620123 - LECTURE SUMMARY 6
SECOND ORDER ODE’S - PART 1
TECHNIQUES FOR SOLVING
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This is a summary of my lectures 14-18, roughly equal to lectures 13-17 in the subject
guide. Note that I am presenting the topics in a different order: use the Introduction and
Summary, and the section and subsection titles to guide you). There are no good references
to Thomas [T] for this material. However, there are tons of good “Introduction to Ordinary
Differential Equations” texts in the library: it would be well worth your while to grab a
couple, to have on your desk for reference.

Note: the notes which were distributed contained some errors. These have been corrected
(and there are some other minor changes), and are flagged by footnotes.

1 Introduction and Summary
As with our general introduction to ODE’s, we begin with some examples of second order
ODE’s.
\[
\frac{d^2x}{dt^2} + kx = 0 \quad (k > 0 \text{ constant}) \quad \text{(Spring Equation)} \quad x = A \cos(\sqrt{k}t) + B \sin(\sqrt{k}t)
\]
\[
\frac{d^2y}{dt^2} + \frac{g}{L} \sin \theta = 0 \quad \text{(Pendulum Equation)} \quad \theta = ???
\]
\[
(t - x^2)y'' - 2xy' + 2y = 0 \quad \text{(Legendre Equation)} \quad y = x \quad \text{(and anything else?)}
\]
\[
y'' + M(x)y' = 0 \quad \text{(Reduction Equation)} \quad y = A + B \int e^{-f M(t)}
\]
As previously, we are pulling these solutions out of thin air: our main task in these notes
is to establish general methods for finding solutions to such ODE’s.

Though it is not the most general form, we consider second order differential equations
to be of the form
\[
y'' = F(x, y, y')
\]
Note that the Legendre Equation can be put in this form, simply by dividing through by
1 – x^2. Of particular importance (both in 123 and the real modelling world) are linear
second order ODE’s:

\[
\begin{align*}
L: & \quad y'' + P(x)y' + Q(x)y = R(x) \\
\text{LH:} & \quad y'' + P(x)y' + Q(x)y = 0
\end{align*}
\]
(standard form)

Note that all of the above equations are linear except the Pendulum Equation; the Legendre
Equation is linear, and can be put in standard form by (again) dividing through by 1 – x^2.
Within the study of linear equations, a very important special case is of homogeneous
equations, where the RHS R(x) = 0.

(Recall that we have two uses for the word “homogeneous”: here, the usage has nothing to
do with spotting 1/x in the equation). The general solution to (LH) is called the comple-
mentary function. The above three linear equations are all homogeneous. If R(x) ≠ 0,
we say the equation is inhomogeneous.

As with first order ODE’s, but much more so, the reality is that most important
second order ODE’s (i.e. the ones you run into in real modeling) simply cannot be solved;
furthermore, most of the important ODE’s which can be solved are linear. Nonetheless,
certain techniques do work (at times) to solve certain ODE’s and we’ll learn some such
techniques. We’ll enumerate the techniques/ODE’s in a moment, but we first make a
couple important pragmatic remarks.

• For a second order ODE, we expect TWO constants of integration.
We’ll discuss this much more later (see §5), but if we think of needing to anti-
differentiate twice to solve a second order ODE, then we can expect two constants
to pop up. This gives us the intuition for when we’ve found enough solutions to stop
searching for more. For example, even if we guess y = x, or even y = Ax, solves
the Legendre equation above, we should suspect that there are other solutions to be
found.

• 123 is an artificial environment - use it to your advantage!
In the general world of ODE’s, anything or nothing might work. But in 123, we
have a small number of techniques (basically six), and (unless your lecturers goofed
or took an overdose of Evil Pills) at least one of them will work. Look through
the possibilities and find it!

We now list the type of ODE’s to come, and very briefly describe the techniques we use
to (attempt to) solve them.
1. ODE’s WHICH ARE FIRST ORDER ODE’S IN DISGUISE (§4)
These fall into two subcategories:
(a) Where the unknown function $y$ doesn’t explicitly occur (§4.1)

The Reduction Equation above is an example of this. We’ll go through it in detail (§4.1: see also §3, where the Reduction Equation is applied). But, briefly, if $y$ doesn’t occur in the ODE then we can introduce $p = y'$ as a new unknown function, and the ODE becomes a first order ODE for $p$.

(b) Where the variable $x$ doesn’t explicitly occur (§4.2)

The Spring Equation and the Pendulum Equation are of this type (though it is more natural to think of the Spring Equation as linear). The technique for solving such an equation is not obvious, but a cute trick transforms it into a first order ODE for a new function (with a new independent variable): see §4.2.

2. LINEAR ODE’S (§§2,3,5,6)

Here, there are subcategories to consider, but in two different ways:
(a) Constant coefficient linear ODES

Here the coefficients $P$ and $Q$ of $y'$ and $y$ are constants, the Spring Equation being a standard example. (We consider the RHS $R(x)$ below). For such ODE’s there are general and simple (sometimes tedious) techniques. See §2 and §6.2.

(b) Linear ODE’s with general coefficients

So, we also consider linear equations with $P(x)$ and $Q(x)$ general functions. Such equations are generally much harder to solve. But a key fact is, if we can find (or are given) one solution to the ODE, then there are methods to find the general solution to the ODE. See §3 and §6.1.

The other splitting into subcategories is:

(i) Linear Homogeneous ODE’s

So, these are equations of the form (LIH), with RHS = $R(x) = 0$. The key fact about such equations is that any linear combination of solutions is again a solution; we’ll see many examples of this, and it is discussed fully in §5.

(ii) Linear Inhomogeneous ODE’s

These can be much more difficult to solve. Here, a key fact is, if we can find one solution to the inhomogeneous equation, then finding the general solution reduces to solving the corresponding homogeneous equation. See §5.

There are thus four subtypes of linear ODE’s we shall consider: first we consider homogeneous linear ODE’s, with (a)(i) in §2 and (b)(i) in §3; we then consider the theory of linear ODE’s in §5; finally, we consider inhomogeneous linear ODE’s in §6, with (b)(ii) in §6.1 and (a)(ii) (with appropriately simple RHS = $R(x)$) in §6.2.

2 Constant Coefficient Linear Homogeneous ODE’s
Here, we’re considering an equation of the form

\[ ay'' + by' + cy = 0 \]

$a, b, c$ constant.

