1. Compute

\[ 4 \int_0^4 \frac{dx}{(x-2)(x+2)} \]

**Solution:**

Since

\[ \frac{4}{(x-2)(x+2)} = \frac{1}{x-2} - \frac{1}{x+2} \]

it is tempting to use the fundamental theorem of calculus to write

\[ 4 \int_0^4 \frac{dx}{(x-2)(x+2)} = \left[ \log |x-2| - \log |x+2| \right]_0^4 \]

\[ = \left[ \log \left( \frac{x-2}{x+2} \right) \right]_0^4 \]

\[ = \log \left( \frac{2}{6} \right) - \log 1 \]

\[ = - \log 3. \]

However, the FTC requires the integrand to be continuous on the interval of integration (i.e., on \([0, 4]\)) and in the present case there is a pole at \(x = 2\).

The correct approach is thus to set

\[ 4 \int_0^4 \frac{dx}{(x-2)(x+2)} = 4 \lim_{a \to 2^-} \int_0^a \frac{dx}{(x-2)(x+2)} + 4 \lim_{b \to 2^+} \int_b^4 \frac{dx}{(x-2)(x+2)}. \]

Since neither limit exists:

\[ \lim_{a \to 2^-} \int_0^a \frac{dx}{(x-2)(x+2)} = \lim_{a \to 2^-} \log \left| \frac{a-2}{a+2} \right|, \]

and something similar for the second integral, the integral diverges.

2. Compute the indefinite integral

\[ 2 \int \sqrt{1+u^2} \, du. \]

You are allowed to use without proof that \((\text{arcsinh}x)' = 1/\sqrt{1+x^2}\) or \(\sinh(2\text{arcsinh}x) = 2x\sqrt{1+x^2}\).

**Solution 1:**
By integration by parts:
\[
\int \sqrt{1 + u^2} \, du = u\sqrt{1 + u^2} - \int \frac{u^2 \, du}{\sqrt{1 + u^2}}
\]
\[
= u\sqrt{1 + u^2} - \int \frac{(1 + u^2 - 1) \, du}{\sqrt{1 + u^2}}
\]
\[
= u\sqrt{1 + u^2} - \int \frac{(1 + u^2) \, du}{\sqrt{1 + u^2}} + \int \frac{1 \, du}{\sqrt{1 + u^2}}
\]
\[
= u\sqrt{1 + u^2} - \sqrt{1 + u^2} \, du + \int \frac{1 \, du}{\sqrt{1 + u^2}}
\]
\[
= u\sqrt{1 + u^2} + \arcsinh u - \int \frac{1 \, du}{\sqrt{1 + u^2}}.
\]

Hence
\[
2 \int \sqrt{1 + u^2} \, du = u\sqrt{1 + u^2} + \arcsinh u + C.
\]

**Solution 2a:**

By the substitution \( u = \sinh v \) (so that \( du = \cosh v \, dv \)) we get
\[
2 \int \sqrt{1 + u^2} \, du = 2 \int \cosh v \sqrt{1 + \sinh v^2} \, dv
\]
\[
= 2 \int \cosh^2 v \, dv
\]
\[
= \frac{1}{2} \int (e^{2v} + 2 + e^{-2v}) \, dv
\]
\[
= \frac{1}{4}(e^{2v} + 4v - e^{-2v}) + C
\]
\[
= v + \frac{1}{2} \sinh(2v) + C
\]
\[
= \arcsinh u + \frac{1}{2} \sinh(2\arcsinh(u)) + C
\]
\[
= \arcsinh u + u\sqrt{1 + u^2} + C.
\]

**Solution 2b:** By the substitution \( u = \sinh v \) (so that \( du = \cosh v \, dv \)) we get
\[
2 \int \sqrt{1 + u^2} \, du = 2 \int \cosh v \sqrt{1 + \sinh v^2} \, dv
\]
\[
= 2 \int \cosh^2 v \, dv
\]
\[
= \int \left(1 + \cosh(2v)\right) \, dv
\]
\[
= v + \frac{1}{2} \sinh(2v) + C.
\]

Now follow Solution 2a.

3. Compute the arc length of the curve
\[
\mathbf{r}(t) = (\cos t, \sin^2 t), \quad t \in [0, 2\pi].
\]
Solution:

Since $\cos(\pi + t) = \cos(\pi - t)$ and $\sin^2(\pi + t) = \sin^2(\pi - t)$ the parametrization traverses the curve twice (from $(1, 0)$ (at $t = 0$) to $(-1, 0)$ (at $t = \pi$) back to $(1, 0)$ at $2\pi$.)

Hence we only need to consider $t \in [0, \pi]$.

