Motivation

Integration is important for

- finding areas, volumes, lengths of curves, . . . ,
- solving differential equations (DEs).

Our motivation is DEs, the main topic of this course.
Definitions

Definition

Given a function \( f(x) \) we say that \( F(x) \) is its **antiderivative** (sometimes called **primitive**) if \( F'(x) = f(x) \).

It should be clear that the antiderivative of a function is not unique; if \( F(x) \) is an antiderivative of \( f(x) \) then so is \( F(x) + C \) with \( C \) an arbitrary constant.

The antiderivative is often written in the form of an **indefinite integral**, and

\[
\int f(x) \, dx
\]

denotes *any* function whose derivative is \( f(x) \).
Example 1
For $\alpha \neq 0$

\[
\int \exp(\alpha x) \, dx = \frac{\exp(\alpha x)}{\alpha} + C
\]
\[
\int \cos(\alpha x) \, dx = \frac{\sin(\alpha x)}{\alpha} + C
\]
\[
\int \sin(\alpha x) \, dx = -\frac{\cos(\alpha x)}{\alpha} + C
\]

Each of these claims can easily be checked by differentiating the right-hand side.
Example 2

For \( n \in \mathbb{Z} \)

\[
\int x^n \, dx = \begin{cases} 
\frac{x^{n+1}}{n+1} + C & n \neq -1 \\
\log|x| + C & n = -1
\end{cases}
\]

This may again be checked by differentiating the right-hand sides. In particular,

\[
\frac{d}{dx} \log|x| = \begin{cases} 
\frac{d}{dx} \log x = \frac{1}{x}, & x > 0 \\
\frac{d}{dx} \log(-x) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}, & x < 0
\end{cases}
\]

Note: Sometimes \( \log x \) is written as \( \ln x \).
Definite integrals

Let $f(x)$ be a continuous function on $x \in [a, b]$.

Then the definite integral

$$\int_{a}^{b} f(x) \, dx$$

is defined as the limit

$$\int_{a}^{b} f(x) \, dx = \lim_{N \to \infty} \sum_{n=1}^{N} (x_n - x_{n-1}) f(\xi_n),$$

where

$$a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b \quad \text{and} \quad \xi_n \in [x_{n-1}, x_n]$$

such that

$$\max_{1 \leq n \leq N} \{x_n - x_{n-1}\} \to 0 \quad \text{as} \quad N \to \infty.$$
The Riemann sum

\[ \sum_{n=1}^{N} (x_n - x_{n-1}) f(\xi_n) \]

corresponds to the sum of the areas of \( N \) rectangles.
From its definition it is clear that the integral
\[ \int_{a}^{b} f(x) \, dx \]
computes the “area” bounded by the x-axis, the lines \( x = a, x = b \)
and the curve \( y = f(x) \), see here.

Note: If \( f(x) < 0 \) for some (or all) \( x \in (a, b) \) the area may be
negative, hence signed area is perhaps a more precise description.
For example,
\[ \int_{0}^{\pi} \sin x \, dx = 2 \]
but
\[ \int_{0}^{2\pi} \sin x \, dx = 2 - 2 = 0. \]
In the definite integral
\[ \int_{a}^{b} f(x) \, dx \]

- \( x \) is called the **integration variable**,
- \( f(x) \) is called the **integrand**,
- \( dx \) is called the **integration element**,
- \( a \) and \( b \) are the **lower and upper terminal**, respectively,
- and \([a, b]\) (or \((a, b)\)) is called the **interval of integration**.

**Note:** The above integral does not depend on \( x \). For this reason \( x \) is often referred to as a “dummy variable”, and
\[ \int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(t) \, dt. \]
There are two equivalent versions of the FTC:

**Theorem (Fundamental theorem of calculus I)**

*For $f$ a continuous function on $[a, b]$ we have*

$$\frac{d}{dx} \int_a^x f(t) \, dt = f(x)$$

*for $x \in [a, b]$*

**Note:** Compare this with $\frac{d}{dx} \int f(x) \, dx = f(x)$. 
**Theorem (Fundamental theorem of calculus II)**

For $f$ a continuous function on $[a, b]$ and $F$ its antiderivative we have

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

**Note:** Often the above is written as

$$\int_a^b f'(x) \, dx = f(b) - f(a)$$

for $f'$ a continuous function on $[a, b]$.

Even (some) ducks know about the FTC, see here.
The second form of the FTC provides an “easy” method for computing definite integrals. All we need to do is to compute the antiderivative of the integrand in two different points. In practice this may not be easy at all, and in the following we shall discuss a number of methods for dealing with this problem.