The most important technique for solving any second order ODE is: guided guessing. Here, we’re motivated by the first order analogue:

\[ ay' + by = 0. \]

Analysing the first order ODE either as a separable or a linear equation, we see that the solutions are exponential functions. We thus guess solutions to (CLH) will also be exponential functions. So, we try a function of the form

\[ y = e^{rx}, \]

where the constant $r$ will be determined. First, we note

\[ y = e^{rx} \quad \Rightarrow \quad \begin{cases} y' = re^{rx} \\ y'' = r^2e^{rx} \end{cases}. \]

Plugging these into to the LHS of (CLH), we find

\[ \text{LHS} = a r^2 e^{rx} + b r e^{rx} + c e^{rx} = e^{rx} \left( a r^2 + b r + c \right). \]

So, $y$ will solve the ODE (i.e. LHS=0=RHS) iff

\[ ar^2 + br + c = 0. \]

We call the quadratic equation (Aux) the auxiliary equation. Of course, we can solve quadratics, and so we’re basically done. But we have to consider the various cases, according to whether the solutions to (Aux) are real, complex or repeated. Further, we have to analyse the results, to consider whether we have genuinely found all solutions to (CLH). The details are worked through in the following examples, and the results are summarised in §2.2.

2.1 Examples

2.1.1 Example: distinct real roots

Consider the ODE

\[ y'' - y' - 6y = 0. \]
In this case, the auxiliary equation is
\[ r^2 - r - 6 = 0, \]
which easily factors to give solutions \( r = -2 \) and \( r = 3 \). Thus, we have found two particular solutions to the ODE:
\[ y_1 = e^{-2x}, \quad y_2 = e^{3x}. \]

The important thing to realise is that any linear combination of these two solutions is also a solution to (1). So, in fact we have a general solution to (1) of the form
\[ y = A_1 + B_2 = Ae^{-2x} + Be^{3x}. \]

This important linearity fact (which holds for any linear homogeneous ODE) is easy to prove: we could prove it for this particular example, but we’ll consider such issues in general, in §5. The companion fact, which is not easy to prove is that (1) is the most general solution to (1); we discuss this as well in §5.

Assuming this theory from §5, we can declare ourselves done: (1) is the most general solution to (1).

2.1.2 Example: purely imaginary roots

We could go straight to general complex roots, but we’ll warm up with a simpler example. Consider the ODE
\[ y'' + y = 0. \]

So, this is a Spring Equation, with \( k = 1 \) (which gives us a clue what to expect). The auxiliary equation is
\[ r^2 + 1 = 0, \]
with complex roots
\[ r = \pm i. \]

So, as long as we’re willing to consider complex-valued solutions, we obtain
\[ y_1' = e^{ix}, \quad y_2' = e^{-ix}. \]

These are complex solutions but are legitimate (for the same reason that \( \frac{d}{dx}(e^{ix}) = xe^{ix} \), even if \( x \) is complex). But what do we do with these, if we want everyday real solutions?
There are two approaches, leading to the same conclusion: we have two particular solutions
\[ y_1 = \cos x, \quad y_2 = \sin x, \]

which leads to the general solution
\[ y = A_1 + B_2 = A \cos x + B \sin x. \]

The first way to see this is to argue that if \( y \) is any complex solution to (1), then \( \text{Re}(y) \) and \( \text{Im}(y) \), the real and imaginary parts of \( y \), are also solutions to (1). This is another basic fact of linearity, which again we delay until §5 (and similar arguments appeared in 121).

Assuming this, we just note
\[ y_1' = e^{ix} = \cos x + i \sin x, \]
and we obtain \( y_1 \) and \( y_2 \) as the real and imaginary parts of \( y_1' \). We can then look at \( y_2' \), but it just gives us the same real solutions.

The second way to see (1) is to stay in Complexland for a little longer, and recognise that the general complex solution to (1) is
\[ y_1 = C e^{ix} + D e^{-ix} = C e^{ix} + D e^{-ix} = C e^{ix} + D e^{-ix}, \quad C, D \text{ complex constants}. \]

But then
\begin{align*}
y_1 &= C(\cos x + i \sin x) + D(\cos x - i \sin x) \\
&= (C + D) \cos x + (Ci - Di) \sin x \\
&= A \cos x + B \sin x \\
&= A = C + D, B = C_1 - D_1.
\end{align*}

It is clear that for any complex \( A \) and \( B \) we can find \( C \) and \( D \) satisfying the two equations \( A = C + D, B = C_1 - D_1 \). So the general complex solution to (1) can be obtained by choosing \( A, B \) complex. Then, the general real solution to (1) is obtained by choosing \( A, B \) real. The latter conclusion is exactly (1).

2.1.3 Example: complex roots

Consider the ODE
\[ y'' + 4y' + 13y = 0. \]

The auxiliary equation is
\[ r^2 + 4r + 13, \]
which has roots
\[ r = -2 \pm 3i. \]

This gives the complex solutions
\[ \begin{cases} y_1' = e^{(-2 + 3i)x} = e^{-2x}(\cos 3x + i \sin 3x) \\
y_2' = e^{(-2 - 3i)x} = e^{-2x}(\cos 3x - i \sin 3x) \end{cases} \]
Arguing exactly as in the previous example, we obtain the particular (real) solutions
\[ (*) \quad y_1 = e^{-2x} \cos(3x), \quad y_1 = e^{-2x} \sin(3x), \]
leading to the general solution to (1):
\[ (1) \quad y = Ae^{-2x} \cos(3x) + Be^{-2x} \sin(3x). \]

2.1.4 Example: repeated roots
Consider the example
\[ (1) \quad y'' - 4y' + 4y = 0. \]
Then the auxiliary equation is
\[ r^2 - 4r + 4 = 0, \]
leading to the one repeated root
\[ r = 2. \]
So, we obtain a particular solution
\[ (*) \quad y_1 = e^{2x}. \]
However, this only leads to more general solutions of the form
\[ y = Ae^{2x}. \]
We definitely expect there to be other solutions. (Of course it does no good to say we found two solutions \(e^{2x}\), leading to a general solution \(Ae^{2x} + Be^{2x}\)).
To find another solution, we need to do something fundamentally different. The trick is, we look for a new solution of the form
\[ (\star) \quad y = y_1w, \]
for some unknown function \(w\); then, as we shall see, for \(y\) to satisfy (1) leads to a very simple ODE for \(w\). This is a general technique called Reduction of Order; we consider it in more generality in §3.

