This is a summary of my lectures 21-24, equal to lectures 21-24 in the subject guide. Note that §11.1 of Thomas is an excellent source for this material.

**CORRECTION:** In §2.3.2 of Handout 7, the argument $t$ has been left out of the solution of the IVP. The solution should read

$$x = \frac{F_0}{m (\omega_0^2 - \omega^2)} \sin \left( \frac{(\omega - \omega_0)}{2} t \right) \sin \left( \frac{(\omega + \omega_0)}{2} t \right).$$

(Thanks, Richard)

1. **Introduction: Examples of Sequences**

Our intuition is that a sequence, either finite or infinite, is just a list of numbers, one after the other, in a definite order. We'll be more precise below, but we'll start with some famous and not so famous examples.

| 2, 5, 8, 11, 14, 17, 20, 23, . . . | Arithmetic | $a_1 = 2, a_{n+1} = a_n + 3$ |
| 3, 6, 12, 24, 48, 96, 192, 384, . . . | Geometric | $a_1 = 3, g_{n+1} = 2g_n$ |
| 1, 1, 2, 3, 5, 8, 13, 21, . . . | Fibonacci | $F_1 = 1, F_2 = 1, F_{n+1} = F_n + F_{n-1}$ |
| 1, 1, -1, -5, -7, 1, 23, 43, . . . | ??? | $C_n =$??? |
| 5, 3, 21, 2 | ??? | $B_n =$??? |
These examples already raise some interesting issues.

- The first three sequences are defined recursively or inductively, where the next term (i.e. number) in the sequence is determined from the previous ones. Of course, for arithmetic and geometric sequences it is easy to find an explicit expression for the \(n\)’th term of the sequence: for our examples above,

\[
a_n = 2 + 3(n - 1) \quad g_n = 3 \cdot 2^{n-1}
\]

On the other hand, an explicit formula for the Fibonacci sequence is not at all obvious: that is, if we want to calculate say \(F_{1000}\), can we do it without just churning out the Fibonacci numbers until we get to the 1000’th one? Perhaps surprisingly, we can:

\[
F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}
\]

This a wild formula (not least because it uses irrational numbers to solve a problem about natural numbers). It is not hard to use induction to prove the formula actually works, but that hardly ends the mystery: what we’d like to know is where the formula comes from. We’ll actually do this later on (Lectures 25 and 26).

- Of course, the rule (if there is one) for a sequence can be very mysterious. The fourth sequence in our list is hardly obvious, but one method of churning out the numbers is

\[
C_1 = 1, \quad C_2 = 1, \quad C_{n+1} = 2C_n - 3C_{n-1}.
\]

So, we can think of it as similar in construction to the Fibonacci sequence. Again we can ask if there is an explicit formula, and again there is:

\[
C_n = \left(\frac{1 + i\sqrt{2}}{6}\right)(1 - i\sqrt{2})^n + \left(\frac{1 - i\sqrt{2}}{6}\right)(1 + i\sqrt{2})^n
\]

This is similar in spirit to the Fibonacci formula (though now we’re using complex numbers to generate natural numbers!). We’ll derive the formula later (Lectures 25 and 26).

- The last sequence in our list is not obvious either, but can be churned out by the formula

\[
B_n = \frac{n^2 - 2^2}{2^2 \cdot n^2} \quad n = 3, 4, 5, \ldots
\]
This actually came up historically, and is now called the Balmer Sequence.\footnote{Actually, it’s usually called the Balmer Series. However, we want to be very careful and consistent in how we use the words “sequence” and “series”.} Joseph Balmer was a physicist trying to understand the spectrum of the Hydrogen atom, the discrete frequencies of light which it emitted. He was basically confronted with the above sequence (though in decimal form and with a constant factor thrown in!), and in 1884 he guessed the pattern. It was decades later that quantum mechanics was used to explain \textit{why} those precise frequencies occurred.

\section{The Definition of an Infinite Sequence}

It is clear that the “pattern” in a sequence may not be obvious. In fact, it is much more extreme than that, and there is a very important point to be made here. To illustrate, consider the following sequence

\[
1, 2, 4, 8, 16, \ldots
\]

If we were asked the next term, we’d automatically reply “32”. But in fact the sequence need not be geometric. In fact the sequence

\[
(*) \hspace{0.5cm} 1, 2, 4, 8, 16, 31, \ldots
\]

naturally arises. What we do is place \(n\) dots around the perimeter of a circle, and we then join all pairs of dots with straight lines. This cuts the inside of the circle into a number of regions: \((*)\) tells us, for each \(n\), what the maximum number of regions is.
But in fact we can make the sequence whatever we like. For example, we could declare that the next number in the sequence is $\pi$, with the sequence up to that stage being

$$1, 2, 4, 8, 16, \pi .$$

How? Here is one formula which works:

$$Q_n = 1 \cdot \frac{(n - 2)(n - 3)(n - 4)(n - 5)(n - 6)}{(1 - 2)(1 - 3)(1 - 4)(1 - 5)(1 - 6)} + 2 \cdot \frac{(n - 1)(n - 2)(n - 3)(n - 5)(n - 6)}{(2 - 1)(2 - 3)(2 - 4)(2 - 5)(2 - 6)} + 4 \cdot \frac{(n - 1)(n - 2)(n - 3)(n - 5)(n - 6)}{(3 - 1)(3 - 2)(3 - 4)(3 - 5)(3 - 6)} + 8 \cdot \frac{(n - 1)(n - 2)(n - 3)(n - 5)(n - 6)}{(4 - 1)(4 - 2)(4 - 3)(4 - 5)(4 - 6)} + 16 \cdot \frac{(n - 1)(n - 2)(n - 3)(n - 4)(n - 6)}{(5 - 1)(5 - 2)(5 - 3)(5 - 4)(5 - 6)} + \pi \cdot \frac{(n - 1)(n - 2)(n - 3)(n - 4)(n - 5)}{(6 - 1)(6 - 2)(6 - 3)(6 - 4)(6 - 5)} .$$

Of course it’s not a beautiful formula,\(^2\) but it works: plugging $n = 1$ gives you 1, $n = 2$ gives you 2, and so on up to $n = 6$. And, it’s clear that we could do the same given any finite sequence of numbers. The point is:

**MATHEMATICALLY, EVERYTHING IS A PATTERN!**

What this means is that there is something misleading about thinking of sequences as patterns or formulas. Of course, *psychologically*, some patterns are “simpler” or “prettier”. But we don’t want to say something like

$$4, 67.3, -\sqrt{2}, \pi, 45$$

is not a sequence just because we think it’s ugly or unmotivated. So,

**FOR US, EVERY “SEQUENCE” IS A SEQUENCE!**

Again, it is misleading to think of sequences as patterns or formulas. And, with this in mind, we can now say precisely what an infinite sequence is. For a given

\(^2\)Well, it depends upon whom you ask. What we’ve constructed here is what’s called an *interpolating polynomial*. It’s an extremely useful concept in other contexts.
sequence we just have to say what the first term is \((n = 1)\), the second term \((n = 2)\), and so on. But this is exactly the concept of a **function**: 

**DEFINITION** An **infinite sequence** is just a function \(a : \mathbb{N} \to \mathbb{R}\) with domain \(\mathbb{N}\), the natural numbers. We write 

\[ a_n = a(n) \]

for the value of the function at \(n\) (i.e. the \(n\)'th term of the sequence). We write 

\[ \{a_n\}_{n=1}^{\infty} \quad \text{or} \quad \{a_n\} \]

for the whole sequence.

### 3 Infinite Sequences from \(\mathbb{R}\)-Functions

Though we have emphasised that infinite sequences are not in general to be thought of as formulas, nonetheless that is the manner in which we obtain most of the **explicit** sequences with which we deal. (The point is, we’ll be dealing later with sequences which are not explicit). In particular, if we have a function \(f\) with domain \(\mathbb{R}\), then we obtain a sequence simply by restricting the domain of \(f\) to \(\mathbb{N}\):

\[
\begin{cases}
  f : \mathbb{R} \to \mathbb{R} \\
  a_n = f(n)
\end{cases}
\Rightarrow \{a_n\} \text{ is an infinite sequence.}
\]

This need not be formulaic (i.e. \(f\) may be given in any abstract manner), but in practice we often obtain sequences (or think about sequences) in such a formulaic manner. However, it is important to keep in mind that even though an \(\mathbb{R}\)-function determines a specific sequence, it doesn’t work in reverse: an infinite sequence does NOT give rise to a specific \(\mathbb{R}\)-function. We illustrate these issues in the examples below.

#### 3.1 Example

As a simple example, the sequence \(a_n = 2 + 3(n - 1)\) can obviously be obtained from the \(\mathbb{R}\)-function \(f(x) = 2 + 3(x - 1) = 1 + 3x\). Note that this gives us two natural ways to picture the sequence \(\{a_n\}\). First, we can graph \(\{a_n\}\) as we would any other function. So, with the current example, the graph is linear, except the sequence consists of only the dots, not the whole line. Secondly, we can simply draw the output/image of the sequence on a number line.

We tend to use the numberline picture more often, but the function picture (i.e. the graph) can also be very useful.
3.2 Example

Consider the sequence

\[ c_n = (-1)^n. \]

There’s a couple things to say about this sequence. First of all, the numberline picture is very dull, since we keep hitting the same two values over and over. The second thing to note is that it is not automatic to think of \( \{c_n\} \) as derived from an \( \mathbb{R} \)-function: certainly it makes no sense to think of \( c_n \) as coming from \( f(x) = (-1)^x \). On the other hand, it is not hard to come up with the function

\[ f(x) = \cos(\pi x). \]

This again indicates that we have to be careful about thinking of sequences as formulas. Though we’ve found a function which fits the sequence, it’s not by formula-matching, it suggests that there may be more than one way to fill in a sequence. This is true, and it is an extremely important point.
3.3 Example

To make the above issues stark, consider the sequence

\[ s_n = \sin(\pi n) . \]

If we blindly follow the formula, we can think of \( s_n \) as being derived from the function

\[ f(x) = \sin(\pi x) \]

This is perfectly legitimate, but it is definitely not the only choice. In particular, we have \( s_n = 0 \) for every \( n \). So, we can just fill in with the zero function

\[ g(x) = 0 . \]

It is important to recognise that the two choices \( f \) and \( g \), and zillions of others, are equally legitimate.

![Figure 3: Filling in \( s_n = \sin(\pi x) \) with both \( f(x) = \sin(\pi x) \) and \( g(x) = 0 \)](image)

4 Limits of Sequences

We now consider the concept of a sequence \( \{a_n\} \) converging to a limit \( a \). This is of course modeled on our notion of limits of functions: the intuition is simply that if \( n \) is huge then \( a_n \) should be very close to \( a \): and the huger \( n \), the closer we’re guaranteed that \( a_n \) is to \( a \). With \( N \) being the measure of the hugeness of \( n \), and \( \epsilon \) measuring the closeness of \( a_n \) to \( a \), we’re led to the following.
Definition: Limit of a Sequence

We say a sequence \( \{a_n\} \) converges to \( a \in \mathbb{R} \) if the following holds

\[
\text{For every } \epsilon > 0 \text{ there is an } N > 0 \text{ such that } \quad n > N \implies |a_n - a| < \epsilon
\]

In this case we write

\[
\lim_{n \to \infty} a_n = a \quad \text{or} \quad a_n \to a
\]

We now make a number of remarks and consequent definitions.

• We’re all familiar with the joys of \( \epsilon - \delta \) proofs, and limits of sequences can be proved by similarly pleasant techniques. As a very simple example, to show \( \frac{1}{n} \to 0 \), for every \( \epsilon > 0 \) we can choose \( N = \frac{1}{\epsilon} \). However, as we detail in the next section, our focus will be a level above, on using limit theorems to compute specific limits. Of course, to initially prove these theorems requires \( \epsilon - N \) arguments: we’ll not give the arguments, as they’re effectively the same as the arguments for the limits of functions.

• Note the two pictures of a sequence \( \{a_n\} \) converging to \( a \). In either case, the best way to interpret/picture the \( \epsilon \)-conclusion is

\[
|a_n - a| < \epsilon \iff a - \epsilon < a_n < a + \epsilon.
\]

Figure 4: Two pictures of \( a_n \to a \)
• Note that a sequence \( \{a_n\} \) can only have one limit. This follows easily from the definition. More importantly, it fits in with our intuition of a convergent sequence, as a sequence honing in on one specific number \( a \).

• We say that a sequence \( \{a_n\} \) **converges** if it converges to some \( a \in \mathbb{R} \). This may seem a rather silly definition, but there will actually be cases where we know a sequence converges, but we don’t know it’s limit! (See the discussion on monotonic sequences in §7). It also enables us to define directly a **divergent sequence** as a sequence which is not convergent. An obvious example is \( c_n = (-1)^n \). Note, however, that even if it is intuitive that \( \{c_n\} \) diverges, it’s not completely obvious how to prove it: in principle, we have to show that \( c_n \not\rightarrow c \) for any \( c \). We’ll give a technique to handle this using the concept of **subsequences**: see §6.

