620123 Applied Mathematics (Advanced)

Applications of first order ODEs

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Population modelling

Given a population of some species, there may be “individuals” who (not all at the same time)

- reproduce,
- die,
- migrate in or out of the system.

In the case of small populations — such as the fish in your fish tank, the lice on your little brother’s head, the people of Wagga Wagga anno 1902 — the modelling of the size of the population is necessarily

- **discrete**: there are no fractional individuals,
- **stochastic** (random): births and deaths do not occur at regular, predictable intervals,

and lies outside the scope of 620123.
In large populations — such as bacteria in your tub of yoghurt, fish in the Indian Ocean, the people of Melbourne anno 2007 — the error of ignoring discreteness and randomness becomes insignificant.

Hence the size $N$ of a large populations may be modelled as being a continuous function of $t$.

Which brings us to Thomas Robert Malthus, see Wikipedia.
Let $N(t)$ be the number of individuals at time $t$ where $t \in [t_0, \infty)$. Assume that

- $\lambda$ is the (constant) birth-rate per unit time
- $\mu$ is the (constant) death-rate per unit time.

Hence

$$\frac{dN}{dt} = \lambda N(t) - \mu N(t) = (\lambda - \mu)N(t).$$

Solving the ODE yields

$$N(t) = C e^{(\lambda - \mu)t}.$$ 

If at the initial time $t_0$ the population is given by $N_0$ we find the Malthusian law of population:

$$N(t) = N_0 e^{(\lambda - \mu)(t-t_0)}.$$
According to the Malthusian law there are three possible scenarios.

1. \( \mu > \lambda \): The population is doomed to extinction.

2. \( \mu < \lambda \): The population is doomed to catastrophic overcrowding, or in Malthus’ words, doomed to “misery and vice”.

3. \( \mu = \lambda \): The population remains constant.

Malthus further concludes that

*the argument is conclusive against the perfectibility of the mass of mankind.*
The world population is currently growing at a rate of approximately 1.25% per annum. If we take as initial condition the start of the new millennium with roughly 6 billion people in 2000, then

\[ N(t) = 6e^{0.0125(t-2000)} \text{ billion} \]

This predicts (counting in billions)

\[
\begin{align*}
N(2100) &= 21 \\
N(2200) &= 73 \\
N(2800) &= 132159
\end{align*}
\]

Ignoring global warming (which may reduce total land area, currently at around 150 million km\(^2\)) this gives about one square metre for each human being in the year 2800.
The Malhusian population law breaks down for very large populations because fertility rates will drop due to overcrowding, limited food resources, lack of water, etc.

A more realistic model is obtained by adding a “damping term” that will slow down growth when populations become (too) large:

\[
\frac{dN}{dt} = rN \left(1 - \frac{N}{\theta}\right),
\]

with \( \theta \gg 0 \).

This model is known as the logistic (demographic) model and was first proposed by Verhulst (1804–1849)
Observe that the logistic ODE
\[ \frac{dN}{dt} = rN \left( 1 - \frac{N}{\theta} \right) \]
has two singular solutions:

1. \( N(t) = 0 \) for all \( t \). The population is extinct.
2. \( N(t) = \theta \) for all \( t \). The population is saturated.

The logistic model is separable and may thus be solved exactly. Before we do so, let us see if we can deduce what the solution will look like without actually solving the ODE.
If \( N \) is very small

\[
\frac{dN}{dt} \approx rN, \quad (\ast)
\]

so that

\[
N(t) = N(t_0) e^{r(t-t_0)},
\]

corresponding to exponential growth.

One could object that very small \( N \) (say \( N = 1 \)) does not make sense, since such populations are doomed to extinction and won’t grow exponentially at all.

(Also, very small populations should not be modelled by non-stochastic continuous models).

More precise would be to take \( N \) sufficiently large (say 100000) but much less than \( \theta \). Then \( N/\theta \ll 1 \) and we again get (\ast).
Because *any* solution which starts with a non-zero initial condition $N(t_0)$ grows exponentially, we say that the singular solution $N(t) = 0$ is an **unstable equilibrium solution**.

![Graph showing exponential growth with equilibrium solution at $N(t) = 0$ for all $t$.]
The (exponential) growth guarantees that eventually solutions will get close to \( N(t) = \theta \).

