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Definitions

Definition

A series is an infinite sum

\[ a_1 + a_2 + \cdots = \sum_{n=1}^{\infty} a_n \]

with \( a_1, a_2, \ldots \) given numbers (real or complex)

Important questions are

- Does the series converge, i.e., is the sum

\[ a_1 + a_2 + a_3 + \cdots \]

a “finite number”?  
- If the series converges, can we find an explicit expression for the sum?

Example

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \]

but \( \sum_{n=1}^{\infty} n = \infty \)
To give a precise definition of the convergence of a series we define the $N$th partial sum $S_N$ of a series as the sum of the first $N$ terms:

$$S_N = a_1 + \cdots + a_N.$$ 

**Note:** This associates a sequence $\{S_N\}$ to each series.

Associated to the series

- $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is the sequence $\{1, 1 + \frac{1}{4}, 1 + \frac{1}{4} + \frac{1}{9}, \ldots \}$,

- $\sum_{n=1}^{\infty} n$ is the sequence $\{1, 1 + 2, 1 + 2 + 3, \ldots \} = \{1, 3, 6, \ldots, N(N+1)/2, \ldots \}$,

- $\sum_{n=0}^{\infty} (-1)^n$ is the sequence $\{1, 0, 1, 0, \ldots \}$. 
**Definition**

A series $\sum a_n$ converges to the sum $S$ if the sequence of partial sums $\{S_N\}$ converges to $S$.

A series which does not converge is said to be **divergent**.

The problem of determining whether a series converges is thus reduced to the problem of determining whether a sequence converges.

For some series we can explicitly compute $S_N$ and then consider $\lim_{N \to \infty} S_N$. For example:

- The series $\sum_{n=1}^{\infty} n$ diverges because the sequence $\{S_N\} = \{1, 3, 6, \ldots, N(N + 1)/2, \ldots\}$ diverges.

- The series $\sum_{n=0}^{\infty} (-1)^n$ diverges because the sequence $\{S_N\} = \{1, 0, 1, 0, \ldots\}$ diverges.

For other series we cannot actually find a closed form expression for $S_N$. This requires more theory to be developed.
Example (Geometric series)

The series
\[ \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for } |x| < 1 \]

and diverges for all other \( x \)

Proof. The partial sum \( S_N \) is given by
\[ S_N = \sum_{n=0}^{N-1} x^n = 1 + x + \cdots + x^{N-1}. \]

When \( x = 1 \) we obviously have \( S_N = N \) so that the series diverges.

In the remainder assume that \( x \neq 1 \).

Multiplying the expression for \( S_N \) by \( 1-x \) yields
\[
(1-x)S_N = (1 + x + \cdots + x^{N-1}) - (x + x^2 + \cdots + x^N)
= 1 - x^N.
\]
Hence

\[ S_N = \frac{1 - x^N}{1 - x}. \]

Since

\[ \lim_{N \to \infty} x^N = 0 \quad \text{for } |x| < 1 \]

we find

\[ \lim_{N \to \infty} S_N = \frac{1}{1 - x}. \]

For all other \( x \) the series diverges.
Telescoping series

Definition

A series \( \sum_{n=1}^{\infty} a_n \) with \( a_n \) of the form \( a_n = b_n - b_{n+1} \) is called a telescoping series.

For telescoping series

\[
S_N = \sum_{n=1}^{N} a_n = \sum_{n=1}^{N} (b_n - b_{n+1})
\]

\[
= (b_1 + b_2 + \cdots + b_N) - (b_2 + b_3 + \cdots + b_{N+1})
\]

\[
= b_1 - b_{N+1}.
\]

Hence a telescoping series converges if the limit \( \lim_{N \to \infty} b_N \) exists.
Example 1

\[ \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \]

**Proof.** By a partial fraction expansion

\[ \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}. \]

Hence

\[ S_N = \sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \frac{1}{n} - \sum_{n=1}^{N} \frac{1}{n+1} \]

\[ = 1 - \frac{1}{N+1}. \]

Since \( \lim_{N \to \infty} \frac{1}{N+1} = 0 \) this yields \( S_N \to 1 \) for \( N \to \infty \) and

\[ \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1. \]
The previous example may be generalized to the next prize challenge:

Show that 

\[
\sum_{n=1}^{\infty} \frac{1}{n(n+1) \cdots (n+p)} = \frac{1}{p \cdot p!}.
\]
Example 2

\[
\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) \text{ diverges}
\]

**Proof.**

\[
S_N = \sum_{n=1}^{N} \log\left(\frac{n+1}{n}\right)
\]

\[
= \sum_{n=1}^{N} \left(\log(n+1) - \log n\right)
\]

\[
= \log(N+1) - \log 1
\]

\[
= \log(N+1).
\]

Since the sequence \( \{S_N\} \) diverges, the series diverges.
Algebra of series

From the algebra of limits for sequences it immediately follows that if \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) are converging series with sums \( A \) and \( B \), then

\[
\sum_{n=1}^{\infty} (a_n + b_n) = A + B
\]

\[
\sum_{n=1}^{\infty} Ca_n = CA
\]

where \( C \) is an arbitrary constant.

We have nothing to say about the much more complicated series

\[
\sum_{n=1}^{\infty} a_n b_n \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{a_n}{b_n}.
\]
Example

Compute \[ \sum_{n=1}^{\infty} \left( \frac{(-1)^n}{4^n} + \frac{6}{n(n+1)} \right) \]

By the geometric series

\[ \sum_{n=1}^{\infty} \left( \frac{-1}{4} \right)^n = \frac{1}{1 - (-1/4)} - 1 = \frac{4}{5} - 1 = -\frac{1}{5}. \]

By Example 1 on telescoping series

\[ \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1. \]
By the algebra of series we thus find

$$\sum_{n=1}^{\infty} \left( \frac{(-1)^n}{4^n} + \frac{6}{n(n + 1)} \right)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} + 6 \sum_{n=1}^{\infty} \frac{1}{n(n + 1)} = -\frac{1}{5} + 6 \cdot 1 = \frac{29}{5}.$$
Note: Great care has to be taken when using the algebra of series. Often
\[
\sum_{n=1}^{\infty} (a_n + b_n) \neq \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.
\]

For example
\[
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \neq \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n + 1}
\]
since the two harmonic series on the right diverge but the series on the left sums to 1.

\[
\sum_{n=1}^{\infty} 0 = \sum_{n=1}^{\infty} (1 - 1) \neq \sum_{n=1}^{\infty} 1 - \sum_{n=1}^{\infty} 1.
\]
Convergence tests

There are many tests to determine whether a series converges or diverges, such as the divergence test, ratio test, comparison test, integral test, Leibniz test and more. We shall discuss a number of these.

We note however that many questions about convergence of series remain open. For example, nobody knows if the following series converges.

The million dollar question

Does the series

$$\sum_{n=1}^{\infty} \frac{\left(\frac{2}{3} + \frac{1}{3} \sin n\right)^n}{n}$$

converge?

Disclaimer: The above does not refer to Dr O’s million!
**The divergence test**

If \( \lim_{n \to \infty} a_n \neq 0 \) then the series \( \sum_{n=1}^{\infty} a_n \) diverges.

**Note:** The converse statement is false, and series with \( a_n \to 0 \) do not necessarily converge. A well-known example of this is the harmonic series to be discussed later.
Proof. We will show that if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \to \infty} a_n = 0$. 

This of course implies that the sum $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n \to \infty} a_n \neq 0$.

If a series converges, the sequence of partial sums converges to a limit:

$$\lim_{N \to \infty} S_N = S.$$ 

Since $S_n - S_{n-1} = a_n$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} (S_n - S_{n-1})$$

$$= \lim_{n \to \infty} S_n - \lim_{n \to \infty} S_{n-1} \quad \text{(by the algebra of limits)}$$

$$= S - S$$

$$= 0.$$
**Example 1**

\[ \sum_{n=1}^{\infty} n^{1/n} \text{ diverges} \]

A standard limit gives \( \lim_{n \to \infty} n^{1/n} = 1 \neq 0 \).
According to the **divergence test** the series diverges.

**Example 2**

The series \[ \frac{1}{3} + \frac{2}{5} + \frac{3}{7} + \cdots \text{ diverges}. \]

We have

\[ a_n = \frac{n}{2n+1} = \frac{1}{2 + 1/n} \to \frac{1}{2} \neq 0. \]

According to the **divergence test** the series diverges.
Example 3 (**Harmonic series**)

The series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges.

