1 Initial Value Problems ([T: §9.1, pp642-643])

In the previous handout, we gave techniques for solving a first order ODE:

\[
y' = F(x, y)
\]

In all cases, we were attempting to obtain the general solution to the ODE, with a \(+c\) appearing somewhere in the answer. However, we didn’t address the issue there of whether we had found all solutions to the ODE; we need to look more carefully at what solutions actually exist.

Further, in applications, it is much more common to be interested in a specific solution, where the \(+c\) is then nailed down to be a specific value. The most common such problem is an initial value problem, where the the function \(y\) is specified to have a certain value \(y_0\) at a given point \(x_0\), an initial condition:

\[
\begin{align*}
y' &= F(x, y) \\
y(x_0) &= y_0
\end{align*}
\]

1.1 Example

Consider the Malthus Equation (Handout 2, p1), where now the value at \(x = x_0\) is specified to be \(y = y_0\):

\[
\begin{align*}
y' &= ky \\
y(x_0) &= y_0
\end{align*}
\]

In Handout 2 (§2.3 and §3.2), we solved the Malthus ODE to give the general solution

\[
y(x) = Ae^{kx}.
\]

But then the initial condition tells us that when \(x = x_0\) then \(y = y_0\). Thus

\[
y_0 = Ae^{kx_0} \implies A = y_0 e^{-kx_0}.
\]

We can then substitute this value for \(A\) into (1), to give the specific solution to the IVP:

\[
y = y_0 e^{k(x-x_0)}.
\]
Then plugging this back into (4), we find

\[ y = \frac{\frac{3}{2}x^2}{1 - \frac{2}{3}x} \]

\[ \Rightarrow y = \frac{3x^2}{4} - \frac{3}{8}x . \]

\[ \diamondsuit \diamondsuit \diamondsuit \]

Note the important practical message of this example:

To solve the IVP, we need the general solution to the ODE, but in evaluating the constant, we can use any earlier convenient equation.

2 Singular Solutions (No Reference)

The idea of a singular solution to an ODE is a solution which does not arise from the "general method". In brief, these arise for us when considering a separable ODE,

\[ y' = f(x)y(g(y)) . \]

If \( g(y^*) = 0 \) then we have to assume \( y \neq y^* \) when solving the ODE (because we divide both sides by \( g(y) \)): however, the constant function \( y = y^* \) clearly solves the ODE.

However, there is some imprecision in the concept of a singular solution, since it can depend upon the method used.

2.1 Example

When solving the Malthus Equation \( y' = y \) as a separable equation (Handout 2, §2.3), we obtained the general solution

\[ y = Ae^x . \]

However, the method excluded \( A = 0 \): in effect, we obtained \( A = \pm c^e \), where \( c \) was an arbitrary constant of integration. However, as we noted, it is obvious that the constant function \( y = 0 \) also solves the ODE. Thus, \( A = 0 \) is also permitted, and we refer to \( y = 0 \) as a singular solution. Note that if we want to solve any IVP for the Malthus equation, we need to include the possibility of this singular solution.

Note also, that there is a sense in which the singular solution for Malthus is not all that singular: first of all, we can obtain the singular solution simply by extending the possibilities for \( A \); secondly, if we treat Malthus as a Cauchy equation (Handout 2, §4.1) then we automatically obtain all solutions, and the issue of \( A = 0 \) doesn’t arise!

2.2 Example

Consider the separable equation

(\( \heartsuit \))\[ y' = y^2 . \]

We can easily solve this ODE:

\[ \int \frac{1}{y^2} dy = \int 1 dx \]

\[ \Rightarrow -\frac{1}{y} = x + c \]

\[ \Rightarrow y = -\frac{1}{x + c} . \]

Here \( c \) is completely arbitrary, but we have still missed a solution: the constant function \( y = 0 \) is also a solution to the ODE. Note that in this case the singular solution does not arise from a new value of the constant of integration. However, if we think of \( x \) being fixed and let \( c \to \infty \), the singular solution can be obtained as the limit of regular solutions.

