620123 Applied Mathematics (Advanced)

Lectures on applications of second order ODEs

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The Tacoma bridge disaster

A spring of (natural) length $L$ is streched a distance $s$ by a weight with mass $m$.
In equilibrium we take the position of the weight (viewed as a point particle) to be $x = 0$, with the positive $x$-axis pointing down.

If we pull down the weight and then release it, it will start to oscillate. If we assume the weight-spring system moves free of any resistance (no air-resistance and no internal friction in the spring) we say the spring is undamped.
To derive an ODE for the undamped spring we must combine Newton’s second law and Hooke’s law.

According to Hooke’s linear spring law the restoring force $T$ exerted by the spring (“trying to unstretch”) is proportional to the distance $d$ it is stretched:

$$T = -kd.$$

Here $k > 0$ is called the spring constant.

The larger the spring constant, the more tightly-coiled the spring.

**Note:** As we all known from experience, Hooke’s law only works if we do not stretch a spring too much.
The Tacoma bridge disaster

Undamped spring
Damped spring
Underdamping
Critical damping
Overdamping
Forced spring
Resonance

\[ L + t \cdot v \cdot e \]

\[ x(t) \]

\[ T \]

\[ s \]

\[ mg \]

\[ mg \]
Using Hooke’s law the total force $F$ is given by

$$F = -k(s + x) + mg,$$

with $x = x(t)$ the displacement out of equilibrium and $g$ ($\approx 9.8 \text{ m/s}^2$) the acceleration on the Earth’s surface.

Before we pulled down the weight the system was in equilibrium, so that

$$ks = mg.$$

We are thus left with

$$F = -kx. \quad (*)$$
According to Newton the total force equals mass times acceleration:

\[ F = mx'' , \quad (**) \]

where \( x' = \frac{dx}{dt} \) and \( x'' = \frac{d^2x}{dt^2} \).

Equating (*) and (**) leads to the equation of motion

\[ mx'' + kx = 0 \]

(which holds throughout the Universe, not just on Earth).
The equation of motion only depends on the ratio of $k$ and $m$, and it is customary to introduce
\[ \omega^2 = \frac{k}{m} \]
where $\omega > 0$ is known as the angular frequency. (For brevity we will drop “angular” in the remainder of these notes.)

Then the equation of motion for the undamped spring is given by
\[ x'' + \omega^2 x = 0 \]
This is a (homogeneous) second order linear ODE with constant coefficients.
The corresponding characteristic equation is given by
\[ \lambda^2 + \omega^2 = 0 \]
with purely imaginary roots
\[ \lambda_1 = i\omega \quad \text{and} \quad \lambda_2 = -i\omega. \]
Two linearly independent solutions are thus

\[ e^{i\omega t} \quad \text{and} \quad e^{-i\omega t} \]

or

\[ \cos(\omega t) \quad \text{and} \quad \sin(\omega t). \]

and the general solution (according to Lemma 2) is

\[ x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t). \]

In most text-books the above solutions is rewritten in terms of a single trigonometric function as

\[ x(t) = A \cos(\omega t - \phi), \]

where the two constants \( c_1 \) and \( c_2 \) have been traded for two new constants \( A > 0 \) and \( \phi \in (-\pi, \pi] \).
Proof.
Using
\[ \cos(a - b) = \cos a \cos b + \sin a \sin b \]
we get
\[ A \cos(\omega t - \phi) = A \cos(\omega t) \cos \phi + A \sin(\omega t) \sin \phi \]
\[ = c_1 \cos(\omega t) + c_2 \sin(\omega t) \]
where
\[ c_1 = A \cos \phi \quad \text{and} \quad c_2 = A \sin \phi, \]
or, conversely, (assuming that \(-\pi/2 \leq \phi \leq \pi/2\))
\[ A = \sqrt{c_1^2 + c_2^2} \quad \text{and} \quad \phi = \arctan\left(\frac{c_2}{c_1}\right). \]
\[ \square \]
The solution
\[ x(t) = A \cos(\omega t - \phi) \]
corresponds to what is called simple harmonic motion.