The harmonic series provides an example of a series that diverges despite the fact that \( \lim a_n \to 0 \).

The standard proof of the divergence of the harmonic series uses the yet to be discussed **integral test**.

Long before calculus was invented d’Oresme (1323–1382) found the following ingenious proof.
Proof. We need to show that the limit $\lim_{N \to \infty} S_N$ does not exist. From the theory of sequences we know that it suffices to show that a subsequence of $\{S_N\}$ does not have a limit.

d’Oresme considered the subsequence $\{S_{2N}\} = \{S_1, S_2, S_4, \ldots\}$. Then

$$S_{2N} - 1 = \frac{1}{2} + \cdots + \frac{1}{2^N}$$

$$= \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$+ \cdots + \left(\frac{1}{2N-1 + 1} + \cdots + \frac{1}{2^N}\right)$$

$$\geq \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$+ \cdots + \left(\frac{1}{2N} + \cdots + \frac{1}{2^N}\right)$$

$$= \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}$$

$$= \frac{1}{2}N.$$

Since $S_{2N} \geq 1 + \frac{1}{2}N$ the subsequence $\{S_{2N}\}$ diverges to $\infty$. □
Let $f : [1, \infty) \to \mathbb{R}$ be a continuous, positive and decreasing function.

**The integral test**

$$\sum_{n=1}^{\infty} f(n) \quad \text{converges} \iff \int_{1}^{\infty} f(x) \, dx \quad \text{exists}$$

**Proof.**

We have

$$f(2) + f(3) + \cdots \leq \int_{1}^{\infty} f(x) \, dx \leq f(1) + f(2) + f(3) + \cdots$$

The integral test may be used to proof convergence/divergence of the generalized harmonic series (or $p$-series).
Generalized harmonic series

\[ \sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges for } p > 1 \\ \text{diverges for } p \leq 1 \end{cases} \]

**Proof.** Assume that \( p \neq 1 \). Then

\[
I(y) := \int_{1}^{y} \frac{dx}{x^p} = \left[ \frac{x^{1-p}}{1-p} \right]_{1}^{y} = \frac{y^{1-p} - 1}{1-p}.
\]

If \( p > 1 \) (so that \( 1 - p < 0 \))

\[
\lim_{y \to \infty} y^{1-p} = 0 \quad \text{and} \quad \lim_{y \to \infty} I(y) = \frac{1}{p - 1}.
\]

If \( p < 1 \) (so that \( 1 - p > 0 \))

\[
\lim_{y \to \infty} y^{1-p} \text{ diverges to } \infty \quad \text{and} \quad \lim_{y \to \infty} I(y) \text{ does not exist.}
\]
Finally we consider $p = 1$. Then

$$I(y) = \int_1^y \frac{dx}{x} = \left[ \log x \right]_1^y = \log y - \log 1 = \log y.$$ 

Hence

$$\lim_{y \to \infty} I(y) \text{ does not exist.}$$

Combining the above findings for the large $y$ limit of $I(y)$ with the integral test completes the proof.
In the following we will consider several convergence tests for series \( \sum_n a_n \) with \( a_n \geq 0 \) for all \( n \) or \( a_n > 0 \) for all \( n \).

It is important to note that convergence and/or divergence of a series is not determined by the first \( N \) elements of the series (with \( N \) arbitrary!).

Hence tests that apply to series with \( a_n > 0 \) (or \( a_n \geq 0 \)) also apply to series for which \( a_n \leq 0 \) (or \( a_n < 0 \)) for a finite number of terms.

Effectively the following series has \( a_n > 0 \):

\[(−3) + (−2) + (−1) + (0) + (1) + (2) + \ldots\]

since only the first four terms are non-positive.
The comparison test (for convergence)

Let $\sum_n a_n$ and $\sum_n t_n$ be series such that $0 \leq a_n \leq t_n$ for all $n$.
Then

$$\sum_n t_n \text{ converges} \implies \sum_n a_n \text{ converges}$$

The series $\sum_n t_n$ is often referred to as a test series. Obviously, if the test series converges to the sum $T$ then the series $\sum_n a_n$ converges to the sum $A$ with $0 \leq A \leq T$.

**Proof.** Let $A_N$ and $T_N$ be the $N$th partial sums of the two series under consideration, and let $T = \lim_{n \to \infty} T_N$.

The sequence $\{A_N\}$ is:

- monotonic, since $a_n \geq 0$ for all $n$.
- bounded by $T$, since $A_N \leq T_N \leq T$ (by $0 \leq a_n \leq t_n$).

By the monotonic sequence theorem the sequence $\{A_N\}$ converges.
The comparison test (for divergence)

Let $\sum_n a_n$ and $\sum_n t_n$ be series such that $0 \leq t_n \leq a_n$ for all $n$. Then

$$\sum_n t_n \text{ diverges} \Rightarrow \sum_n a_n \text{ diverges}$$

**Proof.** Let $A_N$ and $T_N$ be the $N$th partial sums of the two series under consideration.

Since the sequence $\{T_N\}$ is monotonic but does not have limit, it must be unbounded.

But then $\{A_N\}$ must also be an unbounded monotonic sequence, since $0 \leq T_N \leq A_N$.

By the **monotonic sequence theorem** the sequence $\{A_N\}$ diverges.
Example 1

The series \( \sum_{n=1}^{\infty} \frac{1}{3n + 5^n + 1} \) converges

If

\[ a_n = \frac{1}{3n + 5^n + 1} \quad \text{and} \quad t_n = \frac{1}{5^n} \]

then \( 0 \leq a_n \leq t_n \).

But

\[ \sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} \left( \frac{1}{5} \right)^n \]

converges (with sum \( \frac{1}{4} \)) by the geometric series.

By the comparison test \( \sum_n a_n \) converges.
Example 2

The series \( \sum_{n=1}^{\infty} \frac{n^{1/3}}{4n + 1} \) diverges.

If

\[
a_n = \frac{n^{1/3}}{4n + 1} \quad \text{and} \quad t_n = \frac{1}{5n^{2/3}}
\]

then \( 0 \leq t_n \leq a_n \) since

\[
a_n = \frac{n^{1/3}}{4n + 1} \geq \frac{n^{1/3}}{4n + n} = \frac{n^{1/3}}{5n} = t_n.
\]

But

\[
\sum_{n=1}^{\infty} t_n = \frac{1}{5} \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}
\]

diverges (generalized harmonic series with \( p = \frac{2}{3} \)).

By the comparison test \( \sum_n a_n \) diverges.
Note: The divergence test would not have helped in the previous example since

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^{1/3}}{4n + 1} = 0.$$
The “limit” comparison test

Let $\sum_n a_n$ and $\sum_n b_n$ be series such that $a_n > 0$ and $b_n > 0$ for all $n$ and such that $\lim_{n \to \infty} \frac{a_n}{b_n} = c > 0$

Then
- both series converge or
- both series diverge

Proof. Choose $0 < \epsilon < c$.

Since $\lim_{n \to \infty} \frac{a_n}{b_n} = c$ there exists an $N$ such that

$$\left| \frac{a_n}{b_n} - c \right| < \epsilon \quad \text{for} \quad n > N.$$ 

Hence

$$0 < (c - \epsilon) b_n < a_n < (c + \epsilon) b_n \quad \text{for} \quad n > N,$$

and thus

$$0 < (c - \epsilon) \sum_{n=N+1}^{M} b_n < \sum_{n=N+1}^{M} a_n < (c + \epsilon) \sum_{n=N+1}^{M} b_n. \quad (\ast)$$
If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=N+1}^{\infty} a_n$ also converges.

By $(\ast)$ (and $b_n > 0$) $\left( c - \epsilon \right) \sum_{n=N+1}^{\infty} b_n$ also converges.

Hence $\sum_{n=1}^{\infty} b_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges it must diverge to $\infty$ (since $a_n > 0$).

By $(\ast)$ $(c + \epsilon) \sum_{n=N+1}^{\infty} b_n$ also diverges to $\infty$.

Note: In the above we have implicitly used the monotonic sequence theorem to the sequences of partial sums of $\sum a_n$ and $\sum b_n$. 
Example 1

Let \( a_n = \frac{n + \log n}{n^3} \).