In our particular case, we find
\[ y = y_1w = e^{2x}w \quad \Rightarrow \quad \begin{cases} y' = 2e^{2x}w + e^{2x}w' \\ y'' = 4e^{2x}w + 4e^{2x}w' + e^{2x}w''. \end{cases} \]

Plugging these into the LHS of (1), we find
\[ \text{LHS} = y'' - 4y' + 4y \]
\[ = (4e^{2x}w + 4e^{2x}w' + e^{2x}w'') - 4(2e^{2x}w + e^{2x}w') + (2e^{2x}w) \]
\[ = (e^{2x})w'' + (4e^{2x} - e^{2x})w' + (4e^{2x} - 8e^{2x} + 4e^{2x})w \]
\[ = e^{2x}w'' \]
So, we see that \(y\) given by (\(\star\)) satisfies (1) iff \(w'' = 0\); that is, integrating twice, iff
\[ w = A + Bx. \]
This leads to the general solution
\[ (1) \quad y = y_1w = Ae^{2x} + Bxe^{2x}. \]
\(A = 1, B = 0\) gives us our solution \(y_1\) above, and \(A = 0, B = 1\) gives us a new particular solution
\[ (*) \quad y_2 = xe^{2x}. \]

2.2 Summary
For the linear constant coefficient homogeneous equation
\[ (\text{CLIH}) \quad ay'' + by' + cy = 0 \]
we consider solutions of the form
\[ y = e^{rx}. \]
This leads to the auxiliary equation
\[ (\text{Aux}) \quad ar^2 + br + c = 0, \]
which leads to the complementary functions (i.e. solutions)
\[ y = Ae^{rx} + Be^{sx} \quad r_1, r_2 \text{ distinct real roots} \]
\[ y = Ae^{rx} \cos(dx) + Be^{sx} \sin(dx) \quad r = c \pm id \text{ complex roots} \]
\[ y = Ae^{rx} + Bxe^{sx} \quad r \text{ repeated root} \]
As a final remark, note that \(r_1 = 0\) is a possible real root, leading to the constant solution
\[ y_1 = e^{0x} = 1. \]
Thus, it can be regarded as a special case, but is consistent with the above summary.
3 Linear Homogeneous Equations - Reduction of Order

We now consider a general linear homogeneous ODE

\[ y'' + P(x)y' + Q(x)y = 0. \]

There is no general method to solve (LH). However, if we are given one solution \( y_1 \) then there is a method to find the general solution: this technique is called **reduction of order**. So, we try to find a new solution

\[ y = w y_1 \]

and we find the ODE that \( w \) must satisfy in order for \( y \) to solve (LH).

We saw an application of reduction of order in §2.1.4. In that case, the ODE for \( w \) was absolutely trivial (\( w'' = 0 \)). In general we won’t be that lucky, but in fact we always get a reasonably tractable ODE for \( w \) (in fact a first order linear ODE for \( w' \) in disguise: see §4.1 for the general discussion).

As is common with ODE’s, one can either remember the general method or a general formula (and we strongly recommend the former). We’ll give an example of the method, and then go through the general set-up.

3.1 Example: Legendre’s equation

Recall Legendre’s equation

\[ (1 - x^2)y'' - 2xy' + 2y = 0. \] (1)

We guessed (or God gave us) one solution,

\[ y_1 = x. \]

So, to get a general solution, we now try

\[ y = w y_1 = w x. \] (★)

(We could also start with \( y_1 = Ax \), but this won’t result in anything more general). Now,

\[ y = x w \quad \Rightarrow \quad \begin{cases} y' = w + x w' \\ y'' = 2 w' + x w'' \end{cases}. \]

Plugging these into the LHS of (1), we find

\[ \text{LHS} = (1 - x^2)w'' - 2xw' + 2y \]

\[ = (1 - x^2) (2w' + xw'') - 2x (w + xw') + 2w \]

\[ = x (1 - x^2)w'' + (2(1 - x^2) - 2x)w' + (-2x + 2x)w \]

\[ = x (1 - x^2)w'' + (2 - 4x^2)w' \]

So \( y \) given by (★) will solve (1) as long as

\[ x(1 - x^2)w'' + (2 - 4x^2)w' = 0. \]

Now it’s not obvious how to solve this ODE, but notice that \( w \) doesn’t explicitly occur: this is always the case (if the \( w' \)’s don’t cancel out then either \( y_1 \) was not really a solution, or we’ve made an arithmetic mistake). Notice in particular that \( w = A \) is a constant solution to (1): we expect this, since \( y = Ax \) is a solution to the original equation (1).

The fact that \( w \) doesn’t occur means that we can introduce a new variable

\[ p = w'. \]

Then \( p' = w'' \) and (1) becomes

\[ x(1 - x^2)p' + (2 - 4x^2) p = 0. \] (▲)

which we recognise as a first order linear (and separable) equation (hence the name “reduction of order”). It’s not particularly nice, but we know we can solve it (give or take doing integrals). We go through the details in a moment, but we’ll accept for now that the solution to this ODE is

\[ p = \frac{B}{x^2(1-x^2)}. \]

Then, we can use (▲) and integrate to solve for \( w' \):

\[ w' = p = \frac{B}{x^2(1-x^2)}. \]

\[ \Rightarrow w = \int \frac{B}{x^2(1-x^2)} \, dx \]

\[ \Rightarrow w = B \left( \frac{1 + 0x}{x^2} + \frac{1}{2} \log \frac{1 + x}{1 - x} \right) + A \]

(partial fractions)

\[ \Rightarrow w = B \left( -\frac{1}{x} + \frac{1}{2} \log \frac{1 + x}{1 - x} \right) + A \] \quad (A = Bc, \; c \; \text{a constant of integration})

Finally, we use (★) to give

\[ y = y_1 w = B \left( -1 + \frac{x}{2} \log \frac{|1 + x|}{|1 - x|} \right) + Ax. \]

This is the general solution to Legendre’s equation (1). Taking \( A = 1, B = 0 \) gives us our original solution \( y_1 = x \). For a new particular solution, we can take \( A = 0, B = 1 \), giving

\[ y_2 = -1 + \frac{x}{2} \log \frac{|1 + x|}{|1 - x|}. \]

\[ \text{☺☺☺} \]
To finish up, we go through the derivation of (●) from (●●). One can use the general formula (noting Legendre’s equation is not in standard form). However, it’s easier in this case to work through the method, multiplying the LHS of (●●) by an integrating factor \( v \), giving

\[
\text{LHS} = x(1 - x^2)vp' + (2 - 4x^2)v = \frac{d}{dx} \left( x(1 - x^2)w \right)
\]

For this to be the case, we need

\[
\frac{d}{dx} \left( x(1 - x^2)w \right) = (2 - 4x^2) v
\]

\[
\Rightarrow \quad (1 - 3x^2) v + x (1 - x^2) v' = (2 - 4x^2) v
\]

\[
\Rightarrow \quad x (1 - x^2) v' = (1 - x^2) v
\]

\[
\Rightarrow \quad v' = v, \quad x \neq \pm 1
\]

This is trivially solvable to give a particular solution \( v = x \) (one solution will do here). We then multiply (●) by \( v = x \) giving

\[
x^2 (1 - x^2) y'' + (2x - 4x^3) p = 0
\]

\[
\Rightarrow \quad \frac{d}{dx} (x^2 (1 - x^2) y) = 0
\]

\[
\Rightarrow \quad x (1 - x^2) y = B
\]

\[
\Rightarrow \quad p = \frac{B}{x^2 (1 - x^2)}
\]

This is exactly (●), and thus we’re done. \( \Box \)

3.2 General Reduction of Order

We consider the general linear homogeneous equation (in standard form):

\[
\text{(LH)} \quad y'' + P(x)y' + Q(x)y = 0,
\]

and we consider reduction of order in this general case. So, given one solution \( y_1 \), we try to find a new solution

\[
\text{(●●)} \quad y = y_1 w.
\]

Then

\[
y = y_1 w \quad \Rightarrow \quad \begin{cases} y' = y_1 w + y_1 w' \\ y'' = y_1 w + 2y_1 w' + y_1 w'' \end{cases}
\]

Plugging these into the LHS of (LH), we have

\[
\text{LHS} = (y_1^n w + 2y_1 w' + y_1 w'' + P(x)(y_1 w + y_1 w') + Q(x)y_1
\]

\[
= y_1 w'' + (2y_1 + P(x)y_1) w' + (y_1 + P(x)y_1 + Q(x)y_1) w
\]

\[
= y_1 w'' + (2y_1 + P(x)y_1) w' \quad \text{(since } y_1 \text{ is a solution to (LH)).}
\]

Note the key use of (LH) on the last line.