• We also consider the notion of a sequence with limit \( \pm \infty \): this is handled exactly the same way as it is for \( \mathbb{R} \)-functions. So,

\[
\lim_{n \to \infty} a_n = \infty \quad \text{or} \quad a_n \to \infty
\]

if

\[
\text{For every } M > 0 \text{ there is an } N > 0 \text{ such that } \quad n > N \implies a_n > M
\]

So, \( M \) replaces \( \epsilon \), measuring the hugeness of \( a_n \). We say that \( \{a_n\} \) **diverges** to \( \infty \). Analogously, we define the notion of divergence to \( -\infty \).

Note that with these definitions, a sequence with limit \( \pm \infty \) is still divergent. So, a convergent sequence necessarily converges to a real, finite number. This is in some sense just a convention, and not necessarily the best one. As we shall see, there are ways in which \( \infty \) in these definitions behaves like any other number; and, a sequence with \( \pm \infty \) as a limit behaves in ways like a convergent sequence.

### 5 Techniques for Evaluating Limits

In the context of functions, we have several techniques for the practical evaluation of limits. Analogous theorems hold for limits of sequences, and the techniques are applied similarly. We quickly go through these theorems and techniques below. However, notice a simple result gives us most practical sequence results for free:
R-function Theorem:

If \[ \lim_{x\to\infty} f(x) = L \quad \text{and} \quad a_n = f(n) \]

then \[ \lim_{n\to\infty} a_n = L \quad (L \in \mathbb{R}, \pm\infty). \]

The proof of this theorem follows easily from the definitions. Note that it includes the possibility of the function and the sequence having limit \(\pm\infty\). Note also that this Theorem, handy as it is, doesn’t really free us of the work of proving and the limit theorems for sequences directly: as we have seen, there is no direct correspondence between sequences and \(\mathbb{R}\)-functions, and this must be kept in mind. The second example below illustrates this point.

5.0.1 Example

It is getting ahead of ourselves, but we give a simple but typical example of the use of the \(\mathbb{R}\)-function Theorem. Consider the sequence

\[ a_n = \log \frac{n}{n}. \]

This sequence cries out for L’Hôpital’s Rule (see §5.5). However, we cannot directly apply L’Hôpital to sequences: L’Hôpital involves differentiation, and since sequences are defined only on \(\mathbb{N}\), it makes no sense to differentiate them. However, we can define the function \(f(x) = \log \frac{x}{x}\). Then L’Hôpital easily gives us \(f(x) \to 0\), and thus \(a_n \to 0\) as well.

5.0.2 Example

Consider the sequence

\[ s_n = \sin(\pi n). \]

We can recognise that \(s_n = 0\) for all \(n\), and thus trivially \(s_n \to 0\). On the other hand we can fill in with the function \(f(x) = \sin(\pi x)\). Now \(f(x)\) just oscillates between -1 and 1, and thus the \(\lim_{x\to\infty} f(x)\) doesn’t exist.

This example shows that the \(\mathbb{R}\)-function Theorem is not an “if and only if”: even if the function diverges, the sequence may still have a limit.
5.1 Standard Limits

Using the $\mathbb{R}$-function Theorem, or by direct argument, we have a number of “standard” limits. Later, we use these limits without question. However, some of the limits are less standard than others. We label the more contentious limits, and indicate where we discuss these limits further:

<table>
<thead>
<tr>
<th>Limit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{n^p} \to 0 \quad (p &gt; 0)$</td>
<td>easy direct proof</td>
</tr>
<tr>
<td>$\frac{\log n}{n^p} \to 0 \quad (p &gt; 0)$</td>
<td>easy L’Hôpital</td>
</tr>
<tr>
<td>$r^n \to 0 \quad (</td>
<td>r</td>
</tr>
<tr>
<td>$\frac{n^p}{r^n} \to 0 \quad (</td>
<td>r</td>
</tr>
<tr>
<td>$a^n \to 1 \quad (a &gt; 0)$</td>
<td>see §§5.2-5.3</td>
</tr>
<tr>
<td>$n^{\frac{1}{n^p}} \to 1$</td>
<td>L’Hôpital - see §5.2</td>
</tr>
<tr>
<td>$\frac{a^n}{n!} \to 0$</td>
<td>direct proof or see §5.3</td>
</tr>
<tr>
<td>$\left(1 + \frac{a}{n}\right)^n \to e^a$</td>
<td>L’Hôpital - see §5.2</td>
</tr>
<tr>
<td>$n \sin \left(\frac{1}{n}\right) \to 1$</td>
<td>not obvious - see below</td>
</tr>
</tbody>
</table>

The first limit which needs comment is $\frac{n^p}{r^n} \to 0$, which says that geometric terms are “more powerful” than polynomial terms. To take a specific example, we have

$$\frac{n^{37857}}{2^n} \to 0.$$ 

How do we prove this? Direct proofs are possible, but the easiest approach is L’Hôpital. So, we consider the function

$$f(x) = \frac{x^p}{r^x} = \frac{x^p}{e^{x \log r}} \quad \text{(using } r^x = e^{\log(r^x)})$$

We can then repeatedly use L’Hôpital until $x^p$ disappears or becomes a negative power.
The other limit we consider here is \( n \sin \left( \frac{1}{n} \right) \to 0 \). Setting \( \theta = \frac{1}{n} \), this is really just a version of the “standard” limit

\[
\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.
\]

But how do we know this limit? Notice that L'Hôpital is illegal here (why?).\(^3\) And, high school proofs use the geometry of sine, but what we would like is a purely analytic proof (no pictures!). This isolates the real issue: \textit{analytically, we don’t know what sine is!} We’ll leave the discussion here: we simply want to make the point that some limits (and functions) which we usually take for granted are not nearly as simple as presented.\(^4\)

### 5.2 The Continuity Theorem

If we have a sequence \( a_n \to a \) then we can conclude the limit of the sequence \( f(a_n) \), under suitable hypotheses on \( f \). We give two versions, depending upon whether \( a \) is finite or infinite.