To get a feel for the "approach to \( \theta \)" we set

\[
N(t) = \theta (1 - \phi(t)),
\]

where \( \phi \ll 1 \).

Since

\[
\frac{dN}{dt} = -\theta \frac{d\phi}{dt}
\]

we get

\[
\frac{d\phi}{dt} = -r \phi(t)[1 - \phi(t)] \approx -r \phi(t),
\]

so that

\[
\phi(t) \approx \phi(t_0) e^{-r(t-t_0)}.
\]

That is,

\[
N(t) \approx \theta - [\theta - N(t_0)] e^{-r(t-t_0)}.
\]
Populations that start close to saturation $\theta$ continue to grow closer to saturation. We say that $N(t) = \theta$ is an **stable equilibrium solution**.
Let us now solve the logistic model exactly.

Since
\[
\frac{dN}{dt} = rN \left(1 - \frac{N}{\theta}\right)
\]

is separable, we get
\[
\theta \int \frac{dN}{N(\theta - N)} = r \int dt.
\]

By the partial fraction expansion
\[
\frac{\theta}{N(\theta - N)} = \frac{1}{N} + \frac{1}{\theta - N}
\]

this yields
\[
\log|N| - \log|\theta - N| = rt + C.
\]

**Note:** We already known from our previous considerations that
\[0 < N < \theta\] so that the absolute-value signs are not really needed.
Hence
\[ \log\left( \frac{N}{\theta - N} \right) = rt + C \]
and
\[ \frac{N}{\theta - N} = B e^{rt}, \]
where \( B = \exp(C) \).

Since \( N < \theta \) we can multiply both sides by \( \theta - N \). Rearranging terms this gives the general solution
\[ N(t) = \frac{B\theta e^{rt}}{1 + B e^{rt}} = \frac{\theta}{1 + D e^{-rt}}, \]
where \( D = 1/B \).
Finally we fix the constant $D$ by taking $t = t_0$. Then

$$N(t_0) = \frac{\theta}{1 + D e^{-r t_0}}$$

which may be solved for $D$ to give

$$D = \frac{\theta - N(t_0)}{N(t_0)} e^{r t_0}.$$ 

Substituting this back into the general solution yields

$$N(t) = \frac{\theta N(t_0)}{N(t_0) + (\theta - N(t_0)) e^{-r(t-t_0)}}$$
Let us now compare the exact solution to our previous analysis.

1. When $N(t_0) \ll \theta$ and $t - t_0$ small enough, we have

$$N(t_0) \left(1 - e^{-r(t-t_0)}\right) \ll \theta e^{-r(t-t_0)}.$$  

Hence

$$N(t) = \frac{\theta N(t_0)}{N(t_0) + (\theta - N(t_0)) e^{-r(t-t_0)}}$$

$$= \frac{\theta N(t_0)}{N(t_0) \left(1 - e^{-r(t-t_0)}\right) + \theta e^{-r(t-t_0)}}$$

$$\approx \frac{\theta N(t_0)}{\theta e^{-r(t-t_0)}}$$

$$= N(t_0) e^{r(t-t_0)}$$

as before.
When $N(t_0) \approx \theta$ we have
\[
N(t_0) \gg (\theta - N(t_0)) e^{-r(t-t_0)}.
\]

Hence
\[
N(t) = \frac{\theta N(t_0)}{N(t_0) + (\theta - N(t_0)) e^{-r(t-t_0)}}
\]
\[
= \frac{\theta}{1 + \left(\frac{\theta}{N(t_0)} - 1\right) e^{-r(t-t_0)}}
\]
\[
\approx \theta \left(1 - \left(\frac{\theta}{N(t_0)} - 1\right) e^{-r(t-t_0)}\right)
\]
\[
= \theta - \frac{\theta}{N(t_0)}(\theta - N(t_0)) e^{-r(t-t_0)}
\]
\[
\approx \theta - (\theta - N(t_0)) e^{-r(t-t_0)}
\]
as before.

Note: The second equality follows from $1/(1 + x) \approx 1 - x$ for $|x| \ll 1.$
Note: The grey lines correspond to non-physical solutions.
You are a sensible person not spending your money on drinks, fast cars and . . . , but on maths books.