Since \( n \) grows faster than \( \log n \) it is clear that \( a_n \approx \frac{n}{n^3} = \frac{1}{n^2} \) for large \( n \).

Therefore we take \( b_n = \frac{1}{n^2} \).

Then \( a_n > 0, b_n > 0 \) and

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n + \log n}{n} = \lim_{n \to \infty} \left(1 + \frac{\log n}{n}\right) = 1.
\]

Since \( \sum b_n \) converges (generalized harmonic series with \( p = 2 \)), \( \sum a_n \) converges thanks to the limit comparison test.
The previous result can also be obtained using the comparison test for convergence.

Let \( a_n = \frac{n + \log n}{n^3} > 0 \) and \( t_n = \frac{2}{n^2} > 0 \).

\[ \sum t_n \text{ converges as it is twice the generalized harmonic series with } p = 2. \]

But, since \( \log n < n \),

\[ a_n = \frac{n + \log n}{n^3} < \frac{n + n}{n^3} = \frac{2}{n^2} = t_n. \]

Hence the series \( \sum a_n \) converges by the comparison test for convergence.
Example 2

\[ \sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}} \] diverges

Let \( a_n = \frac{1}{n^{1+1/n}} \).

Since \( n^{1/n} \to 1 \) it follows that \( a_n \approx \frac{1}{n} \) for large \( n \).

Therefore we take \( b_n = \frac{1}{n} \).

Then \( a_n > 0, b_n > 0 \) and

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1}{n^{1/n}} = 1.
\]

Since \( \sum b_n \) diverges (harmonic series), \( \sum a_n \) diverges thanks to the limit comparison test.
Ratio test

Let \( a_n > 0 \) for all \( n \) and let \( r = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \) exist (or be \( \infty \))

Then the series \( \sum_{n=1}^{\infty} a_n \)
- converges if \( r < 1 \)
- diverges if \( r > 1 \)
- ??? if \( r = 1 \)

Proof. (of first claim)

Let \( r < R < 1 \). Since \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = r \) there exists an \( N \) such that

\[
\frac{a_{n+1}}{a_n} \leq R \quad \text{for} \quad n \geq N.
\]
Hence

\[ a_{N+1} \leq R \, a_N \]
\[ a_{N+2} \leq R \, a_{N+1} \leq R^2 \, a_N \]
\[ \vdots \]
\[ a_{N+k} \leq \ldots \leq R^k \, a_N \]

and

\[ \sum_{n=N}^{M} a_n \leq a_N (1 + R + \cdots + R^{M-N}) \]
\[ = a_N \frac{1 - R^{M-N+1}}{1 - R}. \]

Therefore

\[ \lim_{M \to \infty} \sum_{n=N}^{M} a_n \leq \frac{a_N}{1 - R}. \]

The proof of the second claim is homework.
As the following examples show, the ratio test is useful for handling series involving factorials and powers.

**Example 1**

\[
\sum_{n=1}^{\infty} \frac{n^{123}}{n!} \quad \text{converges}
\]

It is not easy to find a closed form expression for \( S_N \) or to prove convergence using the comparison test.

Let \( a_n = \frac{n^{123}}{n!} \). Then

\[
\frac{a_{n+1}}{a_n} = \frac{(n + 1)^{123}}{n^{123}} \cdot \frac{n!}{(n + 1)!} = \frac{(1 + \frac{1}{n})^{123}}{n + 1} \to 0 =: r.
\]

Since \( r < 1 \) the series converges by the ratio test.
Example 2

\[
\sum_{n=1}^{\infty} \frac{n}{x^n}
\]

converges for \(x > 1\) and diverges for \(0 < x \leq 1\).

Let \(a_n = \frac{n}{x^n}\).

Then

\[
\frac{a_{n+1}}{a_n} = \frac{n+1}{n x} \rightarrow \frac{1}{x} =: r.
\]

Since \(r < 1\) \((r > 1)\) for \(x > 1\) \((0 < x < 1)\) the series converges (diverges) for such \(x\) by the ratio test.

When \(x = 1\) the series obviously diverges.
The ratio test has little to say about series for which $\frac{a_{n+1}}{a_n} \to 1$.

Take for example

- the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. This series diverges and

$$\frac{a_{n+1}}{a_n} = \frac{n}{n+1} = \frac{1}{1 + \frac{1}{n}} \to 1.$$

- the generalized harmonic series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. This series converges and

$$\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} = \frac{1}{\left(1 + \frac{1}{n}\right)^2} \to 1.$$
Summary

The convergence and divergence tests discussed so far — and when to apply them — may be summarised as follows.

Consider the series $\sum_{n=1}^{\infty} a_n$

- Check if $\lim_{n \to \infty} a_n = 0$. If not the series diverges by the divergence test.

- Check if for sufficiently large $n$ all $a_n \geq 0$ (or all $a_n \leq 0$ and consider $\sum b_n$ with $b_n = -a_n$).

  If yes:

  - Check if $a_n > 0$ and $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} \neq 1$ and apply the ratio test.

  - If ratio test cannot be applied try the comparison tests or the integral test.
Absolute convergence

All test discussed so far (apart from the divergence test) can handle series with \( a_n \geq 0 \) or \( a_n > 0 \) (for \( n \) large enough). In the following we will consider tests for more general series.

**Definition**

\[
\sum_{n=1}^{\infty} a_n \text{ is called absolutely convergent if } \sum_{n=1}^{\infty} |a_n| \text{ converges}
\]

An example of an absolutely convergent series is

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots
\]

since \( \sum_{n=1}^{\infty} |a_n| \) is the generalized harmonic series with \( p = 2 \).
**Absolute convergence theorem**

All absolutely convergent series are convergent

**Proof.** Let $\sum |a_n|$ be convergent. Since $-|a_n| \leq a_n \leq |a_n|$

$$0 \leq a_n + |a_n| \leq 2|a_n|.$$  

Let $b_n = a_n + |a_n|$ and $t_n = 2|a_n|$. Then $0 \leq b_n \leq t_n$ with $\sum_n t_n$ a convergent series.

By the comparison test for convergence the series $\sum b_n$ converges. But then the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} |a_n|$$

is the difference of two convergent series and thus converges by the algebra of series.
Example 1

\[ \sum_{n=1}^{\infty} \frac{\sin n}{n^2} \] converges

Let \( a_n = \frac{\sin n}{n^2} \). Then the sign of \( a_n \) keeps changing when \( n \) increases and earlier test may not be applied.

Now

\[ |a_n| = \frac{|\sin n|}{n^2} \leq \frac{1}{n^2} =: t_n. \]

Since \( 0 \leq |a_n| \leq t_n \) with \( \sum t_n \) a convergent series (generalized harmonic series with \( p = 2 \)) the series \( \sum |a_n| \) converges by the comparison test for convergence.

Since \( \sum |a_n| \) converges, the series \( \sum a_n \) is absolutely convergent and therefore convergent by the absolute convergence theorem.
Example 2

Determine the convergence of \[ \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}} \]

Unlike the geometric series \( \sum_n x^n \) it is not easy to find a closed form expression for the partial sum \( S_N \).

Since the sign of \( x^n \) can be both positive and negative (when \( x < 0 \)) we try to apply the absolute convergence theorem.

First note that by the divergence test the series diverges for \( x > 1 \) or \( x < -1 \) since \( \frac{x^n}{\sqrt{n}} \) does not tend to 0 for such \( x \).

We also see that the series diverges for \( x = 1 \) (generalized harmonic series with \( p = 1/2 \)).
Let $b_n := |a_n| = \frac{|x|^n}{\sqrt{n}}$.

There are several methods to determine the convergence of the series $\sum b_n$.

- Since $0 \leq b_n \leq |x|^n = t_n$ and $\sum t_n$ converges for $-1 < x < 1$ the series $\sum b_n$ converges for $-1 < x < 1$ by the comparison test for convergence.

$$\frac{b_{n+1}}{b_n} = |x| \frac{\sqrt{n+1}}{\sqrt{n}} = |x| \sqrt{1 + \frac{1}{n}} \to |x|.$$  

This is less than 1 for $-1 < x < 1$ so that the series $\sum b_n$ converges for $-1 < x < 1$ by the ratio test.  
(Note: the ratio test assumes $b_n > 0$ but the case $x = 0$ does not pose a problem.)