**The Continuity Theorem (finite version):**

If

\[
\begin{aligned}
a_n &\to a \\
f &\text{is continuous at } a
\end{aligned}
\]

then \( f(a_n) \to f(a) \)

\[\blacklozenge\heartsuit\diamondsuit\spadesuit\]

**The Continuity Theorem (infinite version):**

If

\[
\begin{aligned}
a_n &\to \pm \infty \\
\lim_{x \to \pm \infty} f(x) &= L
\end{aligned}
\]

then \( f(a_n) \to L \)

\[\blacklozenge\heartsuit\diamondsuit\spadesuit\]

Both versions can be easily proved from the definitions of limits for sequences and functions. We now give some simple examples of the use of these Theorems.

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\(^3\) The reason is that calculating the derivative of sine uses the very limit we’re attempting to determine.

\(^4\) The same remark can be made for the function \( e^x \), as it is not obvious how to define \( e \). It is standard to define \( e \) as the limit of \((1 + \frac{1}{n})^n\). But if we do that, then it’s not at all obvious why \( \frac{d}{dx}(e^x) = e^x \). In fact, it’s not even obvious what \( e^x \) means, say for \( x \) irrational.
5.2.1 Example

Consider the “standard” limit

\[ \lim_{n \to \infty} n^{\frac{1}{n}} \]

We can rewrite the \( n \)th term as

\[ n^{\frac{1}{n}} = e^{\log \left( n^{\frac{1}{n}} \right)} = e^{\frac{1}{n} \log n}. \]

Thus, we can think of

\[ n^{\frac{1}{n}} = f(a_n) \quad \text{where} \quad \begin{cases} a_n = \frac{\log n}{n} \\ f(x) = e^x \end{cases} \]

Then \( a_n \to 0 \) by L’Hôpital, and \( f(x) \) is continuous at 0. Thus, by the Continuity Theorem (finite version), we conclude

\[ n^{\frac{1}{n}} \to e^0 = 1. \]

Note that a similar (slightly simpler) argument proves that

\[ a^{\frac{1}{n}} \to 1. \]

5.2.2 Example

Consider the “standard” limit

\[ \left( 1 + \frac{a}{n} \right)^n \]

With the same trick, we can write

\[ \left( 1 + \frac{a}{n} \right)^n = f(a_n) \quad \text{where} \quad \begin{cases} a_n = n \log \left( 1 + \frac{a}{n} \right) \\ f(x) = e^x \end{cases} \]

Writing

\[ g(x) = \frac{\log \left( 1 + \frac{a}{x} \right)}{\frac{1}{x}} \]
L'Hôpital (and a bit of calculation) implies
\[\lim_{x \to \infty} g(x) = a,\]
and thus
\[a_n \to a\]
by the \(\mathbb{R}\)-function Theorem. Then, by the Continuity theorem (finite version)
\[(1 + \frac{a}{n})^n \to e^a.\]

5.2.3 Example

Similar to the §5.2.1, consider
\[\lim_{n \to \infty} n \sqrt[\log n]{1}.\]
With the same trick, we have
\[n \sqrt[\log n]{1} = f(a_n) \quad \text{where} \quad \begin{cases} a_n = \frac{\log n}{\sqrt[\log n]{1}} = \sqrt[\log n]{1} \\ f(x) = e^x \end{cases}\]
Now \(a_n \to \infty\), by the Continuity Theorem (infinite version) with \(b_n = \log n\) and \(g(x) = \sqrt{x}\). Then, using the Continuity Theorem (infinite version) again, we have
\[\lim_{n \to \infty} n \sqrt[\log n]{1} = \lim_{x \to \infty} e^x = \infty.\]

5.3 The Sandwich Rule

The Sandwich Rule for sequences takes the same form, and is proved in the same way, as the Sandwich Rule for functions:

The Sandwich Rule:

If \(\begin{cases} a_n \to L \\ a_n \leq b_n \leq c_n \quad \text{then} \quad b_n \to L \\ c_n \to L \end{cases}\) (\(L \in \mathbb{R}, \pm \infty\))
5.3.1 Example

Consider the “standard” limit
\[ \lim_{n \to \infty} a^{\frac{1}{n}}. \]

If \( a \geq 1 \) then
\[ 1 \leq a^{\frac{1}{n}} \leq n^{\frac{1}{n}} \quad \text{(as long as } n \geq a) \]

Thus, whatever \( a \geq 1 \) is, we can eventually (for \( n \geq a \)) apply the Sandwich Rule to conclude
\[ a^{\frac{1}{n}} \to 1. \]

If \( 0 < a \leq 1 \), we can get the same result by setting \( b = \frac{1}{a} \) and applying standard algebra of limits (see the next section).

5.3.2 Example

Consider the “standard” limit
\[ \lim_{n \to \infty} \frac{a^n}{n!}. \]

As we indicated above, this limit can be calculated directly to be 0. We’ll give here a different proof, based on Stirling’s Formula, which gives the growth of \( n! \) in comparison to \( n^n \). The problem sheets (problem 92), gives a simple version of Stirling’s Formula, and gives the comparison
\[ n! \geq \frac{n^n}{e^{n-1}}. \]
This implies

\[ 0 \leq \frac{a^n}{n!} \leq \frac{1}{e \left( \frac{ae}{n} \right)^n}. \]

Thus, once \( n \geq 2ae \), we can apply the Sandwich Rule (with \( c_n = \frac{1}{2n} \)) to conclude

\[ \frac{a^n}{n!} \to 0. \]

5.4 Algebraic Laws and Algebraic Manipulation

The algebraic limit laws for sequences are proved and used exactly as for the corresponding laws for functions:

**Algebraic Limit Laws:**

If

\[
\begin{align*}
& a_n \to a \\
& b_n \to b \\
& c \in \mathbb{R}
\end{align*}
\]

Then

\[
\begin{align*}
& a_n \pm b_n \to a \pm b \\
& c \cdot a_n \to c \cdot a \\
& a_n \cdot b_n \to a \cdot b \\
& \frac{1}{b_n} \to \frac{1}{b} \\
& \frac{a_n}{b_n} \to \frac{a}{b}
\end{align*}
\]

as long as the RHS is well-defined.

Note that when applying Algebra Laws, *if we find a RHS which is NOT well-defined, that does NOT mean that the limit fails to exist*. It simply means our attempt to apply the algebra laws was invalid. All we can conclude is that we have to look more carefully at the limit. We illustrate this with the examples below. The other point to note is that we are permitting \( a, b = \pm \infty \) in the Algebra Laws, with the same proviso that the RHS makes sense. Again, this is illustrated in the examples below.