We see that \( y \) given by (●●) will solve (LH) iff

\[
y_1 w'' + (2y_1 + P(x)y_1) w' = 0.
\]

This is exactly what we called the Reduction Equation in §1, with

\[
\text{(●●)} \quad M = \frac{2y_1 + P(x)y_1}{y_1}
\]

As above, we see that (●●) is a first order linear (and separable) equation for

\[ p = w', \]

which we can write as

\[ p' + M(x)p = 0. \]

We can then easily solve for \( p \), and substitute back:

\[
p = Be^{-\int M(x)}
\]

\[
\Rightarrow \quad w = \int \left( \int p = A + B e^{-\int M(x)} \right) dx
\]

\[
\Rightarrow \quad y = A y_1 + B y_2 \int e^{-\int M(x)} dx
\]

where \( M \) is given by (●●). Thus, we have obtain a general solution for (LH). Taking \( A = 1, B = 0 \) gives us our original solution \( y_1 \). For a new particular solution, we can take \( A = 0, B = 1 \), giving

\[
y_2 = y_1 \int e^{-\int M(x)} dx
\]

Then the general solution is

\[
y = A y_1 + B y_2.
\]

Note that in these integrals, we have already explicitly taken out the constants of integration. So, any particular antiderivatives will suffice: we needn’t include another \( +c \). \( \Box \)
4 First Order ODE’s In Disguise

As we indicated in §1 these are of two types: when \( y \) does not occur; and when \( x \) does not occur. We’ve seen a number of cases of the first type, so we’ll just make a few general comments and give one more quick example. Then, we’ll show how to deal with the second type, and give some examples.

4.1 Where the unknown function \( y \) doesn’t explicitly occur

Such a differential equation is of the form
\[
y’ = F(x,y)
\]
Typical is the Reduction Equation (§1 and §3.2):
\[
y’ + M(x)y’ = 0 .
\]
See also (1) in §3.1. In each case, the technique is to consider the ODE as a first order ODE in \( p \) where
\[
p = y’
\]
We then (hope to) solve this ODE for \( p \) and then integrate to get \( y \).

From (1), we see that in general the ODE for \( p \) takes the form
\[
p’ = F(x,p) .
\]
That is, we may be hit with any possible first order ODE for \( p \); so any (in 123) or no (real world) technique may work to explicitly get \( p \). (The key point in §3 is that if we start with a linear homogeneous equation for \( y \) then we obtain a linear first order equation for \( p \).

Notice that we’ll always get a \( “+B” \) in the solution to (1), this comes out as the second constant of integration when we integrate \( p \) to get \( y \).1 However, the general solution to (1) will not normally be the linear combination of two particular solutions: this only holds true if (1) is a homogeneous linear equation.

4.1.1 Example

Consider the ODE
\[
y'' - (y')^2 = 1 .
\]
Setting \( p = y’ \), this equation becomes
\[
p’ - p^2 = 1 .
\]
\[\text{In the distributed notes, we wrote that } y = B \text{ is always a solution to (1). This is false, as the next example illustrates.}\]

This is an easily solvable separable ODE:
\[
p’ = 1 + p^2
\]
\[
\int 
\frac{1}{1 + p^2} \, dp = \int 1 \, dx
\]
\[
\arctan p = x + A
\]
\[
p = \tan(x + A) .
\]

Then,\(^2\) from (1),
\[
y’ = p = \tan(x + A)
\]
\[
y = \int \tan(x + A) \, dx
\]
\[
y = - \log|\cos(x + A)| + B . \quad \text{(substituting } u = \cos(x + A))
\]

We remark, as observed above, that the general solution cannot be written as the linear combination of two particular solutions: we can set \( A = 1, B = 0 \) to get a particular solution \( y_1 \), and we can set \( A = 0, B = 1 \) to get a particular solution \( y_2 \). BUT the general solution is NOT then \( Ay_1 + By_2 \).

4.2 Where the variable \( x \) doesn’t explicitly occur

Such a differential equation is of the form
\[
y’’ = F(y,y’)
\]
Examples are the Spring Equation
\[
y'' + ky = 0 \quad k > 0
\]
and the Pendulum Equation
\[
y'' + \sin y = 0 .
\]
\[\text{In the distributed notes, the calculation was complete nonsense. (Thanks, Richard.)}\]
Again we introduce a new variable for $y'$

$\star \quad *p = y' = \frac{dy}{dx}$

However, we think of $p$ as a function of $y$. Then, by the chain rule,

$\star \star \quad y'' = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dy} \left( \frac{dy}{dx} \right) \cdot \frac{dy}{dx} = q \cdot \frac{dy}{dx}$  \hspace{1cm} \text{(by \star)}$

So, in (\$) we can substitute $p$ for $y'$ and $\frac{dy}{dx}$ for $y''$; giving

$(\dagger) \quad \frac{d}{dy} \left[ \frac{dy}{dx} \right] = F(y, p)$

This is then a first order ODE for $q$ as a function of $y$. Assuming we can solve this, we can use $\star \star$ to plug back for $y'$ and solve the first order ODE for $y$ (as a function of $x$).

One seldom uses $\dagger$ in practice; rather we just work with $\star$ and $\star \star$ directly for each particular equation. We now give some examples.

4.2.1 Example: Spring Equation

We’ll use the method to solve

$y'' + y = 0$.

It’s a good illustration, since we know what to expect (and it will also show that linear methods are much better when we have the choice!). From $\star$ and $\star \star$, this equation becomes

$\frac{dq}{dy} + q = 0$,

which is a separable ODE. We can easily solve it to give

$q^2 + y^2 = C^2$.

(Clearly the constant must be nonnegative, so we may as well call it $C^2$). Plugging back from $\star$, this gives us the first order ODE

$$ \left( \frac{dy}{dx} \right)^2 + y^2 = C^2. $$

(Note that this is exactly the Trigonometric Equation we introduced in Handout 2). Solving for $y'$, we again get a separable ODE:

$$ y' = \pm \sqrt{C^2 - y^2} \quad \Rightarrow \int \frac{1}{\sqrt{C^2 - y^2}} \, dy = \pm \int 1 \, dx \quad \Rightarrow \quad \arcsin \left( \frac{y}{C} \right) = \pm x + D \quad \Rightarrow \quad y = C \sin(D \pm x) $$

Expanding the sin term, we get the expected solutions of sin and cos.