Since $\sum b_n$ converges for $-1 < x < 1$, the series $\sum a_n$ converges for $-1 < x < 1$ by the absolute convergence test.
We have not yet dealt with $x = -1$:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = ????$$

All we can say for now is that the above series is not absolutely convergent.

The above results may be summarised as follows:

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}} \left\{ \begin{array}{ll}
\text{converges for } -1 < x < 1 \\
\text{diverges for } x \geq 1 \text{ and } x < -1 \\
??? for x = -1
\end{array} \right.$$
Leibniz test

The series

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \]  

(*)

is an example of an alternating series.

Definition

An alternating series is a series in which the terms alternate in sign.

Definition

A series which is convergent but not absolutely convergent is called conditionally convergent.

The alternating series (\(\ast\)) is an example of a conditionally convergent series.
Leibniz test (or alternating series test)

An alternating series \( \sum_{n=1}^{\infty} (-1)^{n+1} a_n \) converges when

- \( a_n > 0 \) for all \( n \)
- \( a_n \geq a_{n+1} \) for all \( n \)
- \( \lim_{n \to \infty} a_n = 0 \)

**Note:** The sequence \( \{a_n\} \) is a monotonically decreasing with limit 0.
**Proof.** The sequence \( \{S_{2N+1}\} \) is a monotonic decreasing since

\[
S_{2N+3} = S_{2N+1} - a_{2N+2} + a_{2N+3} \leq S_{2N+1}
\]

(by \( a_n \geq a_{n+1} \)).

But

\[
S_{2N+1} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2N-1} - a_{2N}) + a_{2N+1} = \text{sum of positive terms} \geq 0
\]

Hence \( \{S_{2N+1}\} \) is a bounded monotonic sequence and thus converges by the **monotonic sequence theorem**.

The sequence \( \{S_{2N}\} \) is a monotonic increasing since

\[
S_{2N+2} = S_{2N} + a_{2N+1} - a_{2N+2} \geq S_{2N}
\]

(by \( a_n \geq a_{n+1} \)).

But

\[
S_{2N} = a_1 - [(a_2 - a_3) + (a_4 - a_5) + \cdots + (a_{2N-2} - a_{2N-1}) + a_{2N}] = a_1 - \text{sum of positive terms} \leq a_1.
\]

Hence \( \{S_{2N}\} \) is a bounded monotonic sequence and thus converges by the **monotonic sequence theorem**.
From $S_{2N} = S_{2N-1} + a_{2N}$ and the algebra of limits we get

$$
\lim_{N \to \infty} S_{2N} = \lim_{N \to \infty} (S_{2N-1} + a_{2N}) = \lim_{N \to \infty} S_{2N-1} + \lim_{N \to \infty} a_{2N} = \lim_{N \to \infty} S_{2N-1}
$$

since $a_n \to 0$.

Since both the “even” and the “odd” subsequences of $\{S_N\}$ have the same limit, the sequence itself converges.
Example 1

The series \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \] converges.

Multiplying the series by \(-1\) we obtain the alternating series form
\[ \sum_{n=1}^{\infty} (-1)^{n+1} a_n \] with \(a_n = \frac{1}{\sqrt{n}}\).

Hence \(a_n \geq 0\), \(a_{n+1} \leq a_n\) and \(a_n \to 0\).

By the Leibniz test the series converges.

Example 2

The series \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^p} \] converges conditionally for \(0 < p \leq 1\) and absolutely for \(p > 1\).
Conditional convergence

Conditionally convergent series need to be handled with great care. The famous Riemann series theorem says that by rearranging the terms in an alternating series one can make the series converge to any arbitrary sum, or make it diverge.

In contrast, rearranging the terms in an absolutely convergent will not change its sum.

Let us “compute” the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ by rearranging terms.

(Remember, the proper way to compute the sum of a series is to calculate the partial sums $S_N$ and then take the large $N$ limit.)
Since the series conditionally converges it has a sum, which we denote $S$. Then

$$ S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} $$

$$ = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots $$

"=" \((1 - \frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{6} - \frac{1}{8}) + (\frac{1}{5} - \frac{1}{10} - \frac{1}{12}) + \cdots\)

$$ = \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{10} - \frac{1}{12}\right) + \cdots $$

$$ = \frac{1}{2} \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right) $$

$$ = \frac{1}{2} S. $$

But $S \neq 0$ (it is in fact log 2) since

$$ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \cdots \geq 0. $$
Homework problem

“Compute” $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ by grouping $P$ positive terms and $N$ negative terms at each step.

For example, when $P = 2$ and $N = 3$:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

“=” $\left(1 + \frac{1}{3}\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6}\right)$

$$+ \left(\frac{1}{5} + \frac{1}{7}\right) - \left(\frac{1}{8} + \frac{1}{10} + \frac{1}{12}\right) + \cdots$$
General ratio test

Let \( a_n \neq 0 \) for all \( n \) and let \( r = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} \) exist (or be \( \infty \)).

Then the series \( \sum_{n=1}^{\infty} a_n \) converges (absolutely) if \( r < 1 \), diverges if \( r > 1 \), and is undefined if \( r = 1 \).

**Proof.** Apply the ratio test to the series \( \sum_{n=1}^{\infty} |a_n| \) and use the absolute convergence theorem.
Power series

Definition

A **power series** centred around $a$ is a series of the form

$$
\sum_{n=0}^{\infty} c_n(x - a)^n \quad (c_n \text{ independent of } x)
$$

For example, when $a = -1$ and $c_0 = c_1 = 0$, $c_n = \frac{1}{n}$ for $n \geq 2$ we get the power series

$$
\sum_{n=2}^{\infty} \frac{(x + 1)^n}{n}.
$$
If there is set \( D \subseteq \mathbb{R} \) such that the power series converges for \( x \in D \), then we can define the function
\[
f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n, \quad x \in D.
\]

Note that \( D \) contains at least one point, namely \( a \):
\[
f(a) = \sum_{n=0}^{\infty} c_n (a - a)^n = c_0.
\]
Probably the easiest nontrivial example of a power series is the geometric series

\[ \sum_{n=0}^{\infty} x^n. \]

Since we already know that this sums to \( \frac{1}{1-x} \) for \( |x| < 1 \) and diverges elsewhere, we have

\[ f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1. \]

**Note:** The function \( \frac{1}{1-x} \) in fact exists for all \( x \in \mathbb{R} - \{1\} \), but has a pole on the radius of convergence of the geometric series.
Theorem

If \( \sum_{n=0}^{\infty} c_n(x - a)^n \) converges for \( x = b \) \( (b \neq a) \) then it converges absolutely in the interval \( |x - a| < |b - a| \).

Proof. Since \( \sum_{n=0}^{\infty} c_n(b - a)^n \) converges, we have \( c_n(b - a)^n \to 0 \) as \( n \to \infty \) (by the divergence test).

There thus exists a \( K > 0 \) such that

\[ |c_n(b - a)^n| < K \quad \text{for all } n. \]

Now

\[
\sum_{n=0}^{\infty} |c_n(x - a)^n| = \sum_{n=0}^{\infty} |c_n(b - a)^n| \frac{|x - a|^n}{|b - a|^n} \leq \sum_{n=0}^{\infty} K \left| \frac{x - a}{b - a} \right|^n = K \sum_{n=0}^{\infty} \left| \frac{x - a}{b - a} \right|^n.
\]
The last series on the right is a **geometric series** which will converge for
\[
\left| \frac{x - a}{b - a} \right| < 1.
\]

This establishes the claimed absolute convergence for
\[
|x - a| < |b - a|.
\]

The theorem has a very important corollary.

**Corollary**

If \( \sum_{n=0}^{\infty} c_n(x - a)^n \) diverges at \( x = c \) then it diverges for all \( x \) such that \( |x - a| > |c - a| \).
**Proof.** We proof the corollary by contradiction.

Assume there is a $d$ with

$$|d - a| > |c - a| \quad (\ast)$$

such that the series converges for $x = d$.

By the theorem the series would then converge for all $x$ such that $|x - a| < |d - a|$.

But by $(\ast)$ this inequality is certainly satisfied for $x = c$ so that we have convergence for $x = c$.

