THE UNIVERSITY OF MELBOURNE
DEPARTMENT OF MATHEMATICS AND STATISTICS
SEMESTER 2 EXAMINATION, 2005
620-140 INTERMEDIATE MATHEMATICS

Exam duration — Three hours
Reading time — 15 minutes

This paper has 6 pages, including this cover sheet.

Examination Papers with Common Content:

This paper contains some questions in common with the subject 620–141 Mathematics A which examination is being held at the same time.

Instructions to Invigilators:

Initially, students are to receive a 14 page script book.

Authorized Materials:

No calculators or computers are permitted.

No written or printed material may be brought into the examination room.

Instructions to Students:

Write all of your solutions in the booklet(s) provided.

There are 20 questions on this examination paper.

You should attempt to answer all questions.

Each question is worth SIX marks.

The total number of marks for the examination paper is 120.

This paper may be reproduced and lodged with the Baillieu Library.
1. Let

\[ A = \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 \\ 1 & -4 \\ -1 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 0 & -2 \end{bmatrix}. \]

(a) Find the matrix \( C^T B \).
(b) Verify that the matrix \( A \) satisfies \( A^2 = I - A \).
(c) Use the result of part (b) above to find the inverse of \( A \).

[6 marks]

2. Let

\[ A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 5 & -1 & 6 \end{bmatrix}. \]

(a) Use row operations to find the inverse of \( A \).
(b) Use your answer to part (a) above to solve the following system of linear equations:

\begin{align*}
    x - 2y + z &= 1 \\
    2x - 3y + 2z &= 2 \\
    5x - y + 6z &= 4.
\end{align*}

[6 marks]

3. Let

\[ D = \begin{bmatrix} k & k & 2 \\ -1 & 3 & -k \\ 1 & 0 & 1 \end{bmatrix}, \]

where \( k \) is a real number.

(a) Calculate the determinant of \( D \) in the case \( k = 2 \).
(b) Calculate the determinant of \( D \) for any \( k \).
(c) State the values of \( k \) for which the matrix \( D \) is invertible, but DO NOT attempt to find the inverse.

[6 marks]

4. Let \( M \) and \( N \) be two \( 3 \times 3 \) matrices such that \( \det M = 2 \) and \( \det N = -3 \).

(a) Find \( \det(MN) \).
(b) Find \( \det(NTMT) \).
(c) Find \( \det(M^{-1}) \).
(d) Find \( \det(2N) \).

[6 marks]
5. Consider the vectors $\mathbf{u} = (3, -1, 2)$, $\mathbf{v} = (-1, 4, 5)$ and $\mathbf{w} = (1, 2, -1)$.

(a) Evaluate $-2\mathbf{u} + 3\mathbf{v} - \mathbf{w}$.

(b) Find a unit vector that is parallel to $\mathbf{w}$.

(c) Evaluate $\mathbf{u} \cdot \mathbf{v}$.

(d) Is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$ acute or obtuse? Give a reason for your answer.

(e) Evaluate $\mathbf{u} \times \mathbf{v}$.

[6 marks]

6. Consider the points $A(2, -1, 0)$, $B(-1, 1, 5)$, $C(4, -7, -4)$ and $D(1, 2, 2)$.

(a) Write down the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$.

(b) Find the volume of the parallelepiped spanned by the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$.

(c) Explain why the points $A$, $B$, $C$ and $D$ are coplanar.

[6 marks]

7. A plane $P_1$ passes through the point $(3, 2, 2)$ and has normal vector $\mathbf{n} = (1, -1, -1)$. A second plane $P_2$ passes through the three points $(1, 1, 2)$, $(2, 4, 7)$ and $(2, 5, 10)$.

(a) Find the cartesian equation of the plane $P_1$.

(b) Find the cartesian equation of the plane $P_2$.

(c) Find the line of intersection of the two planes $P_1$ and $P_2$.

[6 marks]

8. A line $L_1$ passes through the two points $(1, 2, 3)$ and $(4, 6, 8)$. A second line $L_2$ has cartesian equations $\frac{x}{2} = \frac{y + 1}{3} = \frac{z + 4}{4}$.

(a) Find the cartesian equations for the line $L_1$.

(b) Find the shortest distance between the two lines $L_1$ and $L_2$.

[6 marks]

9. Consider the set of vectors $V = \{(1, -1, 2), (2, 0, 1), (-