4.2.2 Example

Consider the equation

$yy'' + (y')^2 = 0$.

From $\star$ and $\star \star$, this equation becomes

$yy'' + q^2 = 0$.

So, either $q = 0$ (which leads to $y' = 0$ and thus $y = c$), or

$yy'' + y = 0$.

This is a separable equation, which we easily solve to give

$\dagger \quad q = \frac{A}{y}$.

Then, from $\star$,

$\frac{dy}{dx} = \frac{A}{y}.$  \hspace{1cm} \text{(separable equation)}$

$\dagger \quad \frac{y^2}{2} = Ax + B$

$\Rightarrow \quad y = \pm \sqrt{2Ax + D} \quad \Rightarrow \quad y = \pm \sqrt{2Ax + C} \quad \Rightarrow \quad C = 2A, D = 2B.$

Note that the first case $y = c$ above is included as part of the general solution.
4.2.3 Example: Pendulum Equation

Here we consider the equation

\[ y'' + \sin y = 0. \]

From (+) and (++), this equation becomes

\[ \frac{dq}{dy} + \sin y = 0. \]

This is a separable equation, and we find

\[ q^2 = 2\cos y + C. \]

We can apply (+) and solve for \( y' \) to give

\[ y' = \pm \sqrt{2\cos y + C}. \]

This is then a separable equation for \( y \). However, we then run into the integral \( \int \frac{1}{\sqrt{2\cos y + C}} \) and for general \( C \), we cannot evaluate it. Thus, this is as far as we can go.

This is an illustrative example: the method promises that our second order ODE (8) will be reduced to a sequence of first order ODEs, but it doesn't promise anything about the computability of those subsequent ODEs. (However, 123 is an artificial environment ...).

5 IVP’s and Theory

As with the theory of first order ODEs, the appropriate way to discuss existence and uniqueness of solutions is in terms of initial value problems. We discuss this and give examples in §5.1, and in §5.2 we state the main theorem. In §5.3 we discuss the abstract notion of linear operators; we then apply this in §5.4, to characterize the solutions to linear homogeneous equations.

5.1 Initial Value Problems

Consider a general second order ODE

\( y'' = F(x, y, y'). \)

In general, we expect there to be two constants of integration. Thus, to pin down a particular solution, we expect to need two extra conditions. There are actually a number of ways to do this, but we'll just consider the appropriate initial value problem: in this case we specify \( y \) and \( y' \) at some initial \( x_0 \). Thus an initial value problem takes the form

\[
\begin{align*}
  y' &= F(x, y, y') \\
  y(x_0) &= y_0 \\
  y'(x_0) &= z_0.
\end{align*}
\]

5.1.1 Example

Consider the IVP

\[
\begin{align*}
  y'' + (y')^2 &= 0 \\
  y(0) &= -2 \\
  y'(0) &= 3.
\end{align*}
\]

In §4.2.2 we found the general solution to the ODE to be

\( y = \pm \sqrt{2x + D}. \)

Now we can solve the IVP from this equation, but it is much easier to use earlier equations in the derivation of (7). So, we derived the equation

\( y = \pm \sqrt{A - \frac{A^2}{2}}. \)

where \( q = y' \). Then, at \( x = 0 \) we have \( y(0) = -2 \) and \( y'(0) = 3 \), giving

\[ A = -6 \quad \Rightarrow \quad C = -12 \quad \text{(since \( C = 2A \)).} \]

Similarly, we found

\[ \frac{y^2}{2} = A x + B. \]

Again setting \( x = 0 \), we find

\[ 2 = -12 \cdot 0 + B \quad \Rightarrow \quad B = 2 \quad \Rightarrow \quad D = 4 \quad \text{(since \( D = 2B \)).} \]

Thus, we find

\[ y = \pm \sqrt{4 - 12x}. \]

But since \( y(0) < 0 \), we choose the minus sign, and we obtain the solution

\[ y = -\sqrt{4 - 12x}. \]

5.1.2 Example

Consider the linear IVP

\[
\begin{align*}
  y'' - 3y' + 2y &= 0 \\
  y(0) &= 3 \\
  y'(0) &= 5.
\end{align*}
\]

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The LHS is constant coefficient, and we obtain the auxiliary equation \( r^2 - 3r + 2 = 0 \). Thus we obtain the general solution to the ODE

\[
y = Ae^x + Bxe^x.
\]

So the initial condition \( y(0) = 3 \) implies

\[
A + B = 3.
\]

To apply the second initial condition, we differentiate (1), giving

\[
y' = Ae^x + 2Bxe^x.
\]

Then, \( y'(0) = 5 \) gives

\[
A + 2B = 5.
\]

Thus, (1) and (2) gives us two simultaneous linear equations, which are easily solved to give

\[
A = 1, \quad B = 2,
\]

and thus the solution to the initial value problem is

\[
y = e^x + 2xe^x.
\]

Notice that the approach to evaluating \( A \) and \( B \) is completely different to the previous example: \( y \) is not obtained by an antidifferentiation procedure as such, and thus we are simply faced with simultaneous equations to solve. However, the simultaneous equations will always be linear, and so pose no problem.

\[
\Box\Box\Box\Box
\]

5.2 The Existence and Uniqueness Theorem

Consider the IVP

\[
\begin{align*}
y' &= F(x, y, y') \\
y(x_0) &= y_0 \\
y'(x_0) &= z_0.
\end{align*}
\]

We want to state the Existence and Uniqueness Theorem for second order ODE’s. Pretty much it is what we expect: if \( F \) is nice enough then (IVP) will have a unique solution on some small interval \([x_0, x_0 + \delta]\). However, to state the Theorem there’s a slight fiddle; the RHS of the ODE involves \( y' \), and we have to say how \( F \) depends upon \( y' \). To do this, we have to introduce a dummy variable \( z \), which stands for “where the \( y' \) stuff goes”. This is a little confusing: we’ll state the Theorem and then give a couple examples.

Existence and Uniqueness Theorem Suppose \( F, F_y \) and \( F_z \) are continuous in a rectangular box \([x_0, x_0 + \delta] \times [y_0, y_0 + L] \times [z_0, z_0 + M] \). Then there is a \( \delta > 0 \) such that (IVP) has a unique solution on the interval \([x_0, x_0 + \delta] \).

\[
\Box\Box\Box\Box
\]

5.2.1 Example

Consider the IVP from §5.1.1:

\[
\begin{align*}
yy'' + (y')^2 &= 0 \\
y(0) &= -2 \\
y'(0) &= 3.
\end{align*}
\]

Rewriting the ODE as

\[
y'' = -\frac{y}{(y')^2}
\]

the LHS is then of the form \( F(x, y, y') \), where

\[
F(x, y, z) = \frac{y}{z^2}.
\]

So,

\[
F_y = -\frac{1}{z^3}, \quad F_z = 2y.
\]

Now, near the point \((x_0, y_0, z_0) = (0, -2, 3)\) the functions \( F_y, F_z \) and \( F_z \) are all continuous. Thus the Theorem promises a unique solution to the IVP in some small interval. Of course, in §5.1.1 we found a solution, but still, the Theorem is promising us that there is no other.