The constants \( A > 0 \) and \( \phi \in (-\pi, \pi] \) are called the amplitude and phase angle (or phase shift) and are determined by the initial conditions \( x(0) \) and \( x'(0) \).

For simple harmonic motion in action, click here.

The period of the simple harmonic motion is given by
\[ \frac{2\pi}{\omega} \]
Of course, if you try the “spring-weight” system at home it is unlikely to display true simple harmonic motion. If that were possible we could build a perpetuum mobile like

![Inventor (wearing earplugs) reads by light powered by electricity provided by the ceiling transducer of a bouncing ball engine.](image)

_Inventor (wearing earplugs) reads by light powered by electricity provided by the ceiling transducer of a bouncing ball engine._
To make the model more realistic we need to take into account the effect of **damping**. This is caused by air-resistance and mechanical friction of the spring.

The most common model of damping assumes that it corresponds to a force $F_d$, proportional to the velocity of the moving object, and opposite to the direction of movement. Hence

\[ F_d = -\beta x', \]

where $\beta > 0$ is the **damping constant**.

The total force $F$ for the damped spring is thus given by

\[ F = \underbrace{-\beta x'}_{\text{damping force}} + \underbrace{-k(s + x)}_{\text{restoring force}} + \underbrace{mg}_{\text{gravitational force}} \]

As before,

\[ ks = mg \]

so that

\[ F = -\beta x' - kx. \]
Thanks to Newton’s second law this yields the equation of motion

\[ mx'' + \beta x' + kx = 0. \]

This equation only depends on the ratios of \( k/m \) and \( \beta/m \), and it is customary to not only set

\[ \omega^2 = \frac{k}{m} > 0 \]

but also

\[ 2p = \frac{\beta}{m} > 0. \]

Then the equation of motion for the damped spring is given by

\[ x'' + 2p x' + \omega^2 x = 0 \]
This is a (homogeneous) second order linear ODE with constant coefficients.

The corresponding **characteristic equation** is given by

\[ \lambda^2 + 2p\lambda + \omega^2 = 0 \]

which has **discriminant**

\[ D = 4(p^2 - \omega^2). \]

There are three different cases to consider.

1. **\( D < 0 \) (i.e., \( p < \omega \)).**
   This is the **underdamped** or **weakly-damped** spring.

2. **\( D = 0 \) (i.e., \( p = \omega \)).**
   This is the **critically-damped** spring.

3. **\( D > 0 \) (i.e., \( p > \omega \)).**
   This is the **overdamped** or **strongly-damped** spring.
The underdamped or weakly-damped spring

For \( p < \omega \) the characteristic equation

\[
\lambda^2 + 2p\lambda + \omega^2 = 0
\]

has two complex conjugate roots

\[
\lambda_1 = -p + i\sqrt{\omega^2 - p^2} \quad \text{and} \quad \lambda_2 = -p - i\sqrt{\omega^2 - p^2}.
\]

Two linearly independent solutions are thus

\[
e^{-pt} \cos(\omega_p t) \quad \text{and} \quad e^{-pt} \sin(\omega_p t),
\]

where

\[
\omega_p = \sqrt{\omega^2 - p^2} = \omega \sqrt{1 - (p/\omega)^2}.
\]
The general solution is

\[ x(t) = e^{-pt} \left( c_1 \cos(\omega_p t) + c_2 \sin(\omega_p t) \right). \]

Again it is best to write this in terms of a single trigonometric function:

\[ x(t) = A e^{-pt} \cos(\omega_p t - \phi) = A(t) \cos(\omega_p t - \phi). \]

Here \( A(t) \) a time-dependent amplitude:

\[ A(t) = A e^{-pt}. \]

and \( \omega_p \) is the frequency of the underdamped spring.

