The University of Melbourne

Department of Mathematics and Statistics

Semester 2 Examination 2003

620-140 Intermediate Mathematics

Examination Duration: Three hours
Reading time allowed: Fifteen minutes

This paper has six pages including this page.

Authorised Materials:
No materials are authorised. Students entering the examination venue with notes or printed material related to the subject, calculators or computers, or mobile telephones, should stand in their places immediately and surrender these to an invigilator before the instruction to commence writing is given. (In some cases the invigilators may tell candidates to place all such materials beneath their seats instead.)

Instructions to Invigilators:
No special materials are to be supplied. This paper may be removed by the students at the conclusion of the examination.

Instructions to Students:
Write all of your solutions in the booklet provided
Number the questions and question parts clearly, and start each question on a new page.
Use the left pages for rough working. Write material that you wish to be marked on the right pages only.
Read each question completely before starting to answer it. In some cases, you may find that calculations that you perform in one part of a question may be useful in other parts.
You should attempt all questions. Marks and part-marks are shown at the beginning of each question.
The total number of marks possible on this paper is 150. Any student who scores 75 marks or more will have passed the part of the grade based on this paper.

Copies of this paper may be held in the Baillieu Library.
Question 1 \([(2 + 3) + 4 + 3 = 12 \text{ marks}]\)

(a) Let

\[ X = \begin{bmatrix} -1 & -2 \\ 7 & -6 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & 1 & 0 \\ -5 & -1 & 2 \end{bmatrix}. \]

(i) Write down the sizes (that is, the dimensions) of the matrices \( X \) and \( Y \).

(ii) Compute \( X^T Y^T \) where “T” indicates matrix transpose.

(b) Let

\[ A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 1 & 1 & 2 \end{bmatrix}. \]

By reducing the augmented matrix \([A \mid I]\) to row-reduced echelon form show that

\[ A^{-1} = \begin{bmatrix} -5 & -4 & 6 \\ 3 & 2 & -3 \\ 1 & 1 & -1 \end{bmatrix}. \]

(c) Use the information in part (b) to solve the system of linear equations

\[ \begin{align*}
-5x - 4y + 6z &= 3 \\
3x + 2y - 3z &= 5 \\
x + y - z &= -7
\end{align*} \]

Question 2 \([4 + 2 + 2 + 4 = 12 \text{ marks}]\)

(a) Calculate the determinant of the matrix

\[ B = \begin{bmatrix} -1 & 1 & 2 & 3 \\ 0 & 2 & 1 & 5 \\ 0 & 3 & -4 & -2 \\ 0 & -1 & 6 & 0 \end{bmatrix}. \]

(b) Is \( B \) invertible?

Give a reason for your answer.

(c) Evaluate \( \det(-2B) \).

(d) If

\[ C = \begin{bmatrix} 12 & 1 & 7 & 10 \\ 0 & -1 & -16 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \]

evaluate \( \det(B^{-1}C) \).
**Question 3 [2 + 4 + 2 + 2 + 3 = 13 marks]**

Let \( \mathbf{u} = (1, 2, 3), \mathbf{v} = (0, -2, 3), \) and \( \mathbf{w} = (-1, -4, 1). \)

(a) Evaluate \( 3\mathbf{u} + 2\mathbf{v} - 5\mathbf{w}. \)

(b) Find the cosine of the angle between \( \mathbf{u} \) and \( \mathbf{v}. \)
    Are \( \mathbf{u} \) and \( \mathbf{v} \) perpendicular? Give a reason for your answer.

(c) Find a unit vector parallel to \( \mathbf{w}. \)

(d) Find a vector that is perpendicular to both \( \mathbf{u} \) and \( \mathbf{w}. \)

(e) Evaluate the volume of the parallelepiped with sides determined by the vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w}. \)

**Question 4 [7 + 6 + 3 = 16 marks]**

(a) Find the equation of the plane containing the line

\[
(x, y, z) = (3, 1, 2) + t(3, 1, -1), \quad t \in \mathbb{R}.
\]

and the point \( P(2, 0, 2). \)
Express your answer in both vector form and Cartesian form.

(b) Find the point where the following two lines intersect.

line 1 : \( \frac{x - 6}{-6} = \frac{y + 1}{2}, \quad z = 7 \)

and

line 2 : \( x = 3 - s \)
\( y = -2 + s \)
\( z = 1 + 2s \)

(c) Does line 2 intersect the plane with equation

\( 3x + y + z = 8? \)

Give an interpretation of your calculations.
Question 5 [7 + 2 + 1 + 2 + 1 = 13 marks]

(a) Show that the set of vectors
\[ V = \{(1, 0, 2), (-5, 6, 0), (-4, 3, -3)\} \]

is linearly dependent.
Hence, express one of the vectors as a linear combination of the other two.