5.4.1 Example

Consider

\[ \lim_{n \to \infty} \frac{3n^2 + 5n}{2n^2 - 7}. \]
If we apply the Algebra Laws, we have

$$\lim_{n \to \infty} \frac{3n^2 + 5n}{2n^2 - 7} = \frac{\lim_{n \to \infty} (3n^2 + 5n)}{\lim_{n \to \infty} (2n^2 - 7)} = \frac{\infty}{\infty} = ???$$

Since the RHS is undefined, this application of the Algebra Laws is illegal.

Of course, obtaining $\frac{\infty}{\infty}$ indicates that the limit probably can be calculated using L'Hôpital (and it can). Alternatively, we can perform an initial algebraic manipulation, dividing top and bottom by $n^2$:

$$\frac{3n^2 + 5n}{2n^2 - 7} = \frac{3n^2 + 5n}{2n^2 - 7} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \frac{3 + \frac{5}{n}}{2 - \frac{7}{n^2}} - \frac{3}{2},$$

where the last line follows by legal applications of the Algebra Laws.

### 5.4.2 Example

Consider

$$\lim_{n \to \infty} \left( \sqrt{n^2 + 7n} + \sqrt{n^2 + 3} \right).$$

Applying the Algebra Laws (and the Continuity Theorem), we have

$$\sqrt{n^2 + 7n} + \sqrt{n^2 + 3} \to \infty + \infty \to \infty.$$  

Note that there is nothing undefined about $\infty + \infty$.

### 5.4.3 Example

Consider

$$\lim_{n \to \infty} \left( \sqrt{n^2 + 7n} - \sqrt{n^2 + 3} \right).$$

This time, applying the Algebra Laws gives $\infty - \infty$, which is undefined. To avoid this, we first multiply top and bottom by the “conjugate”:

$$\sqrt{n^2 + 7n} - \sqrt{n^2 + 3} = \frac{\sqrt{n^2 + 7n} - \sqrt{n^2 + 3}}{1} \cdot \frac{\sqrt{n^2 + 7n} + \sqrt{n^2 + 3}}{\sqrt{n^2 + 7n} + \sqrt{n^2 + 3}} = \frac{7n - 3}{\sqrt{n^2 + 7n} + \sqrt{n^2 + 3}}.$$  

Now, we can multiply top and bottom by $\frac{1}{n}$. Then, legal application of the Algebra Laws (and Continuity) easily gives

$$\lim_{n \to \infty} \left( \sqrt{n^2 + 7n} - \sqrt{n^2 + 2} \right) = \frac{7}{2}.$$  

$\clubsuit \spadesuit \heartsuit \diamondsuit$
5.5 L’Hôpital’s Rule and Indeterminate Forms

An indeterminate form is any (function or sequence) limit which cannot be evaluated directly by the Algebra Laws. We use expressions such as

\( \frac{0}{0} \) and \( \frac{\infty}{\infty} \)

to indicate the type of indeterminacy. The type indicates in summary why the Algebra Laws fail to work (why, in essence, we can’t simply “plug in” to get the limit). Note that 0 and \( \infty \) in (*) do not refer to numbers but to functions. So, for example

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} \text{ where } f(x), g(x) \to 0.
\]

Of course, the limit here depends upon the specific functions \( f(x) \) and \( g(x) \): that’s exactly the meaning of the label “indeterminate”.

The most powerful tool for evaluating indeterminate forms is:

L’Hôpital’s Rule:

\[
\begin{align*}
\text{If} \quad & \lim_{x \to a} f(x) = 0 \\
& \lim_{x \to a} g(x) = 0 \\
\text{OR} & \\
\lim_{x \to a} f(x) = \infty \\
& \lim_{x \to a} g(x) = \infty
\end{align*}
\]

\[
\text{then } \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \quad \text{as long as the RHS limit exists}
\]

Note that we are including the possibility that \( a = \pm \infty \) (which is of particular interest when studying sequences), as well as the limits being \( \pm \infty \).

Of course, there are other indeterminate forms. Here is a summary, with examples
demonstrating the indeterminacy (with $x \to \infty$):

\[
\begin{array}{ccc}
\infty & \log x & \to 0 \\
\infty & \frac{x}{\log x} & \to \infty \\
0 & \frac{(1/2)^x}{x} & \to \infty \\
0 & \frac{(1/3)^x}{(1/2)^x} & \to 0 \\
0 \cdot \infty & \frac{1}{x} \cdot \log x & \to 0 \\
0 \cdot \infty & \frac{1}{\log x} \cdot x & \to \infty \\
\infty - \infty & 2x - x & \to \infty \\
\infty - \infty & x - 2x & \to -\infty \\
1^\infty & \left(1 + \frac{a}{x}\right)^x & \to e^a \\
\infty^0 & x^{1/x} & \to 1 \\
\infty^0 & x^{\frac{1}{\sqrt{\log x}}} & \to \infty \\
0^0 & \left(\frac{1}{x}\right)^{1/x} & \to 1 \\
0^0 & \left(\frac{1}{x}\right)^{\frac{1}{\sqrt{\log x}}} & \to 0
\end{array}
\]

As we have seen, there are various tricks to evaluating indeterminate forms, and often L’Hôpital is involved. However, we emphasise

\begin{center}
L’Hôpital’s Rule only applies \textit{directly} to forms of type $0^0$ and $\infty^\infty$
\end{center}

Of course it is also important to recognise when there is no indeterminacy, and the following are worth emphasising.

\[
\begin{array}{c}
\infty + \infty = \infty \\
\infty \cdot \infty = \infty \\
1 / \infty = 0 \\
\end{array}
\]

One last form which is worth noting is $\frac{1}{0}$. One is tempted to say that this should equal $\infty$, however it may be either $\pm \infty$, and this uncertainty means it may be neither. For example, the sequence

\[(-1)^n n = \frac{1}{(-1)^n n}\]
does not have either $\infty$ or $-\infty$ as a limit, even though the denominator $\to 0$. However, what can say for sure is:

$$\text{If } a_n \to 0 \text{ then } \frac{1}{a_n} \text{ diverges}$$

6 Subsequences and Non-Existence of Limits

We return to an issue raised earlier, how to show that a given sequence \(\{a_n\}\) does not converge.

6.1 Example

Consider the sequence \(a_n = \frac{(-1)^n n}{n+1}\).

It is “obvious” that \(a_n\) diverges, since it is oscillating between \(-1\) and \(+1\). To nail down this intuition, we introduce the notion of a subsequence. We’ll work through the current example, and give the formal definition of a subsequence below.