Some of the most effective methods of computing definite integrals do in fact circumvent the problem of having to compute an antiderivative. The most powerful such method is based on the residue theorem and allows one to compute integrals like

$$\int_{-\infty}^{\infty} \frac{\cos(tx)}{x^2 + 1} \, dx = \pi e^{-t}, \quad t > 0.$$  

This is outside the scope of 620123. It is however important to realise that some integrals may be explicitly computed even though one can prove that the antiderivative of the integrand cannot be expressed in terms of elementary functions. The most famous example is provided by the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}.$$
Methods of integration

Before you start computing a definite integral always think first.

For example, if $f$ is an odd function, i.e.,

$$f(-x) = -f(x)$$

continuous on $[-a, a]$, then the integral

$$\int_{-a}^{a} f(x) \, dx = 0.$$ 

Note: Be careful to check continuity, because

$$\int_{-a}^{a} x^{-1} \, dx \neq 0.$$ 

(More on this later.)
Example

\[ \int_{-4}^{4} \frac{\sin(x) e^{-x^2}}{x^{24} + 1} \, dx = 0 \]

The function

\[ f(x) = \frac{\sin(x) e^{-x^2}}{x^{24} + 1} \]

is odd, since

\[ f(-x) = \frac{\sin(-x) e^{-(x)^2}}{(-x)^{24} + 1} = -\frac{\sin(x) e^{-x^2}}{x^{24} + 1} = -f(x), \]

and is continuous on \( \mathbb{R} \).
If the integral is simple enough you may immediately recognise the antiderivative of the integrand.

**Example 1**

\[
\int_2^4 \sin x \, dx = \cos 2 - \cos 4
\]

If \( f(x) = \sin x \) then \( F(x) = -\cos x + C \). Hence

\[
\int_2^4 \sin x \, dx = -\left[\cos x\right]_2^4 = \cos 2 - \cos 4.
\]
Example 2

\[
\int_{-e}^{-1} \frac{1}{x} \, dx = -1
\]

If \( f(x) = 1/x \) then \( F(x) = \log|x| + C \). Hence

\[
\int_{-e}^{-1} \frac{1}{x} \, dx = \left[ \log|x| \right]^{-1}_{-e} = \log 1 - \log e = 0 - 1 = -1.
\]
Example 3

\[ \int_0^1 \frac{t \, dt}{t^2 + 1} = \frac{\log 2}{2} \]

The integrand is almost the derivative of \( \log(t^2 + 1) \). Thus

\[ \int_0^1 \frac{t \, dt}{t^2 + 1} = \frac{1}{2} \int_0^1 \frac{2t \, dt}{t^2 + 1} \]

\[ = \frac{1}{2} \int_0^1 \frac{d}{dt} \left( \log(t^2 + 1) \right) \, dt \]

\[ = \frac{1}{2} \left[ \log(t^2 + 1) \right]_0^1 \]

\[ = \frac{\log 2 - \log 1}{2} = \frac{\log 2}{2}. \]

Later you may also try to evaluate the above integral using substitutions.
Example 4

\[ \int_0^1 w(w^2 + 1)^{3/2} \, dw = \frac{2^{5/2} - 1}{5} \]

The integrand is almost the derivative of \((w^2 + 1)^{5/2}\) since

\[ \frac{d}{dw} (w^2 + 1)^{5/2} = \frac{5}{2} (w^2 + 1)^{3/2} \frac{d}{dw} (w^2 + 1) \]
\[ = 5w(w^2 + 1)^{3/2}. \]

Hence

\[ \int_0^1 w(w^2 + 1)^{3/2} \, dw = \frac{1}{5} \left[ (w^2 + 1)^{5/2} \right]_0^1 = \frac{2^{5/2} - 1}{5}. \]

Later you may also try to evaluate the above integral using substitutions.
Note: In each of the four examples one has to check that the integrand is in fact continuous on the interval of integration. Failure to do so may lead to incorrect results.

