $Q$. So, even if the question is asking about the behaviour of $C$, we first find $Q$ and then use (M1) to answer any subsequent question.
3.1 Example

A 500 L tank initially contains 200 L of water, with 10 kg of salt mixed in. Water containing 200 g/L is flowing into the tank at 5 L/min. The mixture in the tank is also flowing out at a rate of 2 L/min. What happens?

We first note that at any time \( t \), the volume \( V \) of water in the tank is

\[
(*) \quad V = 200 + 3t.
\]

Next, the major work, we want to find an IVP for the quantity \( Q \) of salt in the tank at any time \( t \). Here, we are explicitly given the initial condition

\[
(1) \quad Q(0) = 10.
\]

To obtain the ODE, we calculate

\[
\frac{dQ}{dt} = \text{(rate of salt coming in)} - \text{(rate of salt going out)}
\]

\[
= (5 \times 200) - (2 \times C)
\]

\[
= 1000 - 2Q
\]

\[
= 1000 - \frac{2Q}{200 + 3t}
\]

This is a linear equation for \( Q \), which we can easily solve (Handout 2, §3); we obtain an integrating factor

\[
\nu = (200 + 3t)^\frac{1}{2},
\]

which leads to

\[
Q \cdot (200 + 3t)^\frac{1}{2} = \frac{1}{4}(200 + 3t)^2 + c
\]

\[
\Rightarrow Q = \frac{1}{4}(200 + 3t) + \frac{c}{(200 + 3t)^\frac{1}{2}}
\]

Thus we have the general solution to the ODE, and (1) gives

\[
c = -30 \cdot (200)^\frac{1}{2}.
\]

So,

\[
Q = \frac{1}{4}(200 + 3t) - \frac{30 \cdot (200)^\frac{1}{2}}{(200 + 3t)^\frac{1}{2}}
\]

Then, if we want it, we can use (M1) and (\( \star \)) to calculate

\[
C = \frac{1}{4} \cdot \frac{30 \cdot (200)^\frac{1}{2}}{(200 + 3t)^\frac{1}{2}}
\]

We can now use these expressions for \( Q \) and \( C \) to answer any desired question (for example, the concentration of salt when the tank overflows). Note that

\[
C \rightarrow \frac{1}{4} \quad \text{as} \quad t \rightarrow \infty.
\]

This actually makes sense (and is a partial check of our solutions), since in the long-run the salt concentration in the tank should be close to the salt concentration of the water flowing in. \( \blacksquare \)

4 Chemical Reactions and Economic Applications

See OLE's lecture notes [9am stream] for details on these topics.