Writing the first few entries of the sequence, we label the odd and even entries as follows

\[
\begin{array}{cccccc}
\text{a}_{n_1} & \text{a}_{n_2} & \text{a}_{n_3} \\
-\frac{1}{2} & \frac{2}{3} & -\frac{3}{4} & \frac{4}{5} & -\frac{5}{6} & \frac{6}{7} & \ldots \\
\text{a}_{m_1} & \text{a}_{m_2} & \text{a}_{m_3}
\end{array}
\]

Figure 6: The sequence \(a_n = \frac{(-1)^n n}{n+1}\) and two of its subsequences

What we have done here is to isolate two subsequences: \(\{a_{n_k}\}_{k=1}^\infty\) and \(\{a_{m_k}\}_{k=1}^\infty\).
These subsequences respectively pick out the odd and the even entries from \( \{a_n\} \):

\[
\begin{align*}
    a_{n_1} &= a_1 = -\frac{1}{2} & a_{n_2} &= a_3 = -\frac{3}{4} & a_{n_3} &= a_5 = -\frac{5}{6} & (n_k = 2k - 1) \\
    a_{m_1} &= a_2 = \frac{2}{3} & a_{m_2} &= a_4 = \frac{4}{5} & a_{m_3} &= a_6 = \frac{6}{7} & (m_k = 2k)
\end{align*}
\]

Note that \( k \) (not \( n \) or \( m \)) is the “variable” in the subsequences \( \{a_{n_k}\} \) and \( \{a_{m_k}\} \), and it’s the limit as \( k \to \infty \) which determines the behaviour of these subsequences. In fact, we obviously have

\( \lim_{k \to \infty} a_{n_k} = -1 \quad \lim_{k \to \infty} a_{m_k} = 1 \) (†)

We’ll return to this example in a moment.

### 6.2 Definition of a Subsequence

A subsequence \( \{a_{n_k}\} \) of a sequence \( \{a_n\} \) is simply an infinite choice of entries from \( \{a_n\} \), without going backwards. For example,

\( a_2, a_5, a_3, \ldots \)

is not a subsequence: once we’ve chosen \( a_5 \) we can’t choose \( a_3 \). However, this is the only restriction on our choice. (Note if \( a_6 \) happens to equal \( a_3 \), we are still allowed to pick \( a_6 \) after \( a_5 \): it’s where the elements appear in the sequence, not the actual values which matters).

Technically, a subsequence \( \{a_{n_k}\} \) is determined by a function

\[
n : \mathbb{N} \to \mathbb{N}.
\]

Then \( n(1) = n_1 \) is our first choice, \( n(2) = n_2 \) is our second choice, \( n(k) = n_k \) is our \( k \)’th choice, and so one. For example, in §6.1 our two functions were \( n(k) = 2k - 1 \) and \( m(k) = 2k \). Notice then that the “without going backwards” condition is simply stating that the function \( n \) is strictly increasing:

\[
n_{k+1} > n_k.
\]

Note that there doesn’t necessarily need to be a pattern or formula to our subsequence: that is, the function \( n(k) \) determining the subsequence can be any strictly increasing function.

21
The concept of a subsequence is hugely important in mathematics. Here, we just have a restricted use, to analyse the divergence of sequences. The key to this is

Subsequence Theorem:

\[
\begin{align*}
\text{If } & \quad \left\{ a_n \to a \right\} \text{ is a subsequence then } a_{n_k} \to a
\end{align*}
\]

This is a rather intuitive Theorem (and easy to prove): if the \( a_n \) are all getting close to \( a \) then any selective choices of \( a_n \) are also getting close to \( a \). The value of this Theorem is when we turn it around (in logical jargon, taking the contrapositive):

Subsequence Corollary:

\[
\begin{align*}
\text{If } & \quad \left\{ \begin{array}{l}
    a_{n_k} \to L \\
    a_{m_k} \to M
\end{array} \right. \text{ then } \left\{ a_n \right\} \text{ does not have a limit}
\end{align*}
\]

So, if we have two subsequences with different limits, then the whole sequence cannot have a limit (even \( \pm \infty \)). This is rather obvious from the Theorem: if \( \left\{ a_n \right\} \) did have a limit \( K \), then all subsequences would have this same limit \( K \), and our hypotheses clearly rule this out.

It’s a much deeper result, but in fact having subsequences with different limits is actually the only thing which can go wrong. If a sequence \( \left\{ a_n \right\} \) diverges then either \( a_n \to \pm \infty \), or we can find two subsequences with different limits.

### 6.3 Return to Example 6.1

The Subsequence Corollary shows immediately that the sequence \( a_n = \frac{(-1)^n}{n+1} \) diverges. By (†), we have two subsequences with different limits. That’s it!

### 6.4 Example

We give one more simple example, in order to clarify a potential source of confusion. Consider the sequence

\[ c_n = (-1)^n. \]
If we take the odd and even entries of this sequence, we have the obvious:

\[
\begin{align*}
    c_1 &= c_3 = c_5 = \cdots \to -1 \\
    c_2 &= c_4 = c_6 = \cdots \to 1.
\end{align*}
\]

Thus the whole sequence \( \{c_n\} \) diverges. The only point we want to make is that we’ve chosen valid subsequences here, even though \( c_3 = c_1 \) and so on; we’ve chosen later elements in the sequence, and it simply doesn’t matter if the values are the same (or higher or lower).

7 Monotonic Sequences

\textit{NOTE: THIS SECTION IS WEIRD BUT IMPORTANT}

The notion of a monotonic sequence is simple enough. We define \( \{a_n\} \) to be \textbf{increasing} if, for all \( n \in \mathbb{N} \)

\[ a_{n+1} \geq a_n. \]

Note that the definition is slightly weird, since we don’t demand strict inequality. (So, for example, a constant sequence is “increasing”). Similarly, we define a \textbf{decreasing} sequence, and a sequence is \textbf{monotonic} if it is either increasing or decreasing.

7.1 The Monotonic Sequence “Theorem”

The importance of monotonic sequences is the following, deceptively difficult “theorem”.

\textbf{Monotonic Sequence “Theorem”}:

(a) \hspace{1cm} \text{If } \{a_n\} \text{ is monotonic then } a_n \to a \text{ for some } a (\text{possibly } \pm \infty)\]

(b) In particular

\[ \text{If } \{a_n\} \text{ is monotonic and bounded then } \{a_n\} \text{ converges (i.e. to some } a \in \mathbb{R}) \]
To indicate the weirdness of this “Theorem” (and to justify our scare-quotes), notice that we’re proving that sequence has a limit $a$ without giving any clue what $a$ is: there’s something fundamentally mysterious about such such a proof (and it goes to the heart of the mysteriosuness of the real numbers).