On the other hand, if we had an initial condition \( y(0) = 0 \), then \( F \) and its derivatives are not continuous there, and the Theorem would not apply.

\[
\Box\Box\Box\Box
\]

5.2.2 Example

Consider the IVP

\[
\begin{align*}
yy'' + (e^x + e^{7x}\sin(e^x))y' + (e^x + 25\cos x)\tan x y &= 0 \\
y(0) &= 0 \\
y'(0) &= 0.
\end{align*}
\]

We have no hope of solving this ODE, but notice that it is linear and homogeneous, and that the initial conditions are both 0. Thus, it is easy to see that \( y = 0 \) is one solution to the IVP. Then the Theorem promises us that this is the only solution.

\[
\Box\Box\Box\Box
\]
5.3 Linear Operators

Note, this subsection is a bit off the track. It’s a natural way to set up the solving of inhomogeneous ODE’s, but is not examinable per se. If you wish, you can skip straight to the theorems in §5.4.

An operator is just a fancy name for a function whose input (domain) is a function and output (range) is a function. So, if we call the operator $T$, and we input some function $f$, then we write $T(f)$ for the output. Here are some simple and familiar operators, with explicit examples.

\[
\begin{align*}
D(f) &= f' \quad \text{(Differentiation operator)} \\
I(f) &= \int_0^x f(t) \, dt \quad \text{(Integration operator)} \\
S(f) &= f^2 \quad \text{(Squaring operator)}
\end{align*}
\]

This is all pretty straightforward; just note we have to make the Integration operator “definite”, since whatever function we plug in, we must get out one specific function as output (and thus we don’t want the +c). One thing, we’re not worried about the exact domain of the operators, exactly which functions can be used as input: this is an issue, but one we can safely ignore here.

An operator $T$ is linear if for any functions $f$ and $g$, and any constants $A$ and $B$,

\[T(Af +Bg) = A \cdot T(f) + B \cdot T(g)\]

So, $D$ and $I$ are linear operators (a fact we use all the time). However, $S$ is not linear since, for example,

\[S(2\sin x + 3x^2) = (2\sin x + 3x^2)^2 \neq 2(\sin x)^2 + 3(x^2)^2 = 2S(\sin x) + 3S(x^2).
\]

Now suppose we have a linear ODE

\[(L) \quad y'' + P(x)y' + Q(x)y = R(x) \]

We can then define an associated operator

\[L(f) = f'' + P(x)f' + Q(x)f.\]

So, for example, if our ODE is

\[y'' + y = x^2 + 3\]

then the associated operator is

\[L(f) = f'' + f.\]

And then, for example,

\[L(x^3) = 6x + x^3.\]

Notice that the RHS $R(x)$ plays no part in the definition of $L$.

The point of this is the following:

**Fundamental Lemma**

Let $L$ be the operator associated to a linear ODE $(L)$. Then $L$ is linear. That is

\[L(Af + Bg) = A \cdot L(f) + B \cdot L(g) \quad (A,B \text{ constants}).\]

**Proof**

We easily calculate

\[L(Af + Bg) = (Af + Bg)'' + P(x)(Af + Bg)' + Q(x)(Af + Bg)
\]

\[= Af'' + P(x)f' + Q(x)f + Bg'' + P(x)g' + Q(x)g \quad \text{(by the linearity of $D$)}
\]

\[= A \cdot L(f) + B \cdot L(g).\]

5.4 The Solutions of Linear Equations

Here, we consider a linear ODE

\[(L) \quad y'' + P(x)y' + Q(x)y = R(x),\]

and the corresponding homogenous ODE

\[(LH) \quad y'' + P(x)y' + Q(x)y = 0.\]

We are interested in characterising the solutions to both of these ODEs. The analysis of this is easily done with the associated linear operator $L$. In particular we note:

\[
\begin{align*}
\text{If } y \text{ solves (L)} & \iff L(y) = R \\
\text{If } y \text{ solves (LH)} & \iff L(y) = 0.
\end{align*}
\]

This makes it easy to prove the following theorems.

**Homogeneous Theorem**

Suppose $y_1$ and $y_2$ are solutions to (LH). Then any linear combination

\[(i) \quad y = A y_1 + B y_2\]

is a solution to (LH).
Proof
Using (\ref{eq:proof}) and the Fundamental Lemma, we easily calculate
\[ L(Ay_1 + By_2) = AL(y_1) + BL(y_2) = A \cdot 0 + B \cdot 0 = 0. \]
\[\blacktriangledown\blacktriangledown\blacktriangledown\]

Inhomogeneous Theorem Suppose \( y_1 \) and \( y_2 \) are solutions to (LH), and that \( y_p \) is a
(particular) solution to (L). Then any combination
\[ y = y_p + Ay_1 + By_2 \]
is a solution to (L).
Proof
Using (\ref{eq:solution}) and the Fundamental Lemma, we easily calculate
\[ L(y_p + Ay_1 + By_2) = L(y_p) + AL(y_1) + BL(y_2) = R + A \cdot 0 + B \cdot 0 = R. \]
\[\blacktriangledown\blacktriangledown\blacktriangledown\]

A question these theorems don’t answer is whether all solutions are of the forms (1) and (2). By the Existence and Uniqueness Theorem this will be the case if we can solve any IVP in these forms, and this amounts to \( y_1 \) and \( y_2 \) being general enough, what is called independent of each other. In practice, for us, this won’t be an issue; as long as \( y_1 \) and \( y_2 \) aren’t constant multiples of each other then (1) and (2) are the most general solutions to (LH) and (L) respectively. However, it is worth mentioning that there are subtle theoretical issues involved; we won’t go into these here.