Since

\[ \frac{\omega_p}{\omega} = \sqrt{1 - (p/\omega)^2} < 1 \]

the damping of the spring leads to a frequency redshift and an increase in the period.

When \( p \) approaches \( \omega \) the period tends to infinity so that we may expect that the critically-damped spring does not display oscillatory motion.
A plot of $x(t)$ (for $x(0) = 0$ and $x'(0) > 0$).

For the underdamped spring in action, click here.
2. The critically-damped spring

For $p = \omega$ the characteristic equation

$$\lambda^2 + 2p\lambda + \omega^2 = 0$$

has one root of multiplicity two

$$\lambda_1 = -p.$$ 

Two linearly independent solutions are thus

$$e^{-pt} \quad \text{and} \quad t\, e^{-pt}$$

and the general solution is

$$x(t) = e^{-pt}(c_1 + c_2 t).$$
Unlike the underdamped spring it is straightforward to express $c_1$ and $c_2$ in terms of the initial conditions $x(0)$ and $x'(0)$:

$$c_1 = x(0) \quad \text{and} \quad c_2 = x(0)p + x'(0).$$

Therefore

$$x(t) = e^{-pt}(x(0) + x(0)pt + x'(0)t).$$

**Note:** The linear polynomial $x(0) + x(0)pt + x'(0)t$ vanishes for

$$t = -\frac{x(0)}{x(0)p + x'(0)}.$$

This corresponds to a positive time $t$ if $x'(0) < -x(0)p$. 
Depending on the initial conditions, the critically-damped spring behaves like

- The top-most curve corresponds to $x(0)$ and $x'(0)$ both positive, so that the weight is not just pulled down a distance $x(0)$ but is also given a downward kick on release.
- The middle curve corresponds to $x'(0) = 0$ so that we only pull down the weight and then release it.
- The third curve corresponds to $x'(0) < 0$ so that upon release the weight is kicked upward. Since the spring overshoots the equilibrium we must also have that $x'(0) < -x(0)p$. 

The overdamped or strongly damped spring

For $p > \omega$ the characteristic equation

$$\lambda^2 + 2p\lambda + \omega^2 = 0$$

has two distinct real roots

$$\lambda_1 = -p - \sqrt{p^2 - \omega^2} =: -\alpha$$
$$\lambda_2 = -p + \sqrt{p^2 - \omega^2} =: -\beta$$

with $\alpha > \beta > 0$.

Two linearly independent solutions are thus

$$e^{-\alpha t} \quad \text{and} \quad e^{-\beta t}$$

and the general solution is

$$x(t) = c_1 e^{-\alpha t} + c_2 e^{-\beta t}.$$
For generic initial conditions $x(0)$ and $x'(0)$, the constants $c_1$ and $c_2$ are both nonzero.

Assuming this, we have that for sufficiently large $t$

$$x(t)_{\text{critical damping}} \sim c_2 t e^{-pt}$$

and

$$x(t)_{\text{overdamped}} \sim c_2 e^{-\beta t}.$$ 

Since $\beta = p - \sqrt{\ldots} < p$ we see that the overdamped spring takes longer to return to equilibrium than the critically damped spring.
The overdamped spring well-describes a door-closer that is too tight. It may take a long time for the door to actually close because the strong damping restricts (nearly) all movement.

A will try to adjusting a door-closer such that the damping will be as close as possible to critical damping.

As always, Maths Rules!
We are going to make the spring-weight system even more complicated by introducing an additional external, time-dependent force $F_e(t)$ (acting along the $x$-axis only, so that the spring does not exhibit complicated two- or three-dimensional motion.)

In the case of the Tacoma bridge to be discussed later, this external force will be the wind.