To isolate this mysteriousness, we’ll consider how to go about proving the Theorem. In doing so, it is enough to consider an increasing sequence $\{a_n\}$. Of course, $\{a_n\}$ is then bounded below by $a_1$, and there are then two natural cases to consider.

**Case 1:** $\{a_n\}$ is *not* bounded above.

What this means is that, for any $M$ there is an $a_N$ with

$$a_N > M.$$ 

But then, by the increaingness, the terms $a_n$ beyond $a_N$ will be at least as big: that is,

$$n \geq N \implies a_n \geq a_N > M.$$ 

This says exactly that

$$a_n \to \infty,$$

which fits in with the first conclusion of the Theorem.

**Case 2:** $\{a_n\}$ is bounded above.

In this case, there is an $M$, such that, for all $n$

$$a_n \leq M.$$ 

Now, this is where things get weird. *Intuitively*, since the $a_n$ are monotonic,
and since there’s a bound on how big they can get, the $a_n$ must be levelling off to some “limit” $a$. The question is, how can we prove it? The answer is, it depends who you ask.

This kind of existence result is a fundamental property of the real numbers; another example is the Intermediate Value “Theorem”. But these results don’t come for free. At some point, we have to make an assumption, i.e. choose an axiom, about the real numbers. Then, with this assumption we can prove all the other existence results.

The point is, the Monotonic Sequence “Theorem” is as good as any as choice of fundamental axiom. For instance, assuming MST we can prove IVT. And the converse is also true: assuming IVT we can prove MST. But we can’t prove either “Theorem” without some such fundamental assumption.

7.2 Stupid Example

After that weirdness, we’ll try consider some concrete examples. First, a stupid example, but one which illustrates some important points. Consider the sequence

$$a_n = \frac{n + 1}{n}.$$ 

This sequence is obviously decreasing. (Pedantically, we can calculate $a_{n+1} \leq a_n$, or we can show $f'(x) \leq 0$ for $x > 0$, where $f(x) = \frac{x+1}{x}$. But for this sequence, it’s not worth the bother). As well, the sequence is bounded below by -58: that is, for all $n$,

$$a_n \geq -58.$$

\footnote{Recall that IVT says that if a continuous function $f : [a, b] \to \mathbb{R}$ has $f(a) < 0$ and $f(b) > 0$, then there exists a $c \in (a, b)$ with $f(c) = 0$. Note that IVT gives you no clue how to find $c$.}
Thus MST implies that \( a_n \) converges to some \( a \).

Two obvious points about this example. First of all the lower bound -58 was very arbitrary: it needn’t have anything to do with the eventual limit of the sequence. Any bound will do. The second point is that that MST doesn’t tell us anything about what the limit is. In this case, of course we can show explicitly that \( a_n \to 1 \), but MST gives us no clue to this.

7.3 Not-Stupid Example: Recursively Defined Sequences

For a much less obvious example (and typical of a whole family), consider the sequence

\[
\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \ldots
\]

Of course, the next term in the sequence may be \( \pi \)! However, if we consider the obvious (?) pattern, the sequence is defined inductively/recursively by

\[
\begin{cases}
  a_1 = \sqrt{2} \\
  a_{n+1} = \sqrt{2a_n}
\end{cases}
\]

It is not at all obvious that this sequence converges, and if so to what.

One thing we can say is that IF the sequence converges to some \( a \), THEN by the continuity theorem

\[
\begin{align*}
  a_{n+1} &= \sqrt{2a_n} \\
  \downarrow & \quad \downarrow \\
  a &= \sqrt{2a}.
\end{align*}
\]

Thus \( a^2 = 2a \), and so the only conceivable limits are \( a = 0 \) and \( a = 2 \).

How can we show the sequence does in fact converge? This is where the Monotonic Sequence Theorem comes in. If we can show that \( \{a_n\} \) is increasing and bounded above, then we’re guaranteed a limit: then the Continuity Theorem tells us the limit is 2 (since 0 is obviously ruled out by the increasingness).

**Part 1: \( \{a_n\} \) is bounded above**

Note that any bound will do, but it is more natural (and sometimes more helpful) to prove the natural bound. So, we’ll aim to prove \( a_n \leq 2 \). Note that if \( a_n \leq 2 \) then

\[
a_{n+1} = \sqrt{2a_n} \leq \sqrt{2 \cdot 2} = 2.
\]
But then this is exactly the **inductive step** in a proof by induction. The base case is simply the observation that \( a_1 = \sqrt{2} \leq 2 \). Thus \( a_n \leq 2 \) by induction.

**Part 2: \( \{a_n\} \) is increasing**

There are two methods to prove this. The first is an inductive proof, with the base case simply being the observation that \( a_2 = \sqrt{2\sqrt{2}} \geq \sqrt{2} = a_1 \). For the inductive step, we calculate

\[
\begin{align*}
a_{n+1} &= \sqrt{2a_n} \geq \sqrt{2a_{n-1}} & \text{(inductive hypothesis that } a_n \geq a_{n-1}) \\
&= a_n & \text{(definition of } a_n). 
\end{align*}
\]

Thus, by induction, \( a_{n+1} \geq a_n \) for all \( n \).

As a second proof of increasingness, note that

\[
a_{n+1}^2 = 2a_n \quad \implies \quad a_{n+1} = \left( \frac{2}{a_{n+1}} \right) a_n \geq a_n \quad \text{(since } a_{n+1} \leq 2, \text{ by Part 1)}.
\]

Notice this second proof only works if we prove the “right” bound of 2 in Part 1.

So, we’ve proved \( \{a_n\} \) is increasing (twice) and bounded above, and thus we have a complete proof that \( a_n \to 2 \).

♣♦♥♠

### 7.4 Euler’s Constant*

This example is more involved (means less examinable), but in some sense it’s a better illustration of the Monotonic Sequence Theorem in action. The misguiding feature of the previous sequence is that we eventually determined its limit. Sometimes, we simply can’t do that, and the following is such an instance of that.

What we’re interested in is the sequence \( \{a_n\} \) where

\[
a_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n.
\]

We shall show that this sequence converges:

\[
a_n \to \gamma.
\]

This limit \( \gamma \) is called **Euler’s Constant** (not to be confused with the number \( e \) which is occasionally, and erroneously, called Euler’s Number). \( \gamma \) is extremely important in number theory, but essentially nothing is known about it.