Incorrect example

\[ \int_{-1}^{e} \frac{1}{x} \, dx = 1 \]

If \( f(x) = 1/x \) then \( F(x) = \log|x| + C \). Hence

\[ \int_{-1}^{e} \frac{1}{x} \, dx = \left[ \log|x| \right]_{-1}^{e} = \log e - \log 1 = 1 - 0 = 1. \]

What is wrong?
Integration by parts is useful if the integrand is of the form \( f'(x)g(x) \). Since

\[
\frac{d}{dx} \left[ f(x)g(x) \right] = f'(x)g(x) + f(x)g'(x)
\]

we get

\[
\int f'(x)g(x) \, dx = f(x)g(x) - \int f(x)g'(x) \, dx
\]

or

\[
\int_{a}^{b} f'(x)g(x) \, dx = \left[ f(x)g(x) \right]_{a}^{b} - \int_{a}^{b} f(x)g'(x) \, dx
\]
Example 1

\[ \int_{0}^{\pi} t \sin t \, dt = \pi \]

The integrand is of the form \( f'(t)g(t) \) with \( f(t) = -\cos t \) and \( g(t) = t \). Hence

\[
\begin{align*}
\int_{0}^{\pi} t \sin t \, dt &= -\left[ t \cos t \right]_{0}^{\pi} - \int_{0}^{\pi} 1 \cdot (-\cos t) \, dt \\
&= -\pi \cos \pi + \int_{0}^{\pi} \cos t \, dt \\
&= \pi + \left[ \sin t \right]_{0}^{\pi} \\
&= \pi.
\end{align*}
\]
Note: Make sure you choose \( f \) and \( g \) correctly.

Example 1 (again)

Compute \( \int_0^\pi t \sin t \, dt \)

If we make the mistake of identifying the integrand as being of the form \( f'(t)g(t) \) with \( f(t) = t^2/2 \) and \( g(t) = \sin t \) we get

\[
\int_0^\pi t \sin t \, dt = \frac{1}{2} \left[ t^2 \sin t \right]_0^\pi - \frac{1}{2} \int_0^\pi t^2 \sin t \, dt \\
= -\frac{1}{2} \int_0^\pi t^2 \sin t \, dt \\
= ?
\]

Note: Combining the above failure with the previous, more successful calculation implies that

\[
\int_0^\pi t^2 \sin t \, dt = -2\pi.
\]
Example 2

Evaluate \( \int \arctan x \, dx \)

The integrand is of the form \( f'(x)g(x) \) with \( f(x) = x \) and \( g(x) = \arctan x \).

Hence

\[
\int \arctan x \, dx = x \arctan x - \int x \cdot \frac{1}{1 + x^2} \, dx
\]

\[
= x \arctan x - \frac{1}{2} \log(1 + x^2) + C
\]
Example 3

\[ \int x^2 e^x \, dx = x^2 e^x - 2(x - 1) e^x + C \]

A double integration by parts is required:

\[
\int x^2 e^x \, dx = \int x^2 (e^x)' \, dx \\
= x^2 e^x - \int (x^2)' e^x \, dx \\
= x^2 e^x - 2 \int x e^x \, dx \\
= x^2 e^x - 2 \int x (e^x)' \, dx \\
= x^2 e^x - 2x e^x + 2 \int (x)' e^x \, dx \\
= x^2 e^x - 2x e^x + 2 \int e^x \, dx \\
= x^2 e^x - 2x e^x + 2 e^x + C.
\]
Example 4

\[
\int e^{3t} \sin(4t) \, dt = \frac{1}{25} e^{3t} \left(3 \sin(4t) - 4 \cos(4t)\right) + C
\]

Method I. Integration by parts:

\[
\int e^{3t} \sin(4t) \, dt = \frac{1}{3} e^{3t} \sin(4t) - \frac{4}{3} \int e^{3t} \cos(4t) \, dt
\]

\[
= \frac{1}{3} e^{3t} \sin(4t) - \frac{4}{9} e^{3t} \cos(4t) - \frac{16}{9} \int e^{3t} \sin(4t) \, dt.
\]

From this is follows that

\[
\frac{25}{9} \int e^{3t} \sin(4t) \, dt = \frac{1}{9} e^{3t} \left(3 \sin(4t) - 4 \cos(4t)\right) + C.
\]
Method II. Use $e^{i\theta} = \cos \theta + i \sin \theta$:

$$\int e^{3t} \sin(4t) \, dt = \operatorname{Im} \left[ \int e^{(3+4i)t} \, dt \right]$$

$$= \operatorname{Im} \left[ \frac{e^{(3+4i)t}}{3 + 4i} \right] + C$$

$$= \operatorname{Im} \left[ \frac{(3 - 4i)e^{(3+4i)t}}{3^2 + 4^2} \right] + C$$

$$= \frac{1}{25} e^{3t} \operatorname{Im} \left[ (3 - 4i)(\cos(4t) + i \sin(4t)) \right] + C$$

$$= \frac{1}{25} e^{3t} \left( 3 \sin(4t) - 4 \cos(4t) \right) + C.$$
When the integrand is a rational function, partial fraction expansions may help.