(b) Consider the following:
*The set of vectors V span the plane through the origin given by*
\[ 6x + 5y - 3z = 0. \]

Explain the meaning of this statement.

(c) Assuming the truth of the statement given in part (b), does V form a basis for the plane given in part (b)?
Give a reason for your answer.

(d) Using the results from parts (a)–(c), or otherwise, give a basis for the plane given in part (b).

(e) State the dimension of the subspace that is the plane through the origin given in part (b).

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Question 6 [2 + 4 + 2 + 2 = 10 marks]

Let \( T : \mathbb{R}^4 \to \mathbb{R}^2 \) be the linear transformation given by
\[
T(x, y, z, w) = (x + y - z + w, 2x - 5w),
\]
and \( S : \mathbb{R}^2 \to \mathbb{R}^4 \) be the linear transformation given by
\[
S(x, y) = (y, x + y, x, x - y). 
\]

(a) Evaluate \( T(1, 2, 3, 4) \) and \( S(-2, -3) \).

(b) Show that the matrix \( C \), such that
\[
(T \circ S)(x) = Cx, \quad \text{for } x \in \mathbb{R}^2,
\]
is given by
\[
C = \begin{bmatrix} 1 & 1 \\ -5 & 7 \end{bmatrix}.
\]

(c) Evaluate \( (T \circ S)(-3, 4) \).

(d) Is the composite linear transformation \( S \circ T \) defined?
Explain your answer.

Please turn over.
Question 7 [(2 + 4 + 3) + (2 + 3 + 3) = 17 marks]

(a) (i) Verify that the point \( \left( \frac{\pi}{2}, 0 \right) \) lies on the implicitly defined curve given by
\[ y^2 = \cos(x - y). \]

(ii) Show that
\[ \frac{dy}{dx} = \frac{\sin(x - y)}{\sin(x - y) - 2y}. \]

(iii) Find the equation of the tangent to the curve at the point \( \left( \frac{\pi}{2}, 0 \right) \).

(b) (i) Write down \( \cosh x \) and \( \sinh x \) in terms of exponentials.

(ii) Prove that
\[ \cosh 2x = \cosh^2 x + \sinh^2 x. \]

(iii) By differentiating both sides of the expression in part (ii) obtain an expression for \( \sinh 2x \) in terms of \( \cosh x \) and \( \sinh x \).

Question 8 [(4 + 5) + (2 + 2 + 4 + 2) = 19 marks]

(a) (i) Show that the function
\[ f(x) = e^{3x-1} \]
is a one-to-one function.
State the domain and range of \( f(x) \).

(ii) Find the inverse function \( f^{-1}(x) \) of the function in part (i).
State the domain and range of \( f^{-1}(x) \).

(b) (i) Given that \( \cosh^2 x - \sinh^2 x = 1 \), show that \( 1 - \tanh^2 x = \text{sech}^2 x \).

(ii) Show that the derivative of \( y = \tanh x \) is \( \text{sech}^2 x \).

(iii) Show that the derivative of \( y = \text{arctanh} \, x \) is
\[ \frac{dy}{dx} = \frac{1}{1 - x^2}, \quad -1 < x < 1. \]

(iv) Find \( g'(x) \) when
\[ g(x) = \text{arctanh} \, (3x^2). \]
Question 9 \([2 + 4 + 1 + 1 + 3 + 4 = 15\text{ marks}]\)

Consider the function of two variables

\[z = f(x, y) = 3x^3 - 5y^2 + 6xy - x + 7y.\]

(a) Calculate \(f(1, 2)\).

(b) Evaluate \(f_x(1, 2)\) and \(f_y(1, 2)\).

(c) Find \(\nabla f(1, 2)\).

(d) State the direction of steepest descent when \((x, y) = (1, 2)\).

(e) Calculate the directional derivative of \(z = f(x, y)\) when \((x, y) = (1, 2)\) in the direction of \(i + j\).

(f) Find the equation of the tangent plane to \(z = f(x, y)\) when \((x, y) = (1, 2)\).

You may give your answer in either vector form or Cartesian form.

Question 10 \([(4 + 4) + 6 + 4 + 5 = 23\text{ marks}]\)

(a) (i) Express \(z = 2 - 2i\) and \(w = 3 + 3\sqrt{3}i\) in polar form.

(ii) Use part (i) to express \(z^4w\) in Cartesian form.

(b) Evaluate 

\[\frac{d^8}{dx^8}(e^{3x}\sin 3x).\]

(c) Using the identity \((e^{i\theta})^n = e^{i n\theta}\), find an expression for \(\cos 3\theta\) in terms of \(\cos \theta\).

