This is a summary of my lectures 19-20, equal to lectures 18-20 in the subject guide.

**CORRECTION:** The introduction to §6 of Handout 6 is garbled. It should read:

We want to solve

\[ y'' + P(x)y' + Q(x) = R(x). \]  

From §5.4, we know that the general solution is

\[ y = y_p + Ay_1 + By_2, \]  

where \( y_p \) is one particular solution to (L), and \( y_1 \) and \( y_2 \) are (independent) solutions to the complementary equation

\[ y'' + P(x)y' + Q(x) = 0. \]  

(Thanks, Nathan).

## 1 Kinematics

Consider an object of mass \( m \) moving in a straight line, with position \( y(t) \) at time \( t \). It then has velocity \( v = y \) and acceleration \( a = y' \). (The dots indicate derivatives with respect to \( t \)). If the object is subject to a force \( F \), then **Newton’s Law** says

\[ m\ddot{y} = F. \]  

Here we consider two kinds of force applying; a **gravitational force**, and a **frictional force**. If we regard the positive \( y \) direction as up, then both of these forces act negatively, and we can explicitly place the minus signs there by writing (**) as

\[ m\ddot{y} = -F_y - F_f. \]

Now, depending upon whether we’re considering projectiles near the Earth’s surface, or shooting rockets at planets, the gravitational force takes the form

\[ F_y = \begin{cases} \frac{mg}{r} & (\text{constant gravity near the surface of the Earth}) \\ k & (y \text{ measured from the centre of the Earth}). \end{cases} \]

And, it is natural to take the frictional force as a function of the object’s velocity \( y \):

\[ F_f = F_f(y) = \text{constant}. \]

Thus, (**) takes the form

\[ m\ddot{y} = -F_y(y) - F_f(y). \]

Notice that the independent variable \( t \) doesn’t appear, and thus this is a first order ODE in disguise (Handout 6, §4.2). As well, if we assume constant gravity, then \( y \) doesn’t appear either, and the ODE can be handled in that manner as well (Handout 6, §4.1). Finally, it is sometimes assumed that the frictional force \( F_f = cy \) is proportional to the velocity; in such a case, if we assume constant gravity, then (**) is a linear ODE, and can be analysed accordingly (Handout 6, §§2.6).

The point to note is that we have a number of ways of approaching such kinematics equations. The best approach depends upon the precise form of the equation, and the questions being asked.

## 2 Spring Equations

### 2.1 Free Vibrations

Our fundamental assumption for springs is that they are governed by **Hooke’s Law**: the force \( F \) that a spring exerts is proportional to the extension \( x \) of the spring; here, \( x \) is measured from from the rest position (length \( l \)) of the spring. Since the force acts negatively (i.e. in the opposite direction of the extension), Hooke’s Law takes the form

\[ F = -kx \quad k > 0. \]

Thus, By Newton’s Law, the motion of a cat of mass \( m \) on a spring is governed by the ODE

\[ m\ddot{x} + kx = 0 \quad m, k > 0. \]
This is a constant coefficient ODE. And, noting \( m > 0 \) and \( k > 0 \), we immediately have
the general solution
\[
\begin{align*}
x &= A \cos(\omega_0 t) + B \sin(\omega_0 t) \\
\omega_0 &= \sqrt{\frac{k}{m}}
\end{align*}
\]
Such a solution is called \textit{simple harmonic motion}. Notice that the motion is periodic, with \textit{period} \( T \) given by
\[T = \frac{2\pi}{\omega_0}.
\]
Note that \( k \) has units of \( \text{mass} \cdot \text{time}^{-2} \), and thus \( T \) correctly has units of time.

### 2.1.1 Vertical Systems

Consider the situation where a cat of mass \( m \) is hanging vertically from a spring. Then there is also the force of gravity at work, and the spring equation takes the form
\[
m g + k y + m g = 0.
\]
Here, as usual, positive \( y \) indicates the upward direction. Now if the cat is at rest, gravity will extend the cat on the spring downwards a length \( L \), until the spring and gravitational forces balance. Thus
\[m g = k L.
\]
(This also gives a convenient method for computing \( k \): if we attach a cat of known mass \( m \), we can measure the length \( L \) the spring extends, and then use (\textcircled{V}) to determine \( k \).