This contradicts the assumption of the corollary and hence there is no such $d$. 
From the previous theorem and corollary we infer the following.

**Definition**

Each power series \( \sum_{n=0}^{\infty} c_n(x - a)^n \) has a **radius of convergence** \( R \) (which may be 0, a positive real number, or \( \infty \)) such that

\[
\sum_{n=0}^{\infty} c_n(x - a)^n \begin{cases} 
\text{converges for } |x - a| < R \\
\text{diverges for } |x - a| > R
\end{cases}
\]

There is no general conclusion that can be drawn for \( |x - a| = R \).

For the geometric series \( (a = 0 \text{ and } c_n = 1) \) we know that \( R = 1 \) and that the series diverges for \( x = 1 \) and \( x = -1 \).

For the series \( \sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n}} \) we have seen that \( R = 1 \) with convergence at \( x = -1 \) and divergence at \( x = 1 \).
Let

\[ f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n \]

be a power series with radius of convergence \( R > 0 \).

**Theorem**

*For \(|x - a| < R|:*

- \( f(x) \) is continuous and differentiable

- \( f'(x) = \sum_{n=0}^{\infty} c_n \frac{d}{dx} (x - a)^n = \sum_{n=0}^{\infty} n c_n (x - a)^{n-1} \)

- \( F(x) = \int_{a}^{x} f(t) \, dt = \sum_{n=0}^{\infty} c_n \int_{a}^{x} (t - a)^n \, dt = \sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1} \)
Comments:

- The theorem says that we may interchange summation and differentiation or summation and integration. In other words, we may differentiate or integrate term-by-term.

- It can in fact be shown that we can differentiate and integrate an arbitrary number of times.

- Instead of integrating from $a$ to $x$ one can more generally integrate from $A$ to $B$ provided $A$ and $B$ are inside the interval of convergence: $a - R < A \leq B < a + R$.

- If \( \sum_{n=0}^{\infty} c_n(x - a)^n \) diverges for $x = a - R$ (or $x = a + R$) the integrated series may possibly be convergent at these point. If this is the case then $F(a - R) = \lim_{x \to a-R} F(x)$ with limit taken from above (or $F(a + R) = \lim_{x \to a+R} F(x)$ with limit taken from below).
Example 1 (Binomial theorem)

For $p = 0, 1, 2, \ldots$

$$
\sum_{n=0}^{\infty} \binom{n+p}{p} x^n = \frac{1}{(1 - x)^{p+1}} \quad |x| < 1
$$

**Proof.** This can be proved by induction on $p$.

For $p = 0$ the result is true by the geometric series.

Now assume the binomial theorem is true for $p = 0, 1, \ldots, q$. We will show that this implies it is true for $p = q + 1$.

Differentiating both sides of

$$
\sum_{n=0}^{\infty} \binom{n+q}{q} x^n = \frac{1}{(1 - x)^{q+1}}
$$

yields

$$
\sum_{n=1}^{\infty} n \binom{n+q}{q} x^{n-1} = \frac{(q + 1)}{(1 - x)^{q+2}}.
$$
Shifting $n \to n + 1$ on the left and dividing both sides by $q + 1$ gives

$$\sum_{n=0}^{\infty} \frac{n + 1}{q + 1} \binom{n + q + 1}{q} x^n = \frac{1}{(1 - x)^q + 1}.$$ 

Since

$$\frac{n + 1}{q + 1} \binom{n + q + 1}{q} = \frac{n + 1}{q + 1} \frac{(n + q + 1)!}{q!(n + 1)!} = \frac{(n + q + 1)!}{(q + 1)!n!} = \binom{n + q + 1}{q + 1}$$

we finally obtain

$$\sum_{n=0}^{\infty} \binom{n + q + 1}{q + 1} x^n = \frac{1}{(1 - x)^q + 1}.$$ 

This is the binomial theorem for $p = q + 1$ completing the induction.
Example 2

\[
\log(x + 1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} \quad -1 < x \leq 1
\]

This follows by integrating the geometric series

\[
\sum_{n=0}^{\infty} t^n = \frac{1}{1 - t}
\]

from 0 to \(x\):

\[
\Rightarrow \sum_{n=0}^{\infty} \frac{x^{n+1}}{n + 1} = - \log(1 - x).
\]

Replacing \(x\) by \(-x\) gives the desired result for \(|x| < 1\). But we have already seen that for \(x = 1\) the series is conditionally convergent.

Hence the result is true for all \(-1 < x \leq 1\) and \(\log 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\).
Example 3 (Exponential series)

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad x \in \mathbb{R} \]

By the *ratio test* the series on the right converges for all \( x \in \mathbb{R} \):

\[
\frac{a_{n+1}}{a_n} = \frac{x}{n+1} \to 0 =: r < 1.
\]

Denote the function defined by the series as \( E(x) \). Then \( E(0) = 1 \) and

\[
E'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = E(x).
\]

This first order differential equation plus the initial condition have the unique solution \( E(x) = e^x \).
Note that other properties of $e^x$ also follow from its series representation.

For example:

$$e^{x+y} = \sum_{n=0}^{\infty} \frac{(x + y)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} x^m y^{n-m}$$

(Binomial formula)

$$= \sum_{m=0}^{\infty} \frac{x^m}{m!} \sum_{n=m}^{\infty} \frac{y^{n-m}}{(n - m)!}$$

$$= \sum_{m=0}^{\infty} \frac{x^m}{m!} \sum_{n=0}^{\infty} \frac{y^n}{n!}$$

$$= e^x e^y$$
Example 4 (arctan $x$)

$$\text{arctan } x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad |x| \leq 1$$

By the geometric series

$$\sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1 + x^2}, \quad |x| < 1.$$  

Since this is the derivative of arcan $x$ we may integrate to get

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \text{arctan } x, \quad |x| < 1.$$  

But since the series also converges for $x = 1$ and $x = -1$ by the **Leibniz test**, the above is true for all $|x| \leq 1$. 

Since \( \tan \frac{\pi}{4} = 1 \) we have \( \arctan 1 = \frac{\pi}{4} \).

Hence

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.
\]
An important question is:

How do we find the power series representation of a function $f(x)$?

Let $f(x)$ be continuous at $x = a$ and differentiable arbitrarily often at this point.

Suppose we can write

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n, \quad |x - a| < R \quad (\ast)$$

for some $R > 0$.

The problem now is to determine the coefficients $c_n$ and to then show that (\ast) actually exists.
To determine the \( c_n \) first take \( x = a \) in \((\ast)\). This gives

\[
f(a) = \sum_{n=0}^{\infty} c_n(a - a)^n = c_0.
\]

Next differentiate both sides of \((\ast)\) and then take \( x = a \). This gives

\[
f'(a) = \sum_{n=1}^{\infty} nc_n(a - a)^{n-1} = c_1
\]

Differentiating both sides of \((\ast)\) \( k \) times and then taking \( x = a \) gives

\[
f^{(k)}(a) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} c_n(a - a)^{n-k} = k! c_k.
\]

Therefore

\[
c_n = \frac{f^{(n)}(a)}{n!}.
\]
Definition

The **Taylor series** of \( f(x) \) about \( x = a \) is the series

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n
\]

Of course our hope is that for some \( R > 0 \), and \( |x - a| < R \)

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n.
\]

Definition

The **Taylor series** of \( f(x) \) about \( x = 0 \) is called the **Maclaurin series** of \( f(x) \)
The Maclaurin series of $e^x$ is
\[ \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}. \]

We have seen previously that $e^x$ is equal to its Maclaurin series.

