The exam is worth 15\% of your final grade.
Your solutions should be submitted by 5pm on \textbf{WEDNESDAY OCTOBER 4}.

There are EIGHT questions, all of equal weight.
You should give complete explanations in all questions.

1. (a) Consider a first order ODE in the form

\[ M(x, y) \frac{dy}{dx} + N(x, y) = 0. \]

Suppose that there exists a function \( \Phi(x, y) \) such that
\[ \frac{\partial \Phi}{\partial y} = M \quad \text{and} \quad \frac{\partial \Phi}{\partial x} = N. \]

Prove that the general solution of the ODE is
\[ \Phi(x, y) = c. \]

(b) Find the general solution to the ODE
\[ (e^y \cos y + 2 \cos x) \frac{dy}{dx} = 2y \sin x - e^y \sin y. \]

2. Consider the ODE
\[ xy'' - (x+N)y' + Ny = 0, \]
where \( N \) is a natural number.

(a) Show that one solution to the ODE is \( y_1 = e^x \).
(b) Find the general solution to the ODE.

3. A population of rabbits and foxes is living together in a forest. Let \( R(t) \) be the number of rabbits and \( F(t) \) be the number of foxes at any time \( t \). Since foxes enjoy meeting rabbits, but not vice versa, we can model the populations by the \textit{system} of equations
\[ \begin{align*}
\frac{dR}{dt} &= aR - bRF \\
\frac{dF}{dt} &= cF + dRF,
\end{align*} \]
where \( a, b, c, d \) are positive constants.
By considering \( R \) as a function of \( F \), find an ODE for \( R(F) \). Solve the ODE
to give an equation relating \( R \) and \( F \). (You may assume \( R, F > 0 \)).

4. A \textit{generalised logistic equation} is of the form
\[ \frac{dP}{dt} = b(t)P - c(t)P^p \]
where \( b(t) \) and \( c(t) \) are given positive functions.

(a) By setting \( P = \frac{1}{y} \) find the general solution to this ODE. Your answer should involve integrals of \( b \) and \( c \).
(b) Suppose that \( b(t) = b \) is constant. Show that if \( P(0) = P_0 > 0 \) then
\[ \lim_{t \to \infty} P(t) = \frac{b}{\lim_{t \to \infty} c(t)} \]
as long as the limit on the right exists.

\textbf{PLEASE TURN OVER}
5. Consider a second order linear homogeneous ODE
\[ y'' + P(x)y' + Q(x)y = 0. \]
Suppose \( y_1 \) and \( y_2 \) are two solutions to this ODE on an interval \([a, b]\), and define
\[ W = y_1 y_2' - y_2 y_1'. \]
Prove that \( W \) satisfies the ODE
\[ W' + P(x)W = 0. \]
Hence show that \( W \) is either identically zero or never zero.

6. Consider a second order linear inhomogeneous ODE
\[ y'' + P(x)y' + Q(x)y = R(x). \]
Suppose \( y_1 \) and \( y_2 \) are linearly independent solutions of the complementary (homogeneous) equation. Let \( W \) be defined as in the previous question, and suppose \( u_1 \) and \( u_2 \) are defined by
\[
\begin{cases}
  u_1' = -\frac{y_2 R}{W} \\
  u_2' = \frac{y_1 R}{W}.
\end{cases}
\]
(You may assume that \( W \) is never \( 0 \)).

(a) Prove that
\[ u_1 y_1 + u_2 y_2 = 0. \]
(b) Prove that
\[ u_1'' y_1 + u_1' y_1' + u_2'' y_2 + u_2' y_2' = 0. \]
(c) Prove that
\[ y = u_1 y_1 + u_2 y_2 \]
is a solution to \((*)\).
(d) Hence or otherwise, find the general solution to
\[ y'' + y = R(x). \]
Your answer should involve integrals of \( R \).

7. Consider the boundary value problem
\[
\begin{align*}
  y'' + \lambda y &= 0 \\
  y(0) &= 0 \\
  y(\pi) &= 0,
\end{align*}
\]
where \( \lambda \) is a real constant. \( y = 0 \) is always a solution of this problem. Find all \( \lambda \) for which the problem has a non-zero solution.

8. You have poured yourself a cup of hot coffee. You are just about to add some cold milk to your coffee when the phone rings. You’re worried about your coffee being cold when you get off the phone. Should you add the milk before or after you take the phone call?
Assume that Newton’s law of cooling applies, and that the cooling constant \( k \) is the same for milk and coffee. You may assume the milk stays in the fridge (at constant temperature) until you use it. You may also assume that whenever you add the milk to the coffee, the temperature of the mixture is the weighted average of the temperatures of the milk and coffee. (For example, if the milk was at temperature \( M \), the coffee was at temperature \( C \) and if there was twice as much coffee as milk, then the temperature \( T \) of the mixture would be \( T = \frac{M + 2C}{3} \)).

END OF EXAM