The total force $F$ for the damped spring is now given by

$$F = F_e(t) - \beta x' - k(s + x) + mg$$

As before,

$$ks = mg$$

so that

$$F = F_e(t) - \beta x' - kx.$$
Thanks to Newton’s second law this yields the equation of motion

\[ mx'' + \beta x' + kx = F_e(t). \]

Introducing

\[ \omega^2 = \frac{k}{m} > 0 \quad \text{and} \quad 2p = \frac{\beta}{m} > 0 \]

as before, but also setting

\[ f(t) = \frac{F_e(t)}{m} \]

the equation of motion for the forced damped spring is given by

\[ x'' + 2p x' + \omega^2 x = f(t) \]
The previous equation is an inhomogeneous second order ODE with constant coefficients.

From the general theory we know that its solution is

\[ x(t) = x_H(t) + x_P(t) \]

with the complementary function \( x_H(t) \) the solution for the damped but unforced spring and \( x_P(t) \) a particular solution.

In the following we only consider a periodic external force:

\[ f(t) = f_0 \cos \omega_0 t \]

so that

\[ x'' + 2px' + \omega^2 x = f_0 \cos \omega_0 t. \]
To find a particular solution we first consider

\[ z'' + 2pz' + \omega^2 z = f_0 e^{i\omega_0 t}. \]

Setting \( z(t) = u(t) e^{i\omega_0 t} \) so that

\[ z' = (u' + i\omega_0 u) e^{i\omega_0 t}, \]
\[ z'' = (u'' + 2i\omega_0 u' - \omega_0^2 u) e^{i\omega_0 t} \]

we get

\[ u'' + 2(i\omega_0 + p)u' + (\omega^2 - \omega_0^2 + 2i\omega_0 p)u = f_0. \]

From this we may read off the particular solution

\[ u_P = \frac{f_0}{\omega^2 - \omega_0^2 + 2i\omega_0 p}. \]

**Note:** As long as \( p > 0 \) (non-zero damping) we have \( \omega^2 - \omega_0^2 + 2i\omega_0 p \neq 0 \) so that the above particular solution makes sense even when \( \omega_0 = \omega \). This is no longer true when \( p = 0 \).
Therefore

\[ z_P = \frac{f_0 e^{i\omega_0 t}}{\omega^2 - \omega_0^2 + 2i\omega_0 p} \]

and

\[ x_P = \frac{f_0}{(\omega^2 - \omega_0^2)^2 + 4\omega_0^2 p^2} \times \left( (\omega^2 - \omega_0^2) \cos(\omega_0 t) + 2\omega_0 p \sin(\omega_0 t) \right). \]

Again it will be convenient to rewrite the above in terms of a single trigonometric function, so that we get

\[ x_P = \frac{f_0 \cos(\omega_0 t - \xi)}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\omega_0^2 p^2}} \]

with \( \xi \) a phase shift:

\[ \tan \xi = \frac{2\omega_0 p}{\omega^2 - \omega_0^2} \]

and 1/\( \sqrt{\ldots} \) the gain.
**Note:** The amplitude of the particular solution is the amplitude $f_0$ of the external force times the gain.

Now that we have found a particular solution we know that the general solution for the forced damped spring is

$$x(t) = x_H(t) + x_P(t).$$

One can again consider three different cases, corresponding to weak damping, critical damping and strong damping, but the only case of significant interest is the first of these, determined by $p < \omega$. 
For weak damping we have previously worked out that

\[ x_H(t) = A e^{-pt} \cos(\omega_p t - \phi), \]

where \( \omega_p = \sqrt{\omega^2 - p^2} \).