Example 1

\[
\int_0^1 \frac{dy}{y^2 - 4} = -\frac{1}{4} \log 3
\]

Write

\[
\frac{1}{y^2 - 4} = \frac{1}{(y - 2)(y + 2)} = \frac{A}{y - 2} + \frac{B}{y + 2}. \tag{*}
\]

Multiply both sides of (\(\ast\)) by \(y - 2\) and let \(y \to 2\):

\[
A = \left[ \frac{1}{y + 2} \right]_{y=2} = \frac{1}{4}.
\]

Multiply both sides of (\(\ast\)) by \(y + 2\) and let \(y \to -2\):

\[
B = \left[ \frac{1}{y - 2} \right]_{y=-2} = -\frac{1}{4}.
\]
We have thus found that
\[
\frac{1}{y^2 - 4} = \frac{1}{4} \left( \frac{1}{y - 2} - \frac{1}{y + 2} \right).
\]

Hence
\[
\int_{0}^{1} \frac{dy}{y^2 - 4} = \frac{1}{4} \int_{0}^{1} \left( \frac{1}{y - 2} - \frac{1}{y + 2} \right) dy
\]
\[
= \frac{1}{4} \left[ \log|y - 2| - \log|y + 2| \right]_{0}^{1}
\]
\[
= \frac{1}{4} \left[ \log \frac{|y - 2|}{y + 2} \right]_{0}^{1}
\]
\[
= \frac{1}{4} \left( \log \frac{1}{3} - \log 1 \right)
\]
\[
= -\frac{1}{4} \log 3.
\]
Example 2

\[ \int \frac{2x^3}{(x^2 + 1)^2} \, dx = \log(x^2 + 1) + \frac{1}{x^2 + 1} + C \]

Put

\[ \frac{2x^3}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}. \quad (\ast) \]

Do not try to find \( A, B, C \) and \( D \) by putting the right-hand side over a common denominator. Be clever instead:

- Multiplying \((\ast)\) by \( x \) and letting \( x \to \infty \) gives \( A = 2 \).
- Letting \( x = 0 \) in \((\ast)\) gives \( B + D = 0 \).
- Multiplying \((\ast)\) by \((x^2 + 1)^2\) and letting \( x \to \pm i \) gives \( 2i^3 = Ci + D \) and \(-2i^3 = -Ci + D\). This gives \( D = 0 \) and \( C = -2 \).
- Since \( D = 0 \) we find \( B = 0 \).

The integral now follows in the usual way.
Example 3

\[ \int_2^3 \frac{x^4}{(1-x)^3} \, dx = -\frac{63}{8} - 6 \log 2 \]

We have to be careful because

numerator power ≥ denominator power.

Method I: Use

\[ \frac{x^4}{(1-x)^3} = -x - 3 + \frac{6x^2 - 8x + 3}{(1-x)^3} \]

\[ = -x - 3 + \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} \]

to find \( A = 6, \ B = -4 \) and \( C = 1 \).

Hint: Multiply by \( x \) and let \( x \to \infty \); multiply by \( (1-x)^3 \) and let \( x \to 1 \); set \( x = 0 \).
Method II: According to Newton

\[(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k},\]

where \(\binom{n}{k} = \frac{n!}{k!(n-k)!}\).

If we use this with \(b = 1 - x, a = -1\) and \(n = 4\) we get

\[x^4 = (1 - x - 1)^4 = (1 - x)^4 - 4(1 - x)^3 + 6(1 - x)^2 - 4(1 - x) + 1\]

so that

\[\frac{x^4}{(1 - x)^3} = \frac{(1 - x)^4 - 4(1 - x)^3 + 6(1 - x)^2 - 4(1 - x) + 1}{(1 - x)^3}\]

\[= 1 - x - 4 + \frac{6}{1 - x} - \frac{4}{(1 - x)^2} + \frac{1}{(1 - x)^3}\]

\[= -x - 3 + \frac{6}{1 - x} - \frac{4}{(1 - x)^2} + \frac{1}{(1 - x)^3}.\]
Therefore

\[
\int_2^3 \frac{x^4}{(1-x)^3} \, dx
\]

\[
= \int_2^3 \left( -x - 3 + \frac{6}{1-x} - \frac{4}{(1-x)^2} + \frac{1}{(1-x)^3} \right) \, dx
\]

\[
= \left[ -\frac{1}{2}x^2 - 3x - 6 \log|1-x| - \frac{4}{1-x} + \frac{1}{2(1-x)^2} \right]_2^3
\]