The Maclaurin series of $\log(x + 1)$ is
\[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad -1 < x \leq 1. \]

We have seen previously that $\log(x + 1)$ is equal to its Maclaurin series.
Example

Find the Maclaurin series for \( \sin x \)

Let \( f(x) = \sin x \). Then

\[
  f^{(2n)}(x) = (-1)^n \sin x, \quad f^{(2n+1)}(x) = (-1)^n \cos x.
\]

Thus

\[
  f^{(2n)}(0) = 0, \quad f^{(2n+1)}(0) = (-1)^n
\]

and the Maclaurin series of \( \sin x \) is

\[
  \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} x^{2n+1}
\]

Note that by the ratio test this converges for all \( x \in \mathbb{R} \).
When does the Taylor series of \( f(x) \) converge to \( f(x) \)\

**Definition**

The *Nth degree Taylor polynomial* \( p_N(x) \) of \( f(x) \) about \( x = a \) is

\[
\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!} (x - a)^n
\]

For example, for \( \sin x \) about \( x = 0 \) we find

\[
\begin{align*}
p_0(x) &= 0 \\
p_1(x) &= p_2(x) = x \\
p_3(x) &= p_4(x) = x - \frac{x^3}{3!} \\
p_5(x) &= p_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}.
\end{align*}
\]

Our hope is that \( \lim_{N \to \infty} p_N(x) \to f(x) \).
Taylor’s theorem

Let \( p_N \) be the \( N \)th degree Taylor polynomial of \( f(x) \) about \( x = a \)

If \( f^{(N)}(t) \) is continuous for \( t \in [a, x] \) and \( f^{(N+1)}(t) \) exists for \( t \in (a, x) \) then

\[
f(x) = p_N(x) + \frac{f^{(N+1)}(c)}{(N + 1)!} (x - a)^{N+1}
\]

for some \( c \) between \( a \) and \( x \)

Note: The actual value of \( c \) (which cannot be determined exactly except in trivial cases) depends on \( x \). The larger \(|x - a|\) the larger the interval from which \( c \) may be “chosen”.

To prove that a Taylor series converges to \( f(x) \) we need to show that the error term

\[
R_N(x) := \frac{f^{(N+1)}(c)}{(N + 1)!} (x - a)^{N+1}
\]

vanishes when \( N \to \infty \).
Example 1 (Maclaurin series for $\exp x$)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}$$

We already know this result but let's prove it using Taylor's theorem. The function $e^x$ can be differentiated infinitely often so there is no issue with the existence of $p_N(x)$ and $R_N(x)$ for all $N$. According to the theorem

$$e^x = \sum_{n=0}^{N} \frac{x^n}{n!} + R_N(x)$$

with

$$R_N(x) = \frac{e^c}{(N+1)!} x^{N+1}$$

for some $c$ between $a$ and $x$.

Clearly, whatever our choice for $x$ and whatever the corresponding value of $c$,

$$\lim_{N \to \infty} |R_N| = e^c \lim_{N \to \infty} \frac{|x|^{N+1}}{(N+1)!} = 0$$

by one of the standard limits.
Example 2 (Maclaurin series for $\cos x$)

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad x \in \mathbb{R}$$

We can again prove this using Taylor’s theorem, or observe that the series on the right (which converges absolutely for all $x \in \mathbb{R}$) defines a function $C(x)$ which satisfies

$$C(0) = 1, \quad C'(0) = 0 \quad \text{and} \quad C''(x) = -C(x).$$

This second order ODE plus the initial condition have the unique solution $C(x) = \cos x$. 
Applications

Example 1

Compute the integral \( \int \cosh(\sqrt{x}) \, dx \)

One can use the substitution \( y = \sqrt{x} \) and integration by parts (try!) to find

\[
\int \cosh(\sqrt{x}) \, dx = 2\sqrt{x} \sinh(\sqrt{x}) - 2 \cosh(\sqrt{x}) + C.
\]
Instead we could try to use the Maclaurin series for \( \cosh x \) and \( \sinh x \):

\[
\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \quad \sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \quad x \in \mathbb{R}.
\]

Hence

\[
\cosh(\sqrt{x}) = \sum_{n=0}^{\infty} \frac{x^{n}}{(2n)!}, \quad x \geq 0
\]

and

\[
\int \cosh(\sqrt{x}) \, dx = \sum_{n=0}^{\infty} \int \frac{x^{n}}{(2n)!} \, dx = D + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)(2n)!}.
\]
Since
\[ \frac{1}{n+1} = \frac{2}{2n+1} - \frac{2}{(2n+2)(2n+1)}, \]
we get
\[
\int \cosh(\sqrt{x}) \, dx = D + 2 \sum_{n=0}^{\infty} \frac{x^{n+1}}{(2n+1)!} - 2 \sum_{n=0}^{\infty} \frac{x^{n+1}}{(2n+2)!}
\]
\[
= D + 2 \sum_{n=0}^{\infty} \frac{x^{n+1}}{(2n+1)!} - 2 \sum_{n=1}^{\infty} \frac{x^{n}}{(2n)!}
\]
\[
= C + 2\sqrt{x} \sinh(\sqrt{x}) - 2 \cosh(\sqrt{x})
\]
with \( C = D + 2. \)
In the previous example we can compute the integral without the use of power series. However, often series provide the only method for computing an integral.

**Example 2 (The Gaussian integral)**

Compute \( \int_{0}^{1/2} e^{-x^2} \, dx \)

The function \( e^{ax^2} \) (for \( a \neq 0 \)) does not have an antiderivative expressible in terms of elementary functions.

But we can try power series!
Since
\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]
for all \( x \in \mathbb{R} \) we find
\[
e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}.
\]

Hence
\[
\int_a^b e^{-x^2} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_a^b x^{2n} \, dx
\]
\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)n!} \left[ x^{2n+1} \right]_a^b
\]
\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)n!} (b^{2n+1} - a^{2n+1}).
\]
Now take $a = 0$ and $b = \frac{1}{2}$.

Then

$$
\int_0^{1/2} e^{-x^2} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)n!} \left( \frac{1}{2} \right)^{2n+1}
$$

$$
= \frac{1}{2} - \frac{1}{24} + \frac{1}{320} - \frac{1}{5376} + \cdots.
$$

The series alternates with terms decreasing in size. This implies that the error is less than the first omitted term.

For example, if we only keep the first term then $\int \cdots \approx 0.5$ with an error less than $1/24 = 0.04166\ldots$.

If we keep the first two terms then $\int \cdots \approx 0.54166\ldots$ with an error less than $1/320 = 0.003125$. 
Example 3

Estimate $\sqrt{24}$

Since $\sqrt{25} = 5$ we consider $f(x) = \sqrt{x + 25}$ and compute the Maclaurin series of $f$.

Computing derivatives gives

$$f^{(n)}(x) = \frac{(3/2 - n)_n}{(x + 25)^{n-1/2}},$$

with $(a)_n = a(a + 1) \cdots (a + n - 1)$ a Pochhammer symbol.

For example $(3/2 - 5)_5 = (-7/2)(-5/2)(-3/2)(-1/2)(1/2)$.

The Maclaurin series of $f$ is thus

$$f(x) = \sum_{n=0}^{\infty} \frac{(3/2 - n)_n}{5^{2n-1}} \frac{x^n}{n!}, \quad |x| < 25.$$
To compute $\sqrt{24}$ we take $x = -1$ to get

$$\sqrt{24} = \sum_{n=0}^{\infty} \frac{(3/2 - n)_n (-1)^n}{5^{2n-1} n!}$$

$$= 5 - \frac{1}{10} - \frac{1}{1000} - \frac{1}{50000} + \cdots.$$

Now let us assume that we estimate $\sqrt{24}$ by only keeping the first three terms in the series:

$$\sqrt{24} \approx 4.899.$$

According to Taylor’s theorem the error is

$$|R_2(-1)| = \left| \frac{f^{(3)}(c)}{3!} (-1)^3 \right| = \frac{1}{16(c + 25)^{5/2}}$$

with $c$ between $-1$ and 0.
Therefore

\[ |R_2(-1)| \leq \frac{1}{16(24)^{5/2}} < \frac{1}{16(16)^{5/2}} = \frac{1}{4^7} < 0.0001 \]

and

\[ \sqrt{24} = 4.8990 \pm 0.0001. \]

This is to be compared with the actual value

\[ \sqrt{24} = 4.89897948556635619639456814 \ldots \]
Series solutions to ODEs

Example 1

Solve the differential equation \( y'' + 4y = 0 \) around the point \( x = 0 \)

This is a second order ODE with constant coefficients and the usual methods may be applied to find the general solution

\[
y = A \cos(2x) + B \sin(2x).
\]

Instead we could try to use power series as follows.