Therefore

\[ x(t) = A e^{-pt} \cos(\omega_p t - \phi) + \frac{f_0 \cos(\omega_0 t - \xi)}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\omega_0^2 p^2}}. \]

There are two competing oscillations with respective frequencies \( \omega_p \) and \( \omega_0 \).
Since the oscillation with frequency \( \omega_p \) has an amplitude that decays exponentially, eventually the \( \omega_0 \) oscillation “wins” and for large \( t \) (mathematically, \( t \to \infty \))

\[ x(t) \to x_P(t). \]

For this reason \( x_P \) is also referred to as the steady state.
A plot of $x(t)$ (for $x(0) = 0$ and $x'(0) > 0$). After some initial irregularity due to competing frequencies, $x(t)$ is dominated by the particular solution or steady state $x_P(t)$. 
Now that we have learned that for large $t$ the particular solution $x_P$ dominates the complementary function $x_H$, we will investigate the $\omega_0$-dependence of $x_P$.

The amplitude of $x_P$ is given by

$$f_0 \sqrt{(\omega^2 - \omega_0^2)^2 + 4\omega_0^2 p^2}.$$  

To maximize this we need to maximize the gain, i.e., we need to minimize the function $h$ defined by

$$h(\omega_0) = (\omega^2 - \omega_0^2)^2 + 4\omega_0^2 p^2.$$  

Now

$$h'(\omega_0) = 4\omega_0 (\omega_0^2 - \omega^2 + 2p^2)$$

so that $h'(\omega_0) = 0$ gives the solutions

$$\omega_0 = 0 \quad (1) \quad \text{and} \quad \omega_0^2 = \omega^2 - 2p^2 \quad (2)$$

Note: The second solution only corresponds to real $\omega_0$ for $\omega^2 \geq 2p^2$.  

The solution (1) corresponds to a minimum of $h(\omega_0)$ when $2p^2 \geq \omega^2$

but is uninteresting as it does not correspond to a periodic external force, but to $f(t) = f_0$.

The solution (2) yields a minimum of $h(\omega_0)$ for positive $\omega_0$ when $2p^2 < \omega^2$.

In the remainder we will assume this “very weak damping”.
Defining the resonance frequency $\omega_r > 0$ as

$$\omega_r^2 = \omega^2 - 2p^2,$$

we have thus found that the amplitude of $x_P(t)$ attains its maximum (as a function of the frequency $\omega_0$ of the external force) at the resonance frequency.

Moreover, the amplitude at the resonance frequency is given by

$$\frac{f_0}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\omega_0^2p^2}} \bigg|_{\omega_0=\omega_r} = \frac{f_0}{2\sqrt{\omega^2 - p^2}}.$$
Although it is seen that the amplitude of $x_P(t)$ spikes at $\omega_0 = \omega_r$, true resonance (at least mathematically) corresponds to a singularity that arises when damping is absent.
Taking $p = 0$ we find that the previously obtained particular solution simplifies to

$$x_P(t) = \frac{f_0 \cos(\omega_0 t)}{\omega^2 - \omega_0^2}.$$  

This makes perfect sense, except at resonance (i.e., $\omega_0 = \omega$) when the above function has a singularity.

To investigate the nature of the solution at $\omega_0 = \omega$ we have to go back to the equation of motion

$$x'' + \omega^2 x = f_0 \cos(\omega t).$$

From the undamped spring we know that

$$x_H = A \cos(\omega t - \phi).$$

To find a particular solution we first consider

$$z'' + \omega^2 z = f_0 e^{i\omega t}.$$
Setting $z(t) = u(t)e^{i\omega t}$ so that

$$z' = (u' + i\omega u)e^{i\omega t}$$
$$z'' = (u'' + 2i\omega u' - \omega^2 u)e^{i\omega t}$$

we get

$$u'' + 2i\omega u' = f_0.$$  

From this we may read off the particular solution

$$u'_P = \frac{f_0}{2i\omega}$$

so that

$$u_P = \frac{f_0 t}{2i\omega}.$$  

**Note:** Since any particular solution will do, we do not need a constant in the expression for $u_P$. 
We thus find

$$z_P = \frac{f_0 t e^{i\omega t}}{2i\omega}$$

and

$$x_P = \frac{f_0 t \sin(\omega t)}{2\omega}.$$ 

Combining this with the complementary function $x_H$ we find that for the spring in resonance

$$A \cos(\omega t - \phi) + \frac{f_0 t \sin(\omega t)}{2\omega}.$$ 

Since the amplitude of the particular solutions grows linear in time, we find unbounded oscillations and potential disaster!
A plot of $x(t)$ displaying unbounded oscillations.
Although mathematical resonance (i.e., \( p = 0 \) and \( \omega_0 = \omega \)) cannot occur in real-life situations, we can get surprisingly close to it.