5.4.1 Example
The importance of the Inhomogeneous Theorem is that it says that if we can solve the complementary (homogeneous) equation (LH), AND if we can find just one particular solution \( y_p \) to (L), then we can find all solutions to (L). For example, consider the ODE
\[ y'' + y = x^2 + 3. \]
The general solution to the complementary equation \( y'' + y = 0 \) is \( A \cos x + B \sin x \). It’s not obvious how to find any solution to (2) (see §6). But, pulling a rabbit out of a hat, it is easy to check that
\[ y_p = x^2 + 1 \]
is a particular solution to (2). Thus, by the Inhomogeneous Theorem (and the discussion following it), the general solution to (2) is
\[ y = (x^2 + 1) + A \cos x + B \sin x. \]
\[\blacktriangledown\blacktriangledown\blacktriangledown\]

6 Linear Inhomogeneous Equations
We want to solve
\[ y'' + P(x)y' + Q(x)y = R(x). \]
From §5.4, we know that the general solution is
\[ y = y_p + Ay_1 + By_2 \]
where \( y_p \) is one particular solution to \( y_1 \) and \( y_2 \) are (independent) solutions to the comple-
mentary equation
\[ y'' + P(x)y' + Q(x)y = 0. \]
However, this doesn’t tell us how to find \( y_p \); and, recall that we don’t have a general method
for finding \( y_1 \) and \( y_2 \), if (LH) is not constant coefficient. However, in a sense solving (L) is not much worse (in theory) than solving (LH).
We give two methods for solving (L), which we’ll call Method 1 and Method 2. Method 2 is a more general approach, but tends to be more painful in practice: if there is a choice, Method 1 is preferable. However, Method 2 is easier to describe, so we’ll cover that first.

6.1 Method 2: We are given \( y_1 \) - Reduction of Order
This works exactly as for the homogeneous case. If we are given a solution \( y_1 \) to (LH), then we can find the general solution to (L), by trying a solution
\[ y = y_1 w \]
Plugging into (L), this gives a first order linear (but not homogeneous) ODE for \( w' \); so, we can always solve for \( w \). There is a general formula for \( w \), and thus for \( y = y_1 w \), but in practice we just work through the method. One example makes it all pretty clear.

6.1.1 Example
Consider the ODE
\[ x^2 y'' + 7xy' + 5y = x. \]
We have no technique to solve this ODE from scratch. However, if it is pointed out to us that
\[ y_1 = \frac{1}{x} \]
is a solution to the complementary equation
\[ x^2 y'' + 7xy' + 5y = 0, \]

\[\blacktriangledown\blacktriangledown\blacktriangledown\]
then we can use (★) to find the general solution. So
\[ y = y_{1}w = \frac{1}{2}w \implies \begin{cases} y' = \frac{1}{2}w' - \frac{1}{2}w \\ y'' = \frac{1}{2}y'' - \frac{1}{2}w' + \frac{2}{x^2}w. \end{cases} \]

Plugging into (★), we have
\[ \begin{align*}
\text{(★)} & \implies x^2 \left(\frac{1}{2}y'' - \frac{2}{x^2}w' + \frac{2}{x^2}w\right) + 2x \left(\frac{1}{2}w' - \frac{1}{2}w\right) + \frac{5}{2}y = x \\
& \implies xw'' + 5w' = x.
\end{align*} \]

Notice that no \( w \) terms are left in the ODE; this must happen, as long as \( y_{1} \) is a solution to (★★), and we haven’t made any calculation errors.

It is easy to solve this ODE for \( w \); we obtain an integrating factor (in non-standard form) of \( v = x^4 \), and thus we find
\[ \frac{d}{dx} \left( w'x^5 \right) = x^5 \\
\implies w'x^5 = \frac{x^6}{6} + C \\
\implies w' = \frac{x}{6} + \frac{C}{x^5} \\
\implies w = \frac{x^2}{12} + \frac{A}{x^3} + B \quad \text{where } A = \frac{C}{4}.
\]

Then, from (★★), the general solution is
\[ y = y_{1}w = \frac{x^2}{12} + \frac{A}{x^3} + B. \]

Notice that we didn’t apply the Inhomogeneous Theorem (5.5.4) to get this solution, but we can still analyse the solution in those terms. So, the \( B \)-hit \( y_{1} = \frac{1}{2} \) is our original given solution to (★★); similarly, the \( A \)-hit \( y_{2} = \frac{1}{2} \) is another solution to (★★). And, if we put \( A = B = 0 \), we get a particular solution \( y_{p} = \frac{1}{2} \) to (★). (Of course, any values for \( A \) and \( B \) will give a particular solution to (★)).

\[ \blacktriangleleft \odot \odot \blacktriangleleft \]

6.2 Method 1: (LH) is constant coefficient, and \( R(x) \) is nice enough

If (LH) is constant coefficient, then of course we can write down the general solution to (LH). As well, if \( R(x) \) is a linear combination of polynomials, exponentials, sines and cosines, then we can effectively guess the form of a particular solution \( y_{p} \); then, we get the general solution by adding on the solution to the complementary equation. The precise approach depends upon the form of \( R(x) \), so this is best learned by example.

6.2.1 Example: \( R(x) \) is a polynomial

Consider the ODE
\[ \begin{align*}
\text{(★)} & \implies y'' + y = x^2 + 3.
\end{align*} \]

Given the \( R(x) = x^2 + 3 \) is a quadratic, we guess a particular solution \( y_{p} \) to also be a quadratic. (And if the polynomial is degree \( n \), then we guess \( y_{p} \) is degree \( n \)). The point is since we are only after one solution, we can make simplifying choices along the way.

There are now a few ways to find a quadratic \( y_{p} \); in effect there are three unknown coefficients for \( y_{p} \), and (★) allows us to determine them. (And it’s a theorem underling this method that it always works, that there really is a particular solution which is a polynomial). We’ll use a slightly less direct approach, which tends to be a little cleaner in practice.

If we are looking for a solution \( y_{p} \) then of course
\[ \begin{align*}
\text{(1)} & \implies y''_{p} + y_{p} = x^2 + 3.
\end{align*} \]

Differentiating, we find
\[ \begin{align*}
\text{(2)} & \implies y'''_{p} + y'_{p} = 2x.
\end{align*} \]

Differentiating once more,
\[ \begin{align*}
\text{(3)} & \implies y''''_{p} + y''_{p} = 2.
\end{align*} \]

We stop, since the RHS is now a constant. We then choose that
\[ y''_{p} = 2, \]

(should amounts to trying to find a solution which is quadratic). In this case, \( y''_{p} = 0 \), and thus (3) is satisfied.