We want a solution “around the point \( x = 0 \)” and so we try the power series solution

\[
y = \sum_{n=0}^{\infty} c_n x^n.
\]

Twice differentiating gives

\[
y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n.
\]
Substituting the power series in the ODE yields

\[ \sum_{n=0}^{\infty} (n + 2)(n + 1)c_{n+2}x^n + 4 \sum_{n=0}^{\infty} c_n x^n = 0 \]

or

\[ \sum_{n=0}^{\infty} \left[ (n + 2)(n + 1)c_{n+2} + 4c_n \right] x^n = 0. \]

Since \(1, x, x^2, x^3, \ldots\) are linearly independent the above can only hold if the coefficients of \(x^n\) are 0 for all \(n\).

Hence

\[ (n + 2)(n + 1)c_{n+2} + 4c_n = 0. \]
The recurrence may be split into an “even” and “odd” recurrence:

\[ c_{2n} = \frac{-4}{(2n)(2n - 1)} c_{2n-2} \quad \text{and} \quad c_{2n+1} = \frac{-4}{(2n + 1)(2n)} c_{2n-1}. \]

Each of these can be solved by iteration:

\[ c_{2n} = \frac{-4}{(2n)(2n - 1)} c_{2n-2} = \frac{(-4)^2}{(2n)(2n - 1)(2n - 2)(2n - 3)} c_{2n-4} = \ldots = \frac{(-4)^n}{(2n)!} c_0 \]

and

\[ c_{2n+1} = \frac{(-4)^n}{(2n + 1)!} c_1. \]
Since
\[ y = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n} x^{2n} + \sum_{n=0}^{\infty} c_{2n+1} x^{2n+1} \]
this leads to
\[ y = c_0 \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} + \frac{1}{2} c_1 \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} \]
\[ = c_0 \cos(2x) + \frac{1}{2} c_1 \sin(2x). \]
Identifying \( c_0 = A \) and \( c_1 = 2B \) this is in accordance with our earlier solution.
Example 2
Solve the ODE

\[(1 + x^2)y'' + xy' + y = 0\]

This is a second order linear ODE, but since the coefficients (of \(y\), \(y'\) and \(y''\)) are not constant we have no general method of solution.

Hence we try a series solution around \(x = 0\):

\[y = \sum_{n=0}^{\infty} c_n x^n.\]

Then

\[y' = \sum_{n=1}^{\infty} nc_n x^{n-1}\]

and

\[y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}.\]

Substituting this in the ODE yields

\[(1 + x^2)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0.\]
Hence

\[ \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0. \]

Replacing \( n \to n + 2 \) in the first sum (so that \( x^{n-2} \to x^n \)) this becomes

\[ \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0. \]

In the second and third sums we may replace the lower bound on the sum by 0 since the extra terms are identically zero.

Therefore

\[ \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n + \sum_{n=0}^{\infty} n(n-1)c_n x^n + \sum_{n=0}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0. \]
This may be somewhat simplified to

\[
\sum_{n=0}^{\infty} \left[ (n + 2)(n + 1)c_{n+2} + (n^2 + 1)c_n \right] x^n = 0.
\]

Again we use the linear independence of 1, \(x\), \(x^2\), \(x^3\), \ldots to conclude that

\[
c_{n+2} = -\frac{n^2 + 1}{(n + 2)(n + 1)} c_n.
\]

Since we are solving a second order linear ODE we know that we need to find two linearly independent solutions.

The best way to find these is to take two “linearly independent” initial conditions.

The simplest choice is

1. \(c_0 = 1\) and \(c_1 = 0\) (leading to \(y_1(x)\)),
2. \(c_0 = 0\) and \(c_1 = 1\) (leading to \(y_2(x)\)).
First consider $c_0 = 1$ and $c_1 = 0$.
Then $c_{2n+1} = 0$ and

$$c_2 = -\frac{0^2 + 1}{2 \cdot 1}$$

$$c_4 = -\frac{2^2 + 1}{4 \cdot 3} \quad c_2 = \frac{(2^2 + 1)(0^2 + 1)}{4!}$$

$$c_{2n} = \frac{(-1)^n}{(2n)!} \prod_{i=0}^{n-1} \left[(2i)^2 + 1\right]$$

Therefore

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \prod_{i=0}^{n-1} \left[(2i)^2 + 1\right]$$
Next consider $c_0 = 0$ and $c_1 = 1$. Then $c_{2n} = 0$ and

$$c_3 = -\frac{1^2 + 1}{3 \cdot 2},$$

$$c_5 = -\frac{3^2 + 1}{5 \cdot 4} \quad c_3 = \frac{(3^2 + 1)(1^2 + 1)}{5!},$$

$$c_{2n+1} = \frac{(-1)^n}{(2n + 1)!} \prod_{i=0}^{n-1} [(2i + 1)^2 + 1]$$

Therefore

$$y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n + 1)!} \prod_{i=0}^{n-1} [(2i + 1)^2 + 1]$$
The general solution of the ODE is thus

\[ y(x) = Ay_1(x) + By_2(x) \]

**Note:** The series solution to the above problem is only valid for \( |x| \leq 1 \). For example, \( y_1(x) \) is absolutely convergent for \( |x| < 1 \) and conditionally convergent for \( x = 1 \), but divergent for \( |x| > 1 \).

One may actually show (try!) that

\[ y_1(x) = \cos(\text{arcsinh } x) \quad \text{and} \quad y_2(x) = \sin(\text{arcsinh } x). \]

which is valid for all \( x \in \mathbb{R} \).
In the previous two examples we solved ODEs of the form

\[ P(x)y'' + Q(x)y' + R(x)y = 0 \]  \((*)\)

around \(x = 0\) using series.

For linear ODEs of the form \((*)\) with \(P(0) = 0\) the series method is however doomed to fail. Frobenious developed a modification of the series method for dealing with this problem.
**Note:** We will only demonstrate Frobenius’ method through examples. There is quite a bit of underlying theory, and specific conditions on $P$, $Q$ and $R$ need to be imposed for the method to work. We will not give these conditions — which define what is called a regular singular point — here. In all our examples things are “ok” and we are dealing with regular singular points.

**Example 1**

Solve **Bessel’s equation** (of order $\alpha$)

\[ x^2 y'' + x y' + (x^2 - \alpha^2)y = 0 \quad x > 0 \]

where $\alpha$ is a non-negative integer.
Before showing how to solve Bessel’s equation Dr O would like to remark that this is a perfect problem to practise for the exam. It contains

- an ODE,
- series,
- convergence tests,
- a recurrence or difference equation,
- reduction of order,
- an integrating factor,
- heaps of beautiful maths.
Bessel’s equation is of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

with $P(x) = x^2$, so that $P(0) = 0$.

According to Frobenius we should try a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \quad c_0 \neq 0.$$ 

Then

$$y' = \sum_{n=0}^{\infty} (n + r)c_n x^{n+r-1}$$

and

$$y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1)c_n x^{n+r-2}.$$
Substituting the series for $y$, $y'$ and $y''$ in Bessel's equation and dividing all terms by $x^r$ gives

$$
\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^n + \sum_{n=0}^{\infty} (n+r)c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+2} - \alpha^2 \sum_{n=0}^{\infty} c_n x^n = 0.
$$

Shifting $n \rightarrow n - 2$ in the third sum (so that $x^{n+2} \rightarrow x^n$) yields

$$
\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^n + \sum_{n=0}^{\infty} (n+r)c_n x^n + \sum_{n=2}^{\infty} c_{n-2} x^n - \alpha^2 \sum_{n=0}^{\infty} c_n x^n = 0
$$

or

$$(r^2 - \alpha^2)c_0 + [(r+1)^2 - \alpha^2] c_1 x + \sum_{n=2}^{\infty} \left[ \left( (n+r)^2 - \alpha^2 \right) c_n + c_{n-2} \right] x^n = 0.$$
Since $1, x, x^2, x^3, \ldots$ are linearly independent this implies the equations

\[(r^2 - \alpha^2)c_0 = 0\]
\[\left[(r + 1)^2 - \alpha^2\right]c_1 = 0\]
\[\left[(n + r)^2 - \alpha^2\right]c_n + c_{n-2} = 0 \quad n \geq 2.\]

The first equation is known as indicial equation.

Since $c_0 \neq 0$ the indicial equation fixes $r$.

In this particular example, $r = \alpha$. 
Since $r = \alpha$ the other two equations simplify to

\[
\begin{align*}
[(\alpha + 1)^2 - \alpha^2] c_1 &= 0 \\
[(n + \alpha)^2 - \alpha^2] c_n + c_{n-2} &= 0 \quad n \geq 2.
\end{align*}
\]

For $\alpha$ a nonnegative integer $(\alpha + 1)^2 \neq \alpha^2$ so that $c_1 = 0$.