27
The meaning of the sequence \( \{a_n\} \) becomes clearer if we write

\[
\begin{align*}
    b_n &= \log n - \left( \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right) \\
    a_n &= 1 - b_n.
\end{align*}
\]

Figure 9: Obtaining Euler’s Constant \( \gamma \) from the limit of errors in Riemann sums

We can recognise \( b_n \) as pictured, as being the error in a lower Riemann sum for \( f(x) = \frac{1}{x} \) on the interval \([1, n]\): \( b_n \) is exactly the shaded region. This makes it pretty obvious that \( b_n \) is increasing (we’ll prove it without pictures in a moment). So, by MST (first version)

\[ b_n \to \eta. \]

\( \eta \) represents the error in the sum over the whole interval \([1, \infty]\), and then (by Algebra Laws)

\[ \gamma = \lim_{n \to \infty} a_n = 1 - \lim_{n \to \infty} b_n = 1 - \eta. \]

The question is, is \( \eta \) finite or infinite? In fact, we’ll show that \( \{b_n\} \) is bounded above, and thus the second version of MST applies. Thus \( \eta \) is finite, and so \( \gamma \) is finite as well.

Notice that none of this gives us any clue as to what \( \gamma \) is, and we reiterate that basically nothing is known about \( \gamma \).\(^6\)

\(^6\)We can approximate \( \gamma = 0.5772156649 \ldots \). Big whoopdeedoo.
We now fill in the gaps, and show \( \{b_n\} \) is increasing and bounded above. Of course the key behind this is the idea of upper and lower Riemann sums for \( f(x) = \frac{1}{x} \). So, the picture above is basically o.k., though it’s not obvious that \( \{b_n\} \) is bounded above. For both parts, we can use the specific inequalities:

\[
(*) \quad \frac{1}{n+1} \leq \log(n+1) - \log(n) \leq \frac{1}{n}.
\]

To prove \((*)\), note that \( f(x) = \frac{1}{x} \) is decreasing,

\[
n \leq x \leq n+1 \quad \implies \quad \frac{1}{n+1} \leq \frac{1}{x} \leq \frac{1}{n}.
\]

Both inequalities in \((*)\) now come from estimating the integral

\[
\int_{n}^{n+1} \frac{1}{x} \, dx = \log(n+1) - \log(n),
\]

Figure 10: One-rectangle approximation to \( \int_{n}^{n+1} \frac{1}{x} \)
We now have

Part 1: \(\{b_n\}\) is increasing

We have

\[
\begin{align*}
    b_{n+1} - b_n &= \log(n+1) - \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right) - \log n - \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \\
                   &= \log(n+1) - \log n - \frac{1}{n+1} \\
                   &\geq 0,
\end{align*}
\]

where the last line follows from \((\ast)\). Thus \(b_{n+1} \geq b_n\), as desired.

Part 2: \(\{b_n\}\) is bounded above

What we'll actually do here is prove that

\[
(P(n)) \quad b_n \leq 1 - \frac{1}{n}.
\]

The proof is by induction. The base case \(P(1)\) is trivial, since \(b_1 = 0 \leq 0\). For the inductive step, we calculate similarly to the above:

\[
\begin{align*}
    b_{n+1} &= \log(n+1) - \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right) \\
            &= \log n - \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) + \log(n+1) - \log n - \frac{1}{n+1} \\
            &= b_n + \log(n+1) - \log n - \frac{1}{n+1} \\
            &\leq 1 - \frac{1}{n} + \log(n+1) - \log n - \frac{1}{n+1} \quad \text{(inductive hypothesis)} \\
            &= 1 - \frac{1}{n+1} + \left(\log(n+1) - \log n - \frac{1}{n}\right) \\
            &\leq 1 - \frac{1}{n+1} \quad \text{(by \((\ast)\))}
\end{align*}
\]

Thus, by induction \(b_n \leq 1 - \frac{1}{n}\). In particular each \(b_n \leq 1\), and \(\{b_n\}\) is bounded above.

As a final remark, note that we actually proved that \(b_n \to \eta \leq 1\), and thus \(\gamma \geq 0\).
7.5 A Quick Word on Improper Integrals

As we know, an improper integral is evaluated as a limit of Riemann integrals. As a trivial example,

\[
\int_1^\infty \frac{1}{x^2} \, dx = \lim_{N \to \infty} \int_1^N \frac{1}{x^2} \, dx = \lim_{N \to \infty} \left[ -\frac{1}{x} \right]_1^N = \lim_{N \to \infty} \left( 1 - \frac{1}{N} \right) = 1. 
\]

In this case, since the limit exists and is finite, we say the improper integral converges.

On the other hand, we can be given uncomputable integrals, for example

\[
\int_1^\infty \frac{1}{x^2 + \sin^2 x} \, dx. 
\]

What do we do? Well, one thing we can try to do is compare the integrand (i.e. the function we're integrating) to a computable integrand. In this case, we note

\[
\frac{1}{x^2 + \sin^2 x} \leq \frac{1}{x^2}. 
\]

So, from the original calculation, we then declare that \( \int_1^\infty \frac{1}{x^2 + \sin^2 x} \, dx \) converges as well.

That’s all very well, and pretty easy, but what is it we’re really doing here? Think of the upper limit \( N \) defining a function:

\[
f(N) = \int_1^N \frac{1}{x^2 + \sin^2 x} \, dx.
\]

What do we know about this function? The first thing is, since \( \frac{1}{x^2 + \sin^2 x} \geq 0 \), we see that \( f(N) \) is an increasing function of \( N \): integrating over a larger interval gives a larger integral. The second thing is, \( f(N) \) is bounded above: in fact, for any \( N \),

\[
f(N) = \int_1^N \frac{1}{x^2 + \sin^2 x} \, dx \leq \int_1^N \frac{1}{x^2} \, dx \leq \int_1^\infty \frac{1}{x^2} \, dx = 1.
\]

So, what we’re really doing is applying a “Monotonic Function Theorem” to conclude that \( f(N) \) converges to a limit, even we don’t know what that limit is. This is what’s telling us that the improper integral makes sense, that it’s equal to something, even if we can’t explicitly compute the integral. Note also that even if the integral doesn’t
converge it will still make sense, and equal ∞, as long as \( f(N) \) is an increasing function of \( N \) - that is, as long as the function we’re integrating is nonnegative.

As a final remark, we note that Euler’s Constant can be written as a (doubly) improper integral:

\[
\gamma = - \int_0^\infty e^{-x} \log x \, dx.
\]