(d) Find all complex fourth roots of \(1 - i\) by solving the equation 

\[z^4 = 1 - i.\]

Express your answers in the form \(re^{i\theta}\) where \(r\) and \(\theta\) are real numbers.

TOTAL MARKS POSSIBLE ON THIS PAPER: 150 MARKS

END OF PAPER
1. (a) (i) Size of $X$ is $3 \times 2$, size of $Y$ is $2 \times 3$;

(ii) $X^T Y^T = \begin{bmatrix} 7 & -2 \\ -6 & 22 \end{bmatrix}$;

(b) $A^{-1} = \begin{bmatrix} -5 & -4 & 6 \\ 2 & 2 & -3 \\ 1 & 1 & -1 \end{bmatrix}$;

(c) $(x, y, z) = (13, -26, -6)$.

2. (a) $\det B = -96$;

(b) Yes, the determinant is nonzero;

(c) $\det(-2B) = -1536$;

(d) $\det C = -\frac{1}{2}$.

3. (a) $(8, 22, 10)$;

(b) $\cos \theta = \frac{5}{\sqrt{182}}$. No, $\mathbf{u} \cdot \mathbf{v} \neq 0$;

(c) $(-\frac{1}{\sqrt{18}}, -\frac{4}{\sqrt{18}}, \frac{1}{\sqrt{18}})$;

(d) $\mathbf{u} \times \mathbf{v} = (14, -4, -2)$;

(e) Volume $= 2$ units$^3$.

4. (a) Vector form: $(x, y, z) = (3, 1, 2) + s(3, 1, -1) + t(1, 1, 0)$, $s, t \in \mathbb{R}$. Cartesian form: $x - y + 2z = 6$;

(b) $(0, 1, 7)$;

(c) Yes, Line 2 lies within the plane.

5. (a) Working required. $(-5, 6, 0) = 3(1, 0, 2) + 2(-4, 3, -3)$;

(b) Every vector that lies in the plane $6x + 5y - 3z = 0$ can be expressed as a linear combination of the vectors in $V$;

(c) No, $V$ is not linearly independent;

(d) $\{(1, 0, 2), (-5, 6, 0)\}$;

(e) $\dim = 2$.

6. (a) $T(1, 2, 3, 4) = (4, -18)$, $S(-2, -3) = (-3, -5, -2, 1)$;

(b) Working required;

(c) $(T \circ S)(-3, 4) = (1, 43)$;

(d) Yes, having $S : \mathbb{R}^2 \to \mathbb{R}^4$ and $T : \mathbb{R}^4 \to \mathbb{R}^2$ makes $S \circ T : \mathbb{R}^4 \to \mathbb{R}^4$ a well defined linear transformation.

7. (a) (i) Working required;

(ii) Working required;

(iii) $y = x - \frac{\pi}{2}$.

(b) (i) See notes;

(ii) Proof required;

(iii) Proof required.
8. (a) (i) Working required. Domain is \( \mathbb{R} \), range is \((0, \infty)\);
   (ii) \( f^{-1}(x) = \frac{\log x + 1}{3} \). Domain is \((0, \infty)\), range is \( \mathbb{R} \).

(b) (i) Proof required;
   (ii) Proof required;
   (iii) Proof required;
   (iv) \( g'(x) = \frac{6x}{1 - 9x^2} \).

9. (a) \( f(1, 2) = 8 \);

(b) \( f_x(1, 2) = 20, f_y(1, 2) = -7 \);

(c) \( \nabla f(1, 2) = (20, -7) \);

(d) \((-20, 7)\);

(e) \( \hat{u} = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \ D_{\hat{u}} f(1, 2) = \frac{13}{\sqrt{2}} \)

(f) Vector form: \( (x, y, z) = (1, 2, 8) + s(1, 0, 20) + t(0, 1, -7), s, t \in \mathbb{R} \). Cartesian form: \( 20x - 7y - z = -2 \).

10. (a) (i) \( z = \sqrt{8}e^{-\frac{\pi}{4}i}, w = 6e^{\frac{\pi}{2}i} \);
    (ii) \( z^4w = -192 + 192\sqrt{3}i \).

(b) \( 18^4e^{3x} \sin 3x \);

(c) \( \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta \);

(d) \( z = 2^\frac{1}{2}e^{-\frac{\pi}{2}i}, 2^\frac{1}{2}e^{\frac{\pi}{2}i}, 2^\frac{1}{2}e^{\frac{3\pi}{2}i}, 2^\frac{1}{2}e^{\frac{5\pi}{2}i} \).