This lifestyle has saved you $1500, a third of which you wisely spend on more maths books. You put the remaining $1000 in a savings account with a fixed interest rate of 6% per annum.

This means that each year your $1000 will grow by a factor of 1.06. Hence after 12 years you will have roughly doubled your money:

$$1000 \times 1.06^{12} = 1000 \times 2.012 = 2012.$$ 

The adding of earned interest to your investment is known as **compounding**.
One day, just when you are having dinner, you receive a call offering you free financial advice.

You (again wisely) tell them about your investment but not about your love for mathematics.

After hearing you out they make you the following offer. Instead of computing the interest annually, David Tweed Investments will compound interest daily, making your money grow much, much faster. Of course there is a small price to be paid for this and instead of 6% they can only offer you 4% per annum.

Should you accept Tweed’s generous offer?
You consult your maths books about the problem and come up with the following . . .

Let $A(t)$ be the value of your investment at time $t$ with $A(0) = A_0$ the initial amount. (We measure time in years and value in dollars.) If we compound annually we have

$$A(t) = A_0(1 + r)^t,$$

where $r$ is the interest rate.

In our particular problem $r = 0.06$ and $A_0 = 1000$.

If we compound $n$ times a year (in Tweed’s case $n = 365$) with an annual interest rate of $r$ then on each occasion your money grows by the factor

$$1 + \frac{r}{n}.$$

But this occurs $n$ times a year, so that

$$A(t) = A_0\left(1 + \frac{r}{n}\right)^{nt}.$$

The above formula lets you compare Tweed’s offer to your current 6% annual growth.
Before you do so you note that in one of your books there is the nice formula

\[ \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x, \]

so that

\[ A(t) = A_0 e^{rt} \]

in the case of continuous compounding.

You now quickly calculate the following table

<table>
<thead>
<tr>
<th>( t )</th>
<th>((n; r) = (1; 0.06))</th>
<th>((365; 0.04))</th>
<th>((365; 0.06))</th>
<th>continuous comp.</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1124</td>
<td>1083</td>
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<td>3207</td>
<td>2225</td>
<td>3320</td>
<td>3320</td>
</tr>
</tbody>
</table>
According to the table

- You should report David Tweed to the ACIC for trying to rip off innocent people like yourself. What if you hadn’t been such a maths wiz?
- The difference between annual, daily or even continuous compounding is marginal. The dominant factor determining how fast your money will grow is the interest rate.

Because of the above we may as well assume continuous compounding when dealing with financial modelling.

In the present (trivial) model that means that the growth of your money is described by the IVP

\[ \frac{dA}{dt} = rA, \quad A(0) = A_0. \]
An example of an annuity is a saving account into which money is invested at regular intervals without withdrawals.

If the (fixed) interest rate is $r$ and the (fixed) added amount is $M$ per year (added continuously), then the IVP describing the annuity is

$$\frac{dA}{dt} = rA + M, \quad A(0) = A_0.$$ 

The ODE is separable, so that

$$\int \frac{dA}{rA + M} = \int dt$$

Hence

$$\frac{\log(rA + M)}{r} = t + C.$$ 

Note: We do not need $|rA + M|$ because $rA + M > 0$. 
Solving for $A$ yields

$$A(t) = \frac{B e^{rt} - M}{r}.$$ 

Setting $t = 0$ and using that $A(0) = A_0$ gives

$$A_0 = \frac{B - M}{r},$$

so that

$$B = rA_0 + M.$$

The solution to the IVP is thus

$$A(t) = A_0 e^{rt} + \frac{M}{r} (e^{rt} - 1).$$
For example, if we start with a capital of $1000 and add $500 in each year, then

<table>
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<th>$t$</th>
<th>$r=(0.04)$</th>
<th>(0.06)</th>
<th>(0.08)</th>
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<td>15235</td>
<td>19150</td>
<td>24350</td>
</tr>
<tr>
<td>20</td>
<td>17545</td>
<td>22654</td>
<td>29660</td>
</tr>
</tbody>
</table>

Since we have invested $1000 + 20 \times 500 = 11000$, our net profit is $6545$ (if $r = 0.04$), $11654$ (if $r = 0.06$) and $18660$ (if $r = 0.08$). **Note:** If the interest rate is doubled from 4% to 8% our profit almost triples.
We may also interpret the previous solution differently as corresponding to the repayment of a mortgage.