At this point we can just go back to (1), plug in \( y''_{p} = 2 \), and conclude \( y_{p} = x^2 + 1 \); however, the ODE will not always permit such a shortcut, so we’ll indicate a shorter but safer method. Nonetheless, if you spot a shortcut to obtaining \( y_{p} \), you are permitted to use it.
Returning to the general method, we antidifferentiate $y'' = 2$ to give
\[ y' = 2x + C. \]
We have to evaluate $C$. Plugging into (2), and noting $y'' = 0$, we find
\[ 0 + 2x = 2x \implies C = 0. \]
Thus,
\[ y' = 2x. \]
Antidifferentiating once more, we find
\[ y = x^2 + D. \]
Plugging into (1), and noting $y' = 2$, we find
\[ 2 + x^2 + D = x^2 + 3 \implies D = 1. \]
Thus we have found a particular solution
\[ y_p = x^2 + 1. \]
Then the general solution is given by adding on the solution to the complementary equation, giving
\[ y = (x^2 + 1) + A \cos x + B \sin x. \]

### 6.2.2 Example: \( R(x) \) is an exponential (maybe times a polynomial)

Consider the ODE
\[ y'' - y = xe^{2x} \]
The idea in this case is to look for a particular solution of the form
\[ y = e^{2x}w. \]
Then the ODE we get for $w$ will contain no exponential terms. Note this is not really reduction of order (because the ODE for $w$ will not be a first order ODE in disguise), but it’s the same basic trick. So, from (1),
\[ y = e^{2x}w \implies \begin{cases} y' = e^{2x}w' + 2e^{2x}w \\ y'' = e^{2x}w'' + 4e^{2x}w' + 4e^{2x}w. \end{cases} \]

Plugging into (1), this gives,
\[ (e^{2x}w'' + 4e^{2x}w' + 4e^{2x}w) - e^{2x}w = xe^{2x} \]
\[ \implies e^{2x}w'' + 4e^{2x}w' + 3xe^{2x}w = x \]
\[ \implies w'' + 4w' + 3w = x \]
(dividing by $e^{2x}$)

So, now the ODE for $w$ is exactly the type of inhomogeneous equation we can solve, and we have a choice: we can get the general solution for $w$ and plug it back into (1); or we can get one particular solution for $w$. There is no guarantee, but usually the latter method is cleaner, and we’ll apply it here.

Differentiating the ODE (1) for $w$, we have
\[ w'' + 4w' + 3w = 1. \]
The RHs is a constant, so we stop, and we choose
\[ w' = \frac{x}{3}. \]
Then, antidifferentiating,
\[ w = \frac{x^2}{6} + C. \]
So, from (1), and noting $w'' = 0$,
\[ 0 + 4 \cdot \frac{1}{3} + 3 \left( \frac{x}{3} + C \right) = x \implies C = -\frac{4}{9}. \]
Thus
\[ w = \frac{x^2}{6} - \frac{4}{9} \]
is a particular solution for $w$, and that gives
\[ y_p = e^{2x}w = \left( \frac{x^2}{6} - \frac{4}{9} \right)e^{2x} \]
as a particular solution to (1). Adding on the general solution to the complementary equation, we find that the general solution to (1) is
\[ y = \left( \frac{x}{3} - \frac{4}{9} \right)e^{2x} + Ae^{-x} + Be^x. \]
6.2.3 Example: $R(x)$ is a cos or sin (maybe times a polynomial)  
In this case, we use the standard trick of regarding $\cos{x}$ and $\sin{x}$ as the real and imaginary parts of $e^{ix}$. At the end, we then take real and imaginary parts of the solution. (We haven’t gone into it, but this can be justified very easily, using the type of arguments in §5.4).

So, for example, consider the ODE

\[(*)\]  
\[y'' + 2y' + 2y = 5 \sin(3x) .\]

We regard this as the imaginary part of the ODE

\[(**)\]  
\[z'' + 2z' + 2z = 5e^{3ix} .\]

Then to obtain a particular (or general) solution to this ODE, we substitute

\[(\star)\]  
\[z = e^{3ix}w.\]

This gives

\[z = e^{3ix}w \implies \begin{cases} 
    z' = e^{3ix}w' + 3ie^{3ix}w \\
    z'' = e^{3ix}w'' + 6ie^{3ix}w' - 9e^{3ix}w .
\end{cases}\]

Plugging into (**) , we find

\[
\begin{aligned}
    (e^{3ix}w'' + 6ie^{3ix}w' - 9e^{3ix}w) + 2(e^{3ix}w' + 3ie^{3ix}w) + 2e^{3ix}w &= 5e^{3ix} \\
    \implies e^{3ix}w'' + (6ie^{3ix} + 2e^{3ix})w' + (-9e^{3ix} + 3ie^{3ix} + 2e^{3ix})w &= 5e^{3ix} \\
    \implies w'' + (6i + 2)w' + (-7 + 3i)w &= 5.
\end{aligned}
\]

Since the RHS is a constant (polynomial), we can apply the technique of §6.2.1. To get a particular solution we just set

\[
(-7 + 3i)w = 5 \implies w = \frac{-35 - 15i}{58} .
\]

Thus a particular solution to (**) is

\[z_p = e^{3ix}w = e^{3ix}\left(\frac{-35 - 15i}{58}\right) = (\cos(3x) + i \sin(3x))\left(\frac{-35 - 15i}{58}\right).\]

The imaginary part of this is then a particular solution to (*), giving

\[y_p = \Im \{z_p\} = \frac{-15}{58} \cos(3x) - \frac{35}{58} \sin(3x).\]

Adding in the solution to the complementary equation, the general solution to (*) is

\[y = \left(\frac{-15}{58} \cos(3x) - \frac{35}{58} \sin(3x)\right) + Ae^{-x} \cos x + Be^{-x} \sin x .\]

6.2.4 Summary and Variations

In summary, we’re considering a constant coefficient ODE

\[(L)\]  
\[ay'' + by' + cy = R(x) .\]

We need one particular solution $y_p$, and in brief, the technique is

\[
\begin{array}{|c|c|}
\hline
R(x) = p(x) & y_p = \text{a polynomial (differentiate to find the coefficients)} \\
\hline
R(x) = A e^{ix} & y_p = e^{ix}w (\text{reduces to the previous case}) \\
\hline
R(x) = A \sin(kx) & e^{ix} = \cos(kx) + i \sin(kx) (\text{reduces to the previous case}) \\
\hline
\end{array}
\]

We close with a few remarks, indicating variations on these basic methods:

- A product of an exponential with a sin or cos is just a special case of the real or imaginary part of an exponential:
  
  \[e^{ix}\cos(kx) = \Re \left(e^{ix + ikx}\right)\]  
  (and similarly for sin).

- A sum of terms on the RHS can be handled one by one.  
  For example, consider the ODE

  \[(*)\]  
  \[y'' - 5y' + 4y = 3e^x + 4x \sin x .\]

  To solve this we can consider separately the ODE’s

  \[(+1)\]  
  \[y'' - 5y' + 4y = 3e^x .\]

  and

  \[(+2)\]  
  \[y'' - 5y' + 4y = 4x^2 \sin x .\]

  We can then find particular solutions $y_{p1}$ to (+1) and $y_{p2}$ to (+2). But then (by the Fundamental Lemma, §5.3)

  \[y = y_{p1} + y_{p2} .\]

  will be a particular solution to (+).

- Products of sines and cosines can be handled with trig formulas.  
  For example, consider the ODE

  \[y'' - 5y' + 4y = 3 \sin^2 x .\]

  We can write the RHS as

  \[3 \sin^2 x = \frac{3}{2} - \frac{3}{2} \cos(2x) ,\]

  which is then a sum of terms we can handle.

- If $R(x)$ is not of the above types then we MUST use reduction of order.