\[
= -\frac{63}{8} - 6 \log 2.
\]
Theorem

If $g'(t)$ is continuous for $t \in [a, b]$ and if $f(x)$ is continuous for the interval of $x$ values generated by $x = g(t)$ for $t \in [a, b]$, then

$$\int_a^b f(g(t))g'(t) \, dt = \int_{g(a)}^{g(b)} f(x) \, dx$$

Proof. Let $F$ be the antiderivative of $f$. Then by FTC II (twice)

$$\int_a^b f(g(t))g'(t) \, dt = \int_a^b F'(g(t))g'(t) \, dt = \int_a^b \frac{d}{dt} \left( F(g(t)) \right) \, dt$$

$$= \left[ F(g(t)) \right]_a^b = F(g(b)) - F(g(a))$$

$$= \left[ F(x) \right]_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} F'(x) \, dx$$

$$= \int_{g(a)}^{g(b)} f(x) \, dx.$$
In practice, we usually start with what corresponds (in our notation) to the $x$ integral and convert it to the (hopefully simpler) $t$ integral, so that the formula becomes

$$\int_{\alpha}^{\beta} f(x) \, dx = \int_{g^{-1}(\alpha)}^{g^{-1}(\beta)} f(g(t)) g'(t) \, dt.$$  

**Note:** In using the result this way, it is essential that the relation between $t$ and $x$ is a one-to-one correspondence or *bijection*. 
For a change of variables $x = g(t)$, there are 4 steps:

1. Check that the function $g$ is monotonic in the interval of interest, so that the $x$-$t$ relation is a bijection.
2. Change the integration element: $dx = g'(t)\, dt$.
3. Replace $x$ everywhere in the integrand by $g(t)$.
4. Change the terminals on the integral via $t = g^{-1}(x)$. 
Example 1

\[
\int_0^{1/2} \frac{dx}{\sqrt{1 - x^2}} = \frac{\pi}{6}
\]

Put \(x = \sin t\). This provides a bijection from the interval \([0, 1/2]\) to \([0, \pi/6]\). Also \(dx = \cos t \, dt\).

Hence

\[
\int_0^{1/2} \frac{dx}{\sqrt{1 - x^2}} = \int_0^{\pi/6} \frac{\cos t \, dt}{\sqrt{1 - \sin^2 t}}
\]

\[
= \int_0^{\pi/6} \frac{\cos t \, dt}{\sqrt{\cos^2 t}}
\]

\[
= \int_0^{\pi/6} 1 \cdot dt
\]

\[
= \frac{\pi}{6}.
\]
Example 2

\[
\int_1^2 \frac{dx}{x\sqrt{1 + x^2}} = \log(1 + \sqrt{2}) - \log\left(\frac{1 + \sqrt{5}}{2}\right)
\]

Put \(x = \sinh t\). This provides a bijection from the interval \([1, 2]\) to \([\log(1 + \sqrt{2}), \log(2 + \sqrt{5})]\). (Use that \(\sinh(\log a) = (a - a^{-1})/2\).) Also, \(dx = \cosh t\, dt\).

Hence

\[
\int_1^2 \frac{dx}{x\sqrt{1 + x^2}} = \int_{\log(1+\sqrt{2})}^{\log(2+\sqrt{5})} \frac{\cosh t\, dt}{\sinh t \cdot \cosh t} = \int_{\log(1+\sqrt{2})}^{\log(2+\sqrt{5})} \frac{dt}{\sinh t}.
\]

Next put \(w = e^t\). This yields a bijection between \([\log(1 + \sqrt{2}), \log(2 + \sqrt{5})]\) and \([1 + \sqrt{2}, 2 + \sqrt{5}]\). Moreover, \(dw = w\, dt\).
Hence

\[
\int_{\log(1+\sqrt{2})}^{\log(2+\sqrt{5})} \frac{dt}{\sinh t} = 2 \int_{1+\sqrt{2}}^{2+\sqrt{5}} \frac{1}{w - w^{-1}} \frac{dw}{w}
\]

\[
= 2 \int_{1+\sqrt{2}}^{2+\sqrt{5}} \frac{dw}{w^2 - 1}
\]

\[
= \int_{1+\sqrt{2}}^{2+\sqrt{5}} \left( \frac{1}{w - 1} - \frac{1}{w + 1} \right) dw
\]

\[
= \left[ \log \left| \frac{w - 1}{w + 1} \right| \right]_{1+\sqrt{2}}^{2+\sqrt{5}}
\]

\[
= \log \left( \frac{1 + \sqrt{5}}{3 + \sqrt{5}} \right) - \log \left( \frac{\sqrt{2}}{2 + \sqrt{2}} \right)
\]

\[
= \log(1 + \sqrt{2}) - \log \left( \frac{1 + \sqrt{5}}{2} \right).
\]
Example 3