In this case $A_0$ is our initial mortgage, $A(t)$ is our debt at time $t$ (in years), $r$ is the interest (which leads to an increase of $A(t)$) and $-M$ is our annual (continuous) repayment (which leads to a decrease of $A(t)$).

Replacing $M$ by $-R$ (for repayment) we thus have

$$A(t) = A_0 e^{rt} - \frac{R}{r} (e^{rt} - 1).$$

The repayment should be sufficiently large so that $A(t)$ decreases over time. From the IVP

$$\frac{dA}{dt} = rA - R, \quad A(0) = A_0$$

it thus follows that $R > rA_0$. 
Since

\[ rA(t) = R - (R - A_0 r) e^{rt} \]

this will lead to \( A(t) = 0 \) at some fixed time \( t > 0 \): you have paid off your mortgage!

If \( A(0) = 100.000, \ r = 6\% \) and \( R = 8.000 \) it will take approximately 23 years to pay off your loan. Hence you pay the bank 
\( 23 \times 8.000 = 184.000 \), which is the initial loan plus an additional \( 84.000 \).

If \( R = 10.000 \) it will take approximately 15 years to repay your loan, so that the bank receives \( 15 \times 10.000 = 150.000 \), which is the initial loan plus an additional \( 50.000 \).

If \( R = 12.000 \) it will take approximately 11.5 years to repay your loan, so that the bank receives \( 15 \times 12.000 = 138.000 \), which is the initial loan plus an additional \( 38.000 \).
Note

- We again see the effect of compounding. Paying 12.000 a year instead of 8.000 leads to more than a halving of the bank’s profits (and you losses).

- It is instructive to also investigate the effects of a change in interest rates. Homework.

- The above is somewhat of an oversimplification because the value of money decreases over time due to inflation. With an inflation rate of 3% the value of your money (but also that of your debt!) halves in approximately 22 years.
These days DEs are all-pervasive in the modelling of financial markets. The most famous model is that of Black and Scholes for valuing options — and is used in some form or another by all financial institutions around the world.

A call option gives you the right to buy a stock between now and some fixed expiration date at some fixed strike price. There is however no obligation to actually ever buy the stock.

For example, if Telstra is trading for $3.00 today, you may buy a call option for $0.05 per stock with strike price $3.10 and expiration date 18 September 2007.

If Telstra goes up (beyond $3.15) you will make a profit but if Telstra goes down, you will at most have lost $0.05.

Not just stocks themselves have actual value, but also the options based on that stock (just like stocks, options may be traded.)
If $S(t)$ is the value of some stock at time $t$ then the value of the corresponding option is a function of $S$ and $t$:

$$V = V(S, t).$$

According to Black–Scholes the price of a stock may be well described by the differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.$$

Here $\sigma$ is a constant known as the volatility and $r$ is the (constant) interest rate.

Since the Black–Scholes equation is a partial DE and outside the scope of 620123, all prospective bankers, brokers and investment gurus will have to take 620302 “Chance and Options Pricing”.
Environmental modelling

Effluent from a factory flows into a lake in the Blue Mountains.

(photo by Steve Hudson)

The lake feeds a small stream, so that the volume of water in the lake is constant.

At time $t = 0$ (when the park ranger discovers the pollution from the factory) the concentration of the pollutant in the lake is $1\text{g/m}^3$.

The in-flow from the factory into the lake is $10\text{m}^3/\text{s}$ containing $2\text{g/m}^3$ of pollutant.

The volume of water in the lake is $10^8\text{m}^3$. 
Question
Find the concentration of pollutant at time $t$.

Note: Because NSW is in a draught we may ignore the effect of rain in our model.

Let $p(t)$ be the amount (read mass) of pollutant in the lake at time $t$, measured in grams.