The last equation yields the recurrence

\[
n(n + 2\alpha)c_n = -c_{n-2} \quad n \geq 2.
\]

Solving the recurrence gives $c_{2n+1} = 0$ (since $c_1 = 0$) and

\[
c_{2n} = \frac{(-1)^n c_0}{2^{2n} n!(\alpha + 1)_n}
\]

(recall that $(a)_n = a(a+1) \cdots (a+n-1)$).
We have thus obtained the series solution

\[ y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n c_0}{2^{2n} n!(\alpha + 1)_n} x^{2n+\alpha}. \]

There is a standard normalisation of this solution given by

\[ c_0 = \frac{1}{2^{\alpha} \alpha!}. \]

Then

\[ y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n + \alpha)!} \left( \frac{x}{2} \right)^{2n+\alpha} =: J_{\alpha}(x) \]

\( J_{\alpha}(x) \) is known as the Bessel function of the first kind and is one of Dr O's favourite functions. It is very important indeed!
A plot of $J_0$, $J_1$ and $J_2$. 
Frobenius’ method has led to only one solution to Bessel’s differential equation. To find the general solution we try reduction of order. That is, we try to find a solution of the form

\[ y(x) = J_\alpha(x)z(x) \]

with \( z(x) \) to be determined.

By the product rule we find

\[ y' = J'_\alpha z + J_\alpha z' \]

and

\[ y'' = J''_\alpha z + 2J'_\alpha z' + J_\alpha z''. \]

The ODE for \( z(x) \) is thus

\[ x^2 \left( J''_\alpha z + 2J'_\alpha z' + J_\alpha z'' \right) + x \left( J'_\alpha z + J_\alpha z' \right) + (x^2 - \alpha^2)J_\alpha z = 0. \]
This may also be written as
\[
\left( x^2 J''_\alpha + xJ'_\alpha + (x^2 - \alpha^2) J_\alpha \right) z + x \left( xJ_\alpha z'' + (2xJ'_\alpha + J_\alpha) z' \right) = 0.
\]

Since $J_\alpha$ satisfies Bessel's equation the first term vanishes and we are left with a first order ODE for $z'$:
\[
xJ_\alpha z'' + (2xJ'_\alpha + J_\alpha) z' = 0
\]

To solve this we first write the ODE in canonical form
\[
z'' + \left( 2 \frac{J'_\alpha}{J_\alpha} + \frac{1}{x} \right) z' = 0.
\]

As integrating factor we thus get
\[
I(x) = \exp \left( \int \left( 2 \frac{J'_\alpha}{J_\alpha} + \frac{1}{x} \right) dx \right)
\]
\[
= \exp \left( 2 \log J_\alpha + \log x \right)
\]
\[
= xJ^2_\alpha.
\]
Hence

\[(z' x J_\alpha^2)' = 0.\]

Integrating once yields

\[z' x J_\alpha^2 = C\]

or

\[z' = \frac{C}{x J_\alpha^2}.\]

Integrating again this becomes

\[z = C \int \frac{dx}{x J_\alpha^2}.\]

Since \(y = J_\alpha z\) we finally obtain the general solution

\[y(x) = C J_\alpha(x) \int \frac{dx}{x J_\alpha^2(x)}.\]
Finally we need to investigate the convergence of the Bessel function $J_\alpha(x)$.

Since

$$J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n + \alpha)!} \left(\frac{x}{2}\right)^{2n+\alpha}$$

contains factorials and powers, we apply the ratio test.

If $J_\alpha(x) = \sum a_n$, then

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{(n + 1)(n + \alpha + 1)} \left| \frac{x}{2} \right|^2 \to 0.$$ 

Hence the Bessel function is absolutely convergent for all $x$. 

Note: When solving the indicial equation $r^2 - \alpha^2 = 0$ we have ignored the solution $r = -\alpha$.
If one repeats the above calculations for $r = -\alpha$ then one obtains the Bessel function $J_{-\alpha}$ instead of $J_\alpha$.
Since $J_{-\alpha} = (-1)^\alpha J_\alpha$ this second solution is not linearly independent and therefore of little interest.
Example 2

Solve \( xy'' - y = 0 \)

Again this is an ODE of the form

\[ P(x)y'' + Q(x)y' + R(x)y = 0 \]

with \( P(0) = 0 \).

According to Frobenius we should try a solution of the form

\[ y = \sum_{n=0}^{\infty} c_n x^{n+r} \quad c_0 \neq 0. \]
Then
\[ y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1)c_n x^{n+r-2} \]
leading to
\[ \sum_{n=0}^{\infty} (n + r)(n + r - 1)c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0. \]

Shifting \( n \to n + 1 \) in the first sum this is
\[ \sum_{n=-1}^{\infty} (n + r + 1)(n + r)c_{n+1} x^n - \sum_{n=0}^{\infty} c_n x^n = 0 \]
or
\[ r(r - 1)c_0 x^{-1} + \sum_{n=0}^{\infty} \left[ (n + r + 1)(n + r)c_{n+1} - c_n \right] x^n = 0. \]
The **indicial equation** is thus

\[ r(r - 1)c_0 = 0 \]

which implies that (since \( c_0 \neq 0 \)) \( r = 0 \) or \( r = 1 \).

Let us first assume that \( r = 0 \).

Then we find the recurrence

\[ (n + 1)nc_{n+1} = c_n. \]

For \( n = 0 \) this says, \( c_0 = 0 \), contradicting the assumption that \( c_0 \neq 0 \).
Next assume that $r = 1$.
Then we find the recurrence

$$(n + 2)(n + 1)c_{n+1} = c_n$$

which is solved by

$$c_n = \frac{1}{(n + 1)!n!}c_0.$$ 

Hence a solution to the ODE is given by

$$y_1(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n + 1)!n!} = \sum_{n=1}^{\infty} \frac{x^n}{n!(n - 1)!}.$$ 

By the ratio test it follows that this solution is absolutely convergent for all $x$. 
To find the general solution we again try reduction of order. That is, we try to find a solution of the form

$$y(x) = y_1(x)z(x)$$

with $z(x)$ to be determined.

By the product rule we find

$$y'' = y_1''z + 2y_1'z' + y_1z''.$$ 

The ODE for $z(x)$ is thus

$$x(y_1''z + 2y_1'z' + y_1z'') - y_1z = 0$$
This may also be written as

\[(x y_1'' - y_1)z + x(2y_1'z' + y_1z'') = 0.\]

Since \(y_1\) satisfies the ODE the first term vanishes and we are left with a first order ODE for \(z'\):

\[2y_1'z' + y_1z'' = 0.\]

To solve this we first write the ODE in canonical form

\[z'' + 2\frac{y_1'}{y_1}z' = 0.\]

As integrating factor we thus get

\[I(x) = \exp\left(\int \left(2\frac{y_1'}{y_1}\right) \, dx\right)\]

\[= \exp\left(2 \log y_1\right)\]

\[= y_1^2.\]
Hence
\[(z'y_1^2)' = 0.\]

Integrating once yields
\[z'y_1^2 = C\]
or
\[z' = \frac{C}{y_1^2}.\]

Integrating again this becomes
\[z = C \int \frac{dx}{y_1^2(x)}.\]

Since \(y = y_1z\) we finally obtain the general solution
\[y(x) = C y_1(x) \int \frac{dx}{y_1^2(x)}.\]
Note that we can expand

$$\frac{1}{y_1^2(x)} = \frac{1}{x^2} \left(1 - x + \frac{7}{12}x^2 - \frac{19}{72}x^3 + \cdots \right).$$

Hence

$$y(x) = C y_1(x) \int \frac{dx}{y_1^2(x)}$$

$$= -C \left(1 + \frac{1}{2}x - \frac{1}{2}x^2 - \frac{11}{72}x^3 + \cdots \right)$$

$$+ x(D - C \log x) \left(1 + \frac{1}{2}x + \frac{1}{12}x^2 + \cdots \right).$$

**Homework:** Use this to compute $xy'' - y$. 
Definitions
Telescoping series
Algebra of series
Convergence tests
Absolute convergence
Leibniz test
Conditional convergence
General ratio test
Power series
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