2.3 Example

Consider the ODE

(\( \clubsuit \))\[ y' = \frac{y^3 + 2xy}{x^2} , \]

for which we obtained (Handout 2, §4.1) the general solution

(\( \spadesuit \))\[ y = \frac{Ax^2}{1 - Ax} . \]

As for the Malthus equation, the calculation gives \( A = \pm c^e \), but \( A = 0 \) gives the genuine solution \( y = 0 \): this is a singular solution.

There is another singular solution, which is not so obvious. Recall that the method of solving (\( \spadesuit \)) is to substitute \( v = \frac{1}{x} \), which gives rise to the ODE

\[ \frac{dv}{dx} = v^2 + v . \]

This is a separable ODE, and so it can give rise to singular solutions. One such solution is \( v = 0 \), leading to \( y = 0 \), which we have already noted. The other singular solution is \( v = -1 \), giving rise to a second singular solution to (\( \spadesuit \)):

\[ y = -x . \]
If we divide top and bottom of (★) by $A$, we can write the general solution as
$$y = \frac{x^2}{\pi} - x.$$ Then, in this form, the singular solution $y = -x$ can be obtained by letting $A \to \infty$.

### 3 The Existence and Uniqueness Theorem (No Reference)

Most ODE’s have infinitely many solutions: when do we have enough? The appropriate question is phrased in terms of IVP’s: when does there exist a solution to the IVP

$$(\text{IVP}) \quad \begin{cases} y' = F(x, y) \\ y(x_0) = y_0. \end{cases}$$

and when is that solution unique?

For a specific ODE which we can solve, all solutions may be apparent. For example, for the Malthus equation $y' = y$, we explicitly found a solution $y = y_0 e^{k(x-x_0)}$ for any initial condition $x = x_0, y = y_0$ ([1], noting that we need to include the “singular” solution $y = 0$ to cover all possible initial conditions); moreover, the solution procedure (either as a separable or as a linear ODE) guarantees that we have found all possible solutions.

We can see the completeness of this collection of solutions, by graphing all the solutions together: in this context, we call the graphs of the solutions integral curves.

![Integral Curves](image)

The fact that there exists a solution to the initial condition $x = x_0, y = y_0$ corresponds to there being an integral curve passing through the point $(x_0, y_0)$. The fact that the solutions to the IVP’s are unique is illustrated by the fact that none of the integral curves cross.

### 3.1 Example

The situation is much less clear if we cannot solve the ODE. For example consider the IVP

$$(\text{IVP}) \quad \begin{cases} y' = x^2 + y^2 \\ y(1) = 3 \end{cases}$$

We cannot solve this IVP (or at the very least, it’s not obvious how to solve it). But we can still ask, does there exist a solution to the IVP, even if we cannot explicitly write it down? The answer is (sort of) “yes”, as long as the ODE is of a reasonable form:

### 3.2 Statement of The Existence and Uniqueness Theorem

Consider the IVP

$$(\text{IVP}) \quad \begin{cases} y' = F(x, y) \\ y(x_0) = y_0 \end{cases}$$

Consider a rectangle

$R = [x_0, a] \times [y_0 - b, y_0 + b].$

and suppose that

- $F(x, y)$ is continuous on $R$.
- $\frac{\partial F}{\partial y}(x, y)$ is continuous on $R$.

Then there is a $\delta > 0$ such that (IVP) has a unique solution on $[x_0, x_0 + \delta]$.

We consider some examples below, but we first make a few remarks.

(a) The Theorem is not easy to prove, and is beyond the scope of 620123 (even if we had time). We hope to make some comments on the proof later.

(b) Even if the function $F$ is continuous on a huge rectangle (even everywhere), the Theorem only promises that solution to the IVP exists for an interval of some, perhaps tiny, length $\delta$. This may seem a deficiency, but the examples below show that the Theorem has to be worded this way.

(c) Though $\delta$ may be quite small (smaller than $a$), one can in fact obtain an explicit estimate for $\delta$. Suppose $M$ is an upper bound for $|F(x, y)|$ on $R$. That is,

- $|F(x, y)| \leq M$ for all $(x, y) \in R$.