- Since the in-flow from the factory into the lake is $10\text{m}^3/\text{s}$ containing $2\text{g}/\text{m}^3$ of pollutant, there is 20 grams of pollutant flowing into the lake per second at any time $t$.
- The out-flow of pollutant (in grams) at time $t$ is the concentration of pollutant in the lake at time $t$ times the out-flow. Since the concentration is $p(t)/V$ and since the out-flow is $10\text{m}^3/\text{s}$ this is

$$10 \times \frac{p(t)}{10^8} = p(t) \times 10^{-7}$$

grams per second.
In a small time-interval from $t$ to $t + \Delta t$ the change in mass of the pollutant in the lake is therefore

$$\Delta p = (\text{pollutant in}) - (\text{pollutant out})$$

$$= (20 - 10^{-7} p(t)) \Delta t.$$

Taking the limit $\Delta t \to 0$ we thus find the ODE

$$\frac{dp}{dt} = 20 - 10^{-7} p.$$

This is a separable ODE so that

$$\int \frac{dp}{20 - 10^{-7} p} = \int dt$$

and

$$-10^7 \log|20 - 10^{-7} p| = t + C.$$
After exponentiating this yields

\[ p(t) = 2 \times 10^8 + A e^{-10^{-7}t}. \]

At \( t = 0 \) the concentration of pollutant is 1g/m\(^3\). This corresponds to \( p(0) = 1 \times 10^8 \).

Therefore

\[ A = -1 \times 10^8, \]

leading to the final solution

\[ p(t) = 10^8 \left( 2 - e^{-10^{-7}t} \right) \]

If \( c(t) \) is the concentration of pollutant at time \( t \) (so that \( c(t) = p(t)/V = p(t)/10^8 \)) we thus find

\[ c(t) = 2 - e^{-10^{-7}t}. \]
This is the solution you should have expected: At time 0 the concentration is 1g/m$^3$ and for very large times (mathematically, $t \to \infty$) the concentration of pollutant will be that of the in-flow, i.e., 2g/m$^3$.

Can you determine for how long the factory has been polluting the lake?

The ranger is very alarmed by the above result because all fish will die if the concentration of pollutant exceeds 1.02g/m$^3$, which will happen

$$10^7 \log(0.98) \approx 202027$$

seconds into the future, which is approximately 140 days.
The ranger therefore partially blocks the in-flow so that only $8 \text{m}^3/\text{s}$ of effluent flows into the lake.

**Question**

How much longer will the fish survive?

The main difference with the previous situation is that the amount of water (volume) of the lake is no longer constant. Every second the lake is losing $2 \text{m}^3$ of water. Therefore

$$V(t) = 10^8 - 2t.$$
Repeating the previous analysis we now get the following.

- Since the in-flow from the factory into the lake is $8 \text{m}^3/\text{s}$ containing $2 \text{g}/\text{m}^3$ of pollutant, there is 16 grams of pollutant flowing into the lake per second at any time $t$. Before this was 20 grams per second.

- The out-flow of pollutant (in grams) at time $t$ is the concentration of pollutant in the lake at time $t$ times the out-flow. Since the concentration is $p(t)/V(t)$ and since the out-flow is $10 \text{m}^3/\text{s}$ this is

\[
\frac{10p(t)}{10^8 - 2t}
\]

grams per second. Before this was $p(t) \times 10^{-7}$. 
In a small time-interval from \( t \) to \( t + \Delta t \) the change in mass of the pollutant in the lake is therefore

\[
\Delta p = (\text{pollutant in}) - (\text{pollutant out}) \\
= \left( 16 - \frac{10p(t)}{10^8 - 2t} \right) \Delta t.
\]

Taking the limit \( \Delta t \to 0 \) we thus find

\[
\frac{dp}{dt} = \left( 16 - \frac{10p}{10^8 - 2t} \right).
\]

This ODE is no longer separable, but it is linear with

\[
P(t) = \frac{10}{10^8 - 2t} \quad \text{and} \quad Q(t) = 16.
\]
As integrating factor we may take

\[ I(t) = \exp \left( \int \frac{10 \, dt}{10^8 - 2t} \right) \]
\[ = \exp \left( -5 \log(10^8 - 2t) \right) \]
\[ = \frac{1}{(10^8 - 2t)^5}. \]

Hence

\[ \left( \frac{p}{(10^8 - 2t)^5} \right)' = \frac{16}{(10^8 - 2t)^5} \]

and

\[ \frac{p}{(10^8 - 2t)^5} = \frac{2}{(10^8 - 2t)^4} + C. \]

The solution to the linear ODE is thus

\[ p(t) = 2(10^8 - 2t) + C(10^8 - 2t)^5. \]
The initial condition is the same as before: \( p(0) = 10^8 \), leading to \( C = -10^{-32} \).