The most famous and destructive example (although not undisputed) is the 1940 Tacoma bridge disaster in which the third-largest suspension bridge in the USA (with the ominous nick-name “Galloping Gertie”) collapsed due to forced winds, causing the bridge to oscillate out of control.
Consider the following LRC circuit
Because of the applied voltage (battery, generator, power-point) electrical charge $Q$ (units: Coulomb) is moved around the circuit, leading to a current $I$ (units: Ampere):

$$I = \frac{dQ}{dt}.$$

- According to Ohm’s law, the voltage drop across a resistance $R$ (units: Ohm) in the circuit is $RI$. 
The voltage drop across an inductance $L$ (units: Henry) — creating a magnetic field — is

$$L \frac{dl}{dt}.$$
The voltage drop across a capacitance $C$ (units: Farads) is $Q/C$.

According to Kirchhoff’s second law the sum of the voltage drops across the circuit is equal to the applied external voltage $V$. 
Therefore

\[ V(t) = L \frac{dI}{dt} + RI + \frac{Q}{C}. \]

Since

\[ I = \frac{dQ}{dt} \]

this yields the second order ODE

\[ V(t) = L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C}. \]

Introducing

\[ p := \frac{R}{2L}, \quad \omega^2 := \frac{Q}{LC}, \quad f(t) := \frac{V(t)}{L} \]

this gives

\[ Q'' + 2p Q' + \omega^2 Q = f(t) \]

Compare this with the equation of motion of the forced, damped spring:

\[ x'' + 2p x' + \omega^2 x = f(t). \]
Mathematically the LRC circuit behaves exactly as the forced, damped spring!

To produce an alternating current (AC) we need the external source to be of the form

\[ f(t) = f_0 \cos(\omega_0 t) \]

(i.e., \( V(t) = V_0 \cos(\omega_0 t) \))

The particular solution \( Q_P(t) \) for this choice for \( f(t) \) may be copied from our earlier solution of the spring. Instead, however, we will do what most electrical engineers do and treat electrical circuits as being complex. Hence we consider

\[ q'' + 2p q' + \omega^2 q = f_0 e^{i\omega_0 t}. \]

A particular solution is now given by

\[ i\omega_0 q_P(t) = \frac{f_0 e^{i\omega_0 t}}{2p + i(\omega_0 - \frac{\omega^2}{\omega_0})} = \frac{V_0 e^{i\omega_0 t}}{R + i(\omega_0 L - \frac{Q}{\omega_0 C})}. \]
The complex quantities

\[ Z = R + i\left(\omega_0 L - \frac{Q}{\omega_0 C}\right) \]

and \( \frac{1}{Z} \) are known as the impedance and admittance, respectively.

Moreover

\[ \Re(Z^{-1}) \quad \text{is the conductance} \]

and

\[ \Im(Z^{-1}) \quad \text{is the susceptance} \]

For more on all of the above ask your local electrical engineer but not Dr O! He is severely technologically challenged.
The amplitude of the current going through the circuit is largest when

\[ \omega_0 = \left( \frac{Q}{LC} - \frac{R^2}{2L^2} \right)^{1/2}. \]

This is again called the resonance frequency.

Unlike the spring system where resonance is generally undesirable, electrical engineers really like resonance. For example, it may be used to tune your radio so that you can listen to your favourite programs on 774 ABC Melbourne.