If $L$ is the required length, then

$$L = \int_0^\pi |\mathbf{r}'(t)| dt.$$ 

Now

$$\mathbf{r}'(t) = (-\sin t, 2\sin t \cos t)$$

so that

$$|\mathbf{r}'(t)|^2 = (-\sin t)^2 + (2\sin t \cos t)^2 = \sin^2 t(1 + 4\cos^2 t).$$

Since $\sin t \geq 0$ for $t \in [0, \pi]$

$$|\mathbf{r}'(t)| = |\sin t|\sqrt{1 + 4\cos^2 t} = \sin t\sqrt{1 + 4\cos^2 t}.$$ Therefore

$$L = \int_0^\pi \sin t \sqrt{1 + 4\cos^2 t} dt.$$ 

By the substitution $u = 2\cos t$ (so that $du = -2\sin t dt$) the above integral becomes

$$L = \frac{1}{2} \int_{-2}^{2} \sqrt{1 + u^2} du = \frac{1}{2} \int_{-2}^{2} \sqrt{1 + u^2} du = \int_{-2}^{2} \sqrt{1 + u^2} du.$$ 

Using Question 2, this yields

$$L = \frac{1}{2} \left[U\sqrt{1+u^2} + \text{arcsinh}u\right]_{0}^{2} = \sqrt{5} + \frac{1}{2}\text{arcsinh}2.$$ 

Alternatively, the problem may be approached using Cartesian coordinates.

Since $x(t) = \cos t$ we have $y(t) = \sin^2 t = 1 - \cos^2 t = 1 - x^2(t)$. In other words we are looking at the parabola

$$y = f(x) = 1 - x^2$$

for $x \in [-1, 1]$.

The length therefore is

$$L = \int_{-1}^{1} \sqrt{1 + 4x^2} dx = \frac{1}{2} \int_{-2}^{2} \sqrt{1 + u^2} du.$$ 

The rest proceeds as before.
4. Solve the IVP

\[(x + y)y' = x - \frac{y^2}{x}, \quad y(1) = 1.\]

**Solution:**

Since

\[x - \frac{y^2}{x} = \frac{x^2 - y^2}{x} = \frac{(x - y)(x + y)}{x} = \left(1 - \frac{y}{x}\right)(x + y)\]

the ODE may be recognized as being of homogeneous type:

\[y' = 1 - \frac{y}{x}.\]

(Note that dividing left and right by \(x + y\) is harmless because the initial condition rules out \(y = -x\).)

The standard substitution to solve the above is \(y = vx\), etc. Not following the rules is always fun, so let us instead try \(y = v/x\), i.e., \(v = xy\). Then \(v' = xy' + y\) so that the ODE becomes

\[v' = x\]

with general solution

\[v(x) = \frac{1}{2}x^2 + C.\]

In terms of \(y\) this yields

\[y(x) = \frac{1}{2}x + \frac{C}{x}.\]

Finally we fit the inition condition \(y(1) = 1\) to find \(C = 1/2\). Hence the solution to the IVP is

\[y(x) = \frac{1}{2}(x + x^{-1}).\]

5. Solve the ODE

\[x^2(1 - x)y' = (y - 1)(x + 2).\]

**Solution:**

The ODE is separable, so that we get

\[\int \frac{ydy}{y - 1} = \int \frac{(x + 2)dx}{x^2(1 - x)}.\]

To compute the \(y\)-integral we proceed as follows:

\[\int \frac{ydy}{y - 1} = \int \frac{(y - 1 + 1)dy}{y - 1} = \int \left(1 + \frac{1}{y - 1}\right)dy = y + \log|y - 1| + C.\]

For the \(x\)-integral we need to make the continued fraction expansion:

\[\frac{x + 2}{x^2(1 - x)} = \frac{3}{x} + \frac{2}{x^2} + \frac{3}{1 - x}.\]
\[
\int \frac{(x + 2)\,dx}{x^2(1 - x)} = 3 \log |x| - \frac{2}{x} - 3 \log |1 - x| + C = 3 \log \left| \frac{x}{1 - x} \right| - \frac{2}{x} + C.
\]

The general solution to the ODE is thus
\[
y + \log |y - 1| = 3 \log \left| \frac{x}{1 - x} \right| - \frac{2}{x} + C.
\]

This is, however, not all because it readily follows that the ODE also has the singular solution \( y = 1 \).

6. (a) Show that \( y_p(x) = \frac{1}{x} \) is a particular solution to the ODE
\[
y' = -\frac{1}{x^2} - \frac{y}{x} + y^2.
\]

(b) Use the substitution
\[
y = \frac{1}{x} + \frac{1}{v} \quad (*)
\]

(with \( v = v(x) \)) to find the general solution.

Solution:
Since \( y_p'(x) = -\frac{1}{x^2} \) the substitution of \( y_p \) into the ODE gives
\[
-\frac{1}{x^2} = -\frac{1}{x^2} - \frac{1}{x^2} + \left( \frac{1}{x} \right)^2.
\]

Obviously this is true.

Making the substitution (*) gives
\[
y' = -\frac{1}{x^2} - \frac{v'}{v^2}. \quad (**)\]

Using (*) and (**) to eliminate \( y \) from the ODE we obtain the linear ODE
\[
v' + \frac{v}{x} = -1.
\]

Multiplying the ODE by the integrating factor
\[
I(x) = \exp \left( \int \frac{dx}{x} \right) = \exp(\log x) = x
\]
yields\[
(vx)' = -x.
\]

Integrating we thus find
\[
v(x) = C - \frac{x}{2}.
\]
By (*) we finally get

\[ y(x) = \frac{1}{x} + \frac{1}{C \frac{x}{x - 2}}. \]

Note that unlike linear ODEs it is not true that the general solution is of the form \( y(x) = y_p(x) + C f(x) \). In the above example the only way to recover the particular solution by taking the limit \( |C| \to \infty \).