Then it turns out that we can always take

$$(\triangledown) \quad \delta = \min \left( \frac{b}{M}, \frac{a}{2M} \right).$$
3.3 Example

Consider the IVP we introduced above

\[
\begin{aligned}
y(t) &= x^2 + y^2 \\
y(0) &= 1
\end{aligned}
\]

Here, \(F(x, y) = y^2\), and so obviously \(F\) and \(F_y\) are continuous for all \(x\) and \(y\). Thus The Existence and Uniqueness Theorem promises that \((\square)\) has a unique solution on some interval \([\alpha, \beta]\).

Note that we don’t have to choose a specific rectangle to make this claim. However, choosing such a rectangle gives a definite (though maybe not the largest possible) value for \(\delta\). For example, suppose we choose (rather arbitrarily) the rectangle \(R = [0.3] \times [0.2]\), i.e. \(a = 3\) and \(b = 1\). Then \(|F_y| = |2y|\) has a maximum value of \(4\) on \(R\). Thus, we can choose

\[\delta = \min\left(\frac{b}{M}, \frac{a}{M}\right) = \min\left(3, \frac{1}{4}\right) = \frac{1}{4}\]

\[\square\square\square\]

3.4 Example

Consider the IVP

\[
\begin{aligned}
y' &= y^2 \\
y(0) &= 1
\end{aligned}
\]

Our Theorem promises us existence of a solution to the IVP on some interval \([0, \delta]\), but we can also just solve the IVP. In §2.2 we derived the general solution to the IVP,

\[y = \frac{1}{x + c}\]

The initial condition then gives \(c = -1\), and thus the solution to the IVP is

\[y = \frac{1}{1 - x}\]

Notice that, even though \(F(x, y) = y^2\) is nice everywhere, the solution to the IVP only exists on the interval \([0, 1]\), before exploding. Thus, whatever rectangle we choose to apply the Theorem on, we’ll have to end up with \(\delta < 1\). (In fact, this is the only thing that can go wrong: as long as the solution to an IVP stays within the given rectangle \(R\), the solution can be continued further. Choosing \(\delta\) according to \((\square)\) guarantees this).

Notice that the solution to the IVP \((\square)\) will similarly explode, since

\[y' = x^2 + y^2 \geq y^2\]

(Such comparisons of solutions to ODE’s can be made rigorous.)

\[\square\square\square\]

3.5 Example

Consider the IVP

\[
\begin{aligned}
y' &= \sqrt{y} \\
y(0) &= 0
\end{aligned}
\]

Here, \(F(x, y) = \sqrt{y}\), and the Theorem doesn’t apply: \(F\) is only continuous for \(y \to 0^+\) (though this can be dealt with), and \(F_y = \frac{1}{2\sqrt{y}}\) is fundamentally discontinuous at \((0,0)\).

To solve the ODE, we calculate

\[\int \frac{1}{\sqrt{y}} \, dy = \int 1 \, dx\]

\[2\sqrt{y} = x + c\]

\[y = \left(\frac{x + c}{2}\right)^2\]

The initial condition then gives \(c = 0\), and we have

\[y = \frac{x^2}{4}\]

is a solution to the IVP for \(x \geq 0\).

However, it is easy to see that \(y = 0\) is also a (singular) solution to the IVP. Thus, in this case, there exists a solution to the IVP, but the solution is not unique.

\[\square\square\square\]
4 Direction Fields and Isoclines ([T: §9.1, pp644-645])

If we cannot solve an ODE (and sometimes even if we can), it can be helpful to try to picture together how all the solutions behave. To illustrate this, consider our unsolvable ODE from above:

\[ y' = x^2 + y^2. \]

Of course \( y' = \frac{dy}{dx} \) indicates the slope of the given solution to the ODE. This means, for instance, that at any point \((x,y)\) on the unit circle

\[ F(x, y) = x^2 + y^2 = 1 \]

the slope of integral curve will be 1 at that point. We indicate this by drawing little arrows of slope 1 along the unit circle. Similarly, for any constant \( m \)

\[ x^2 + y^2 = m \implies \frac{dy}{dx}(x, y) = m. \]

Drawing a few of these “curves of constant slope”, with the attached arrows, we can get a sense of how the integral curves must travel. The curves of constant slope (i.e. the circles for this example) are called isoclines, and the collection of indicative arrows is called the direction field of the ODE.