The final solution therefore is

\[
p(t) = 2(10^8 - 2t) - 10^{-32}(10^8 - 2t)^5
\]

**Note:** This only makes sense for \( 0 \leq t \leq 5 \times 10^7 \), since for larger times (that is, beyond approximately 95 years) the lake will be empty.

Of course what we are really after is the concentration of pollutant, i.e., \( p(t)/V(t) \). This is given by

\[
c(t) = 2 - 10^{-32}(10^8 - 2t)^4.
\]

We can now answer the question about the survival of the fish.

To this end we take \( c = 1.02 \) to find a corresponding time of approximately 251897 seconds. This is roughly 175 days giving the fish an extra 35 days respite.
Newton’s law of cooling

Newton’s law of cooling says that

\[ \text{The rate of change of the temperature of a “body” is proportional to the temperature difference between the “body” and its surrounding medium.} \]

If \( T \) is the temperature of the body and \( T_m \) the temperature of the medium then, according to Newton,

\[ \frac{dT}{dt} \sim T - T_m. \]

To make \( \frac{dT}{dt} \) negative in the case of actual cooling (we can also use the law for heating) we need a negative proportionality constant.

Hence

\[ \frac{dT}{dt} = -k(T - T_m), \]

with \( k > 0 \).
If at time $t_0$ the temperature of the body is $T_0$ then we get the IVP

$$\frac{dT}{dt} = -k(T - T_m), \quad T(t_0) = T_0.$$ 

The ODE is separable, so that

$$\int \frac{dT}{T - T_m} = -k \int dt$$

and

$$\log|T - T_m| = -kt + C.$$ 

After exponentiating this yields

$$T = T_m + Ae^{-kt}.$$ 

Since $T(t_0) = T_0$ it follows that

$$A = (T_0 - T_m)e^{kt}.$$
The final solution to the IVP is thus

\[ T(t) = T_m + (T_0 - T_m)e^{-k(t-t_0)} \]

**Note:** For large \( t \) (mathematically, for \( t \to \infty \)) the body will reach the same temperature as its surrounding medium.
Example

It is a 30°C day and you have to give a 9 o’clock lecture. Before you start you desperately need a tea (real (wo)men don’t drink coffee). You order a tea at B’s around the corner at 8:50 but it is literally boiling and you burn your tongue. You try again at 8:55 but the tea is still 95°C and too hot to drink. If tea is drinkable at 60°C, will you have time to have your fix before the class starts at 9:05? If not, and you have to wait till after the lecture, what will the temperature of your tea be at 10 o’clock?

A physicist would try to answer the question using
Being smarter than that you solve the problem using Newton’s law

\[ T(t) = T_m + (T_0 - T_m) e^{-k(t-t_0)}, \]

with \( T_m = 30 \) and \( T_0 = 100 \).

If you reset you stopwatch to 0 at 8:50 (so that \( t_0 = 0 \) instead of 8:50) then (measuring time in minutes)

\[ T(5) = 30 + (100 - 30) e^{-5k}. \]

Since \( T(5) = 95 \) this fixes \( k \) as

\[ k = \frac{1}{5} \log\left(\frac{14}{13}\right) \approx 0.01482. \]
This may be used to calculate your tea’s temperature at 9:05 (which for us is $t = 15$):

$$T(15) = 30 + 70 e^{-0.01482 \times 15} \approx 86^\circ C.$$ 

A bit of a bummer and you will have to wait till after your lecture. Then

$$T(70) = 30 + 70 e^{-0.01482 \times 70} \approx 55^\circ C.$$ 

Tea at last . . .

![Tea cup image]
It is your little brother’s birthday and you promised him some fireworks:

This being illegal you have to make compound $C$ yourself using the readily available $A$ and $B$.

Disclaimer: Dr O strongly disapproves of your plans. It is irresponsible and dangerous behaviour, and a much better present would be a maths book.
In your chemistry book you read that it takes 4 kg of $A$ and 1 kg of $B$ to make 5 kg of $C$.

You, being a bit wasteful, mix 2 kg of $A$ with 1 kg of $B$. After 10 minutes you have obtained 0.1 kg of $C$.