Compute the indefinite integral \( \int \frac{dx}{x^2 + 2x + 5} \)

We note that \( x^2 + 2x + 5 = (x + 1)^2 + 4 = 4\left(\left(\frac{x + 1}{2}\right)^2 + 1\right) \). This suggests the substitution \( y = \frac{x + 1}{2} \) (so that \( dx = 2\,dy \)). Therefore

\[
\int \frac{dx}{x^2 + 2x + 5} = \frac{1}{2} \int \frac{dy}{y^2 + 1} = \frac{1}{2} \arctan y + C = \frac{1}{2} \arctan\left(\frac{x + 1}{2}\right) + C.
\]

**Note:** Do not give your answer in terms of \( y \).

Try the same integral using partial fractions and

\[ x^2 + 2x + 5 = (x - 1 - 2i)(x - 1 + 2i). \]
Improper integrals

We will consider two types of improper integrals. In the first type the integrand has one or more singularities in the interval of integration (often at one (or both) of the terminals). In the second type the upper/lower terminal is $\pm\infty$.

All improper integrals are defined by a limiting procedure.

**Example 1**

$$\int_0^1 \frac{dx}{x^{1/3}} = \frac{3}{2}$$

The integrand is not well-defined at the lower terminal $x = 0$. By definition

$$\int_0^1 \frac{dx}{x^{1/3}} := \lim_{a \to 0^+} \int_a^1 \frac{dx}{x^{1/3}}$$

$$= \frac{3}{2} \lim_{a \to 0^+} \left[ x^{2/3} \right]_a^1$$

$$= \frac{3}{2} \lim_{a \to 0^+} (1 - a^{2/3}) = \frac{3}{2}.$$
Example 2

\[
\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi
\]

\[
\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = 2 \int_{0}^{1} \frac{dx}{\sqrt{1-x^2}}
\]

\[
:= 2 \lim_{a \to 1^-} \left[ \arcsin x \right]_{0}^{a}
\]

\[
= 2 \lim_{a \to 1^-} \arcsin a
\]

\[
= \pi.
\]
Example 3

\[ \int_{-1}^{1} \frac{dx}{x} \text{ does not exist} \]

This is an important example. The singularity occurs at \( x = 0 \) and one needs to separately consider \( x > 0 \) and \( x < 0 \). Otherwise an incorrect result arises.

Correct treatment:

\[ \int_{-1}^{1} \frac{dx}{x} = \int_{-1}^{0} \frac{dx}{x} + \int_{0}^{1} \frac{dx}{x} . \]

But

\[ \int_{0}^{1} \frac{dx}{x} := \lim_{a \to 0^+} \int_{a}^{1} \frac{dx}{x} \]

\[ = - \lim_{a \to 0^+} \log a \]

which does not exist. The same applies to the \( \int_{-1}^{0} \) integral.
Incorrect treatment:

\[
\int_{-1}^{1} \frac{dx}{x} = \int_{-1}^{0} \frac{dx}{x} + \int_{0}^{1} \frac{dx}{x} = \lim_{a \to 0^+} \left[ \int_{-1}^{-a} \frac{dx}{x} + \int_{a}^{1} \frac{dx}{x} \right]
\]

\[
= \lim_{a \to 0^+} \left[ -\int_{a}^{1} \frac{dx}{x} + \int_{a}^{1} \frac{dx}{x} \right] = \lim_{a \to 0^+} 0 = 0.
\]

Note: Although \(\int_{-1}^{-a} \frac{dx}{x} + \int_{a}^{1} \frac{dx}{x} = 0\) for all \(a > 0\) the integral \(\int_{-1}^{1} \frac{dx}{x}\) does not exist. We don’t allow the “cancelation of infinities”.
Example 4

\[
\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \pi
\]

\[
\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} := \lim_{a \to \infty} \int_{-a}^{0} \frac{dx}{1 + x^2} + \lim_{b \to \infty} \int_{0}^{b} \frac{dx}{1 + x^2}
\]

\[
= \lim_{a \to \infty} \int_{0}^{a} \frac{dx}{1 + x^2} + \lim_{b \to \infty} \int_{0}^{b} \frac{dx}{1 + x^2}
\]

\[
= 2 \lim_{a \to \infty} \int_{0}^{a} \frac{dx}{1 + x^2}
\]

\[
= 2 \lim_{a \to \infty} \left[ \arctan x \right]_{0}^{a}
\]