But this suggests that we introduce a new variable
\[
x = y + L.
\]
That is, \( x \) is measuring the extension from the new rest position of the cat. Then, by (\textcircled{I}),
\[k y + m g = k x - k L + m g = k x.
\]
And of course \( \ddot{y} = \ddot{x} \). Thus, (VS) transforms exactly into (S) with respect to the new variable \( x \) given by (\textcircled{I}).

The conclusion is, whether the spring-cat system is vertical or horizontal, the ODE (S) with solution (\textcircled{I}) governs the motion, as long as \( x \) is measured from the relevant rest position.

### 2.2 Damped Free Vibrations

In reality, a spring-cat system will experience at least some frictional force due to the internal workings of the spring and the motion of the cat. If we assume that this force is proportional to the cat’s velocity, we are led to the equation
\[
m \ddot{x} + c \dot{x} + k x = 0,
\]
which has solutions
\[
\textcircled{Q} \quad r_1, r_2 = -\frac{c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m}.
\]

We then have three scenarios, with three possible types of solutions to (DS):
\[
\begin{align*}
\textcircled{C} \quad \&^2 - 4mk > 0 & \begin{cases} x = Ae^{r_1 t} + Be^{r_2 t} \\
r_1, r_2 \text{ given by } \textcircled{Q} \end{cases} & \text{(overdamped)} \\
\textcircled{L} \quad \&^2 - 4mk = 0 & x = Ae^{-\frac{ct}{2m}} + Be^{-\frac{ct}{2m}} & \text{(critically damped)} \\
\textcircled{U} \quad \&^2 - 4mk < 0 & \begin{cases} x = e^{-\frac{ct}{2m}} (A \cos(\sqrt{mk} t) + B \sin(\sqrt{mk} t)) \\
o_0 = \frac{\sqrt{4mk - c^2}}{2m} \end{cases} & \text{(underdamped)}
\end{align*}
\]
We make a few simple observations about these solutions.

- The underdamped system still exhibits some periodic behaviour, with period
  \[ T = \frac{2\pi}{\omega_0}. \]
  Of course, this reduces to the period for the undamped spring if \( c = 0 \). Note that, the behaviour is not truly periodic, since the amplitude decays exponentially.

- For the critically damped and overdamped springs, there is no periodic behaviour. There will be at most one crossing of the axis, depending upon the constants \( m, k \) and \( c \), and upon the initial conditions.

- Notice that in all three cases, we have
  \[ \lim_{t \to \infty} x(t) = 0 \]
  For the underdamped system this follows from the Sandwich Rule, and for the critically damped system this follows from L'Hôpital's Rule. For the overdamped system, we just have to note that \( r_1, r_2 < 0 \); this follows easily from \( \langle \triangledown \rangle \), since \( \sqrt{c^2 - 4mk} < \sqrt{c^2} = c \).

\[ \blacktriangleleft \triangledown \blacktriangledown \]

2.3 Periodic Forced Vibrations

We can now consider a (damped or undamped) spring system which is also subject to an external time-dependent force:

(FS) \[ mx + cx + kx = F(t). \]

For any \( F(t) \), we can solve this ODE (give or take doing integrals) by reduction of order (Handout 6, [6.1]). As well, if \( F(t) \) is nice (i.e. a linear combination of polynomials, sin and cos, and exponentials) then we can more readily compute a particular solution to (FS) (Handout 6, [6.2]): this can then be combined with the general solution to (FS), to give the general solution to (FS).

Here, we'll just consider a special case, where the external force is periodic:

(PS) \[ mx + cx + kx = F_0 \cos(\omega t), \]

where \( F_0 > 0 \) and \( \omega > 0 \) are constant. (More general periodic forces can be handled by considering sums of sin and cos terms: this is edging into the very important theory of Fourier series).

We'll need to consider subcases below, but for now we can look for a particular solution \( x_p \) of (PS) by our standard approach (Handout 6, [6.2]). We want a particular solution

(PS) \[ mx_p + cx_p + kx_p = F_0 \cos(\omega t). \]

The first step is to complexify (PS), giving

(CPS) \[ m\dot{x}_p + c\dot{x}_p + kx_p = F_0 e^{i\omega t}. \]

We then do the reduction of order trick:

\[ z_p = e^{i\omega t} \dot{x}_p \implies \begin{cases} \dot{z}_p = e^{i\omega t} \ddot{x}_p + i\omega e^{i\omega t} \dot{x}_p \\ \ddot{z}_p = e^{i\omega t} \dddot{x}_p + 2i\omega e^{i\omega t} \dot{x}_p - \omega^2 e^{i\omega t} x_p \end{cases}. \]

Plugging into (CPS), rearranging, and cancelling out the common factor of \( e^{i\omega t} \), we obtain

(\bullet) \[ m\dot{z}_p + (2m\omega + c) \ddot{z}_p + (k - m\omega^2 + cw) z_p = F_0. \]

In general, we can set \( z_p \) equal to the appropriate constant, but this won’t work directly if the coefficient of \( z \) in (\bullet) is zero. This naturally leads us to consider the subcases below.
2.3.1 Case 1: \( c \neq 0 \) (steady state solutions)

If we have a truly damped spring, then we always have a particular solution to (\( \bullet \)) of the form

\[
x_p = \frac{F_0}{k - m\omega^2 + ci} = F_0 \left( \frac{k - m\omega^2}{(k - m\omega^2)^2 + c^2\omega^2} \right).
\]

Setting \( x_p = \text{Re} \left( v_p e^{i\omega t} \right) \), this then gives us the particular solution

(\( \triangledown \))

\[
x_p = \frac{F_0}{(k - m\omega^2)^2 + c^2\omega^2} \left( (k - m\omega^2) \cos(\omega t) + c\omega \sin(\omega t) \right)
\]

We then get the general solution to (PS) to be

\[
x = x_p + Ax_1 + Bx_2,
\]

where \( Ax_1 + Bx_2 \) is given by (\( \triangledown \)).

Notice that \( x_p \) is a genuinely periodic solution to (PS), with period \( \omega \). On the other hand, by (\( \triangledown \)), \( x_1 \) and \( x_2 \) decay to 0. Thus, in the long-run, \( x \approx x_p \), no matter the initial conditions of the system. Thus, the particular solution \( x_p \) is called the **steady state solution** of the system.

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2.3.2 Case 2(a): \( c = 0 \) and \( \omega \neq \omega_0 \) (the phenomenon of beats)

Note that

\[
\omega = \omega_0 \iff k = m\omega^2.
\]

Thus, if \( \omega \neq \omega_0 \) then again (\( \bullet \)) gives rise to a particular solution (\( \triangledown \)) for \( v_p \). With \( c = 0 \) the solution simplifies: fiddling slightly, we obtain the general solution

(\( \star \))

\[
x = A\cos(\omega_0 t) + B\sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)
\]

We see that the solution in general involves two periods, \( \omega \) and \( \omega_0 \). This can lead to the phenomenon of **beats**. To see this, let’s consider the solution with the initial conditions

\[
x(0) = 0 \quad x'(0) = 0.
\]

This easily leads to

\[
A = -\frac{F_0}{m(\omega_0^2 - \omega^2)} \quad B = 0,
\]

and thus we get the specific solution

\[
x = \frac{F_0}{m(\omega_0^2 - \omega^2)} [\cos(\omega_0 t) - \cos(\omega t)].
\]

If we now apply the trig identities

\[
\cos(\theta \pm \phi) = \cos(\theta) \cos(\phi) \mp \sin(\theta) \sin(\phi),
\]

we can rewrite this solution as

\[
x = \frac{F_0}{m(\omega_0^2 - \omega^2)} \sin \left( \frac{\omega - \omega_0}{2} \right) \sin \left( \frac{\omega + \omega_0}{2} \right).
\]

Beats occur if \( \omega \approx \omega_0 \). As a consequence, we hear a high frequency \( \omega + \omega_0 \) with an amplitude varying slowly as \( \sin(\omega - \omega_0) \). This is exactly what one hears, when two clear musical notes which are very close in frequency are played together.