**Question**

How long will it take before you have the required 1 kg of $C$?

Provided you keep mixing really well, a reasonable assumption is that the reaction rate will be proportional to the amount of $A$ times the amount of $B$.

If we denote by $a(t)$, $b(t)$ and $c(t)$ the amounts of $A$, $B$ and $C$ (measured in kilograms) at times $t$ (measured in minutes), we thus have

$$\frac{dc}{dt} = k a(t)b(t).$$
This is not yet very useful because we do not know what \( a(t) \) and \( b(t) \) are.

At time \( t = 0 \) we of course know that

\[
    a(0) = 2, \quad b(0) = 1 \quad \text{and} \quad c(0) = 0.
\]

To say something about \( a(t) \) and \( b(t) \) at other times we observe that if we have \( c(t) \) of compound \( C \) at time \( t \) this has “used up” \( 4c(t)/5 \) of \( A \) and \( c(t)/5 \) of \( B \).

Therefore

\[
    a(t) = a(0) - \frac{4}{5} c(t)
\]

and

\[
    b(t) = b(0) - \frac{1}{5} c(t).
\]
Note that if we sum the above two equations we get

\[ a(t) + b(t) = a(0) + b(0) - c(t) \]

or

\[ a(t) + b(t) + c(t) = a(0) + b(0). \]

In the absence of nuclear reactions, the total mass should remain constant.

**Note:** Had we started with a nonzero amount \( C \) the conservation of mass would of course have implied the more general

\[ a(t) + b(t) + c(t) = a(0) + b(0) + c(0). \]
Substituting

\[ a(t) = a(0) - \frac{4}{5} c(t) \quad \text{and} \quad b(t) = b(0) - \frac{1}{5} c(t) \]

into

\[ \frac{dc}{dt} = k a(t)b(t) \]

— and using that \( a(0) = 2 \) and \( b(0) = 1 \) — gives

\[ \frac{dc}{dt} = k \left( 2 - \frac{4}{5} c \right) \left( 1 - \frac{1}{5} c \right) \]

We rewrite this slightly by replacing \( 2k/5 \) by \( r \). Then

\[ \frac{dc}{dt} = \frac{1}{5} r (5 - 2c)(5 - c) . \]
The ODE is separable, so that
\[ 5 \int \frac{dc}{(5 - 2c)(5 - c)} = r \int dt. \]

By the partial fraction expansion
\[ \frac{1}{(5 - 2c)(5 - c)} = \frac{1}{5} \left( \frac{2}{5 - 2c} - \frac{1}{5 - c} \right) \]
this yields
\[ \log|5 - c| - \log|5 - 2c| = rt + \gamma. \]

After exponentiating this yields
\[ \frac{5 - c}{5 - 2c} = \beta e^{rt}. \]

Finally solving for \( c \) gives
\[ c(t) = \frac{5\beta e^{rt} - 5}{2\beta e^{rt} - 1} = \frac{5\beta - 5 e^{-rt}}{2\beta - e^{-rt}}. \]
To fix $\beta$ we need the initial condition $c(0) = 0$. Then

$$0 = \frac{5\beta - 5}{2\beta - 1}$$

so that $\beta = 1$.

The final solution is thus

$$c(t) = \frac{5 - 5 e^{-rt}}{2 - e^{-rt}}$$

**Note:** Someone as clever as you could perhaps have guessed this answer: When $t = 0$ it gives $c = 0$ and when $t \to \infty$ it gives $c = \frac{5}{2}$ as it should. (Why?)
We are now in the position to answer the question.

It is in fact easiest not to use the final solution but the earlier result (*) with $\beta = 1$:

$$e^{rt} = \frac{5 - c(t)}{5 - 2c(t)}.$$

At $t = 10$ we have $c = 0.1$. Hence

$$e^{10r} = \frac{5 - c(10)}{5 - 2c(10)} = \frac{4.9}{4.8}$$

This gives

$$r = \frac{1}{10} \log(49/48) \approx 0.002062.$$

The time when you have $c = 1$ is thus determined by

$$e^{0.002062t} = \frac{5 - 1}{5 - 2} = \frac{4}{3}$$

and gives $t \approx 140$, i.e., two hours and twenty minutes.