\[
= 2 \lim_{a \to \infty} \arctan a
\]

\[
= 2 \cdot \frac{\pi}{2}.
\]
Note: We could of course have started with

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = 2 \int_{0}^{\infty} \frac{dx}{1 + x^2}$$

so that only a single limit is required.
For $x > 0$ the **Gamma function** is defined as

$$
\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt.
$$

**Note:** The above definition in fact makes sense for complex $x$ such that $\text{Re}(x) > 0$.

**Example 5**

For $n$ a nonnegative integer $\Gamma(n + 1) = n!$

Note that for $x \geq 1$

$$
\Gamma(x) := \lim_{b \to \infty} \int_0^b t^{x-1} e^{-t} \, dt
$$

since the integrand is regular at the lower terminal.
Assuming $x \geq 1$ and using integration by parts we have

$$\Gamma(x + 1) = \lim_{b \to \infty} \int_0^b t^x e^{-t} \, dt$$

$$= \lim_{b \to \infty} \left( -\left[ t^x e^{-t} \right]_0^b + x \int_0^b t^{x-1} e^{-t} \, dt \right)$$

$$= \lim_{b \to \infty} \left( -b^x e^{-b} + x \int_0^b t^{x-1} e^{-t} \, dt \right)$$

$$= - \lim_{b \to \infty} b^x e^{-b} + x \lim_{b \to \infty} \int_0^b t^{x-1} e^{-t} \, dt$$

$$= x \Gamma(x).$$

Since $\Gamma(1) = 1$ (check) the statement $\Gamma(n + 1) = n!$ follows by induction on $n$ (check).
More generally we have for \( n \) a nonnegative integer that

\[
\frac{\Gamma(x + n)}{\Gamma(x)} = x(x + 1) \cdots (x + n - 1) =: (x)_n
\]

where \((x)_n\) is a Pochhammer symbol. Since \( \Gamma(1) = 1 \) it follows that \((1)_n = n!\).

**Note:** The gamma function has many nice properties. A famous result is the Euler reflection formula

\[
\Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin(\pi x)}
\]

for \(-1 < x < 1\).

This may in fact be used to extend the \( \Gamma \) function to all \( \mathbb{R} - \{0, -1, \ldots\} \). It may also be used to prove that \( \Gamma(1/2) = \sqrt{\pi} \).
The existence of improper integrals can often be tested quickly using the **comparison test for integrals**. There are many different formulations of this test, depending on the interval of integration and on the signature of the integrand. Here we will give just one form, but you should be able to apply obvious variations of the next result.

**Theorem (Comparison test)**

If \( f(x) \geq g(x) \geq 0 \) for all \( x \in (0, \infty) \) then

\[
\int_0^\infty f(x) \, dx \text{ exists } \Rightarrow \int_0^\infty g(x) \, dx \text{ exists}
\]

\[
\int_0^\infty g(x) \, dx \text{ does not exist } \Rightarrow \int_0^\infty f(x) \, dx \text{ does not exist}
\]

**Proof.** Left as an exercise.

Often, instead of saying “the improper integral exists” we say “the improper integral **converges**”. Similarly, when the limiting process defining the improper integral does not converge we say that “the improper integral **diverges**”.


Example

- Does \( \int_0^\infty \frac{e^{-x}}{1 + x} \, dx \) converge?
- Does \( \int_0^1 \frac{e^{-x}}{\sqrt{x}} \, dx \) converge?

The first integrand is positive for \( x > -1 \), and for all \( b > 0 \)

\[
\int_0^b \frac{e^{-x}}{1 + x} \, dx < \int_0^b e^{-x} \, dx = 1 - e^{-b} \leq 1.
\]

Hence the first integral converges

The second integrand is positive for \( x > 0 \), and for all \( 0 < a < 1 \)

\[
\int_a^1 \frac{e^{-x}}{\sqrt{x}} \, dx \leq \int_a^1 \frac{dx}{\sqrt{x}} = [2\sqrt{x}]_a^1 = 2 - 2\sqrt{a} < 2.
\]

Hence the second integral converges.
What to do in the case of an alternating improper integral

\[ I = \int_{0}^{\infty} f(x) \, dx \]

Here \( f(x) \) is a function that changes sign infinitely often, so that no \( x_0 \) exists such that \( f(x) > 0 \) (or \( f(x) < 0 \)) for \( x > x_0 \)?

**Definition**

The improper integral

\[ \int_{0}^{\infty} f(x) \, dx \]

is called **absolutely convergent** if

\[ \int_{0}^{\infty} |f(x)| \, dx \]

converges

It should be clear that **absolutely convergent** integrals are convergent.
We can now answer the previous question in the case of absolutely convergent integrals.