2.3.3 Case 2(b): \( c = 0 \) and \( \omega = \omega_0 \) (the phenomenon of resonance)

If \( c = 0 \) and \( \omega = \omega_0 \), then we obtain a fundamentally different solution. Returning to (\( \bullet \)), our particular solution \( v_p \) must now satisfy the equation

\[
m v_p'' + 2m\omega_0 v_p' + F_0 = 0.
\]

In this case, we cannot choose \( v_p \) a constant. This is a general phenomenon, which occurs when the RHS is also a solution of the homogeneous equation.

However, we can clearly choose \( v_p \) a constant, leading to the specific solution

\[
v_p = \frac{F_0 t}{2m\omega_0} = \frac{-F_0 t}{2m\omega_0}
\]

Notice that \( v_p \) also has a free complex constant of integration which we’ve set to 0: we discuss this below. In any case, we obtain a particular solution

\[
x_p = \text{Re} \left( z_p \right) = \text{Re} \left( e^{i\omega_0 t} v_p \right) = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t).
\]

The general solution is

(\( \triangledown \))

\[
x = A\cos(\omega_0 t) + B\sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t \sin(\omega_0 t).
\]
Notice that in this case including the free +c in \( v_p \) will just give us extra terms solving the complementary homogeneous equation.

The equation (\( L \)) exhibits the phenomenon of resonance. There are bounded periodic terms, but \( x_p \) has amplitude growing linearly with time, and no matter the initial conditions, \( x \) will oscillate without bound.

Clearly, this could be catastrophic to our system, and it is to be considered whenever we have a spring system subject to periodic vibrations. It is the reason, for instance, why soldiers are ordered to break step, to not march in unison, when crossing a bridge. (However, as we have mentioned, the collapse of the Tacoma Narrows Bridge was due to a more complicated spring phenomenon.)

3 Electrical Circuits

We shall discuss this very little; the main point is that the equation for the charge \( q \) flowing around a circuit has exactly the same form as the spring system:

\[
L \frac{dq}{dt} + R q + \frac{1}{C} q = E(t)
\]

(E)

Or course we have to explain how the terms in the equation arise, and the interpretation of the solutions will differ. But the solutions themselves will be of exactly the same form.

In a basic \( L - R - C \) circuit, we consider the flow of the charge \( q \) (measured in Coulombs). The current

\[
I = \frac{dq}{dt}
\]

is the derivative of \( q \), measured in Amps (=Coulombs per second). Associated with this current flow, the voltage \( V \) (measured in Volts = Joules/Coulomb) changes across the circuit. Kirchhoff’s Law says that around a circuit the sums of the voltage changes must be zero; equation (E) is exactly expressing Kirchhoff’s Law (analogous to the acceleration in a spring system arising from the sums of the forces). We just have to itemize the various elements in the circuit, contributing the various terms to (E).

- A resistor with resistance \( R \) (measured in ohms) gives rise to a voltage change

\[
V_R = IR = qR.
\]

- A capacitor with capacitance \( C \) (measured in farads) gives rise to a voltage change

\[
V_C = \frac{1}{C} q.
\]

In fact, \( q(t) \) in (E) refers to the charge on the capacitor at any time \( t \).

- An inductor with inductance \( L \) (measured in henrys) gives rise to a voltage change

\[
V_L = LI = L \frac{dq}{dt}
\]

- the impressed voltage is any external voltage source

\[
E = E(t)
\]

Commonly \( E \) is either constant or periodic. But, in any case, we can look to solve (E) exactly as for the various spring systems.