**Example**

Does \( \int_{0}^{\infty} \frac{e^{-x} \sin x}{1 + x} \) converge?  

Because of the sine-function we are dealing with an alternating integral. Hence we consider

\[
\int_{0}^{\infty} \frac{e^{-x} |\sin x|}{1 + x} \ dx \quad (**) .
\]

Since \( \frac{e^{-x} |\sin x|}{1 + x} \leq \frac{e^{-x}}{1 + x} \) and since

\[
\int_{0}^{\infty} \frac{e^{-x}}{1 + x} \ dx
\]

converges, the integral (**) converges. Therefore (*) converges (absolutely).
**Note:** There are integrals that are convergent but not absolutely convergent. A well-known example is the **Dirichlet integral**

\[
\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2}.
\]
Arc length

If \( x(t) \), \( y(t) \) and \( z(t) \) are continuous functions of a real parameter \( t \) then

\[
\mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}
\]

defines a (continuous) curve in \( \mathbb{R}^3 \) parametrized by \( t \).

If \( x \), \( y \), \( z \) are differentiable functions of \( t \), then that portion of the curve corresponding to the small parameter interval \([t, t + \delta t]\) is well-approximated by a straight line.
The length of this segment of the curve is

\[ |\mathbf{r}(t + \delta t) - \mathbf{r}(t)| = |\mathbf{r}'(t)\delta t + \cdots| = |\mathbf{r}'(t)|\delta t + \cdots, \]

provided that we parametrize the curve so that in performing the calculation, \( \delta t > 0 \).

Thus the length \( L \) of the curve corresponding to the parameter interval \( t_1 \leq t \leq t_2 \) is

\[ L = \int_{t_1}^{t_2} |\mathbf{r}'(t)| \, dt \]

**Note:** Some theory on existence of a well-defined length — based on polygonal arc approximations — is suppressed here.
For a curve in $\mathbb{R}^2$ described in Cartesian coordinates by the equation $y = f(x)$, we can use $x$ as the parametrization parameter (instead of $t$).

Then

$$r(x) = x\mathbf{i} + y\mathbf{j} = x\mathbf{i} + f(x)\mathbf{j}$$

so that

$$r'(x) = \mathbf{i} + f'(x)\mathbf{j} \quad \text{and} \quad |r'(x)| = \sqrt{1 + f'(x)^2}.$$ 

Hence we obtain the formula

$$L = \int_{x_1}^{x_2} \sqrt{1 + f'(x)^2} \, dx = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.$$
Example 1

Find the length of the curve

\[ y = \frac{x^4}{16} + \frac{1}{2x^2} \quad \text{for} \quad 2 \leq x \leq 3 \]

We have \( f(x) = \frac{x^4}{16} + \frac{1}{2x^2} \) and \( f'(x) = \frac{x^3}{4} - \frac{1}{x^3} \).

Therefore

\[ 1 + f'(x)^2 = 1 + \frac{x^6}{16} - \frac{1}{2} + \frac{1}{x^6} = \left( \frac{x^3}{4} + \frac{1}{x^3} \right)^2 \]

and

\[ L = \int_2^3 \left( \frac{x^3}{4} + \frac{1}{x^3} \right) \, dx = \left[ \frac{x^4}{16} - \frac{1}{2x^2} \right]_2^3 = \frac{595}{144}. \]
Example 2

Find the length of a circular arc of angle $\theta$ and radius $r$

We will solve this problem in two different ways. **Parametrically** we have

$$ r(t) = r \cos t \mathbf{i} + r \sin t \mathbf{j} = r(\cos t, \sin t), \quad t \in [0, \theta]. $$

Hence

$$ r'(t) = r(-\sin t, \cos t), \quad |r'(t)| = r $$

and

$$ L = \int_{0}^{\theta} r \, dt = r \theta. $$
Using **Cartesians** we have \( y = f(x) = \sqrt{r^2 - x^2} \).
(For \( x \in [-r, r] \) this describes a semi-circle.)

Hence

\[
\frac{d^2}{dx^2} = \frac{x^2}{r^2 - x^2} \quad \text{and} \quad 1 + \frac{d^2}{dx^2} = \frac{r^2}{r^2 - x^2}.
\]

and

\[
L = \int_r^{r \cos \theta} \frac{r \, dx}{\sqrt{r^2 - x^2}}.
\]

By the substitution \( x = r \cos t \) this again becomes

\[
L = \int_0^\theta r \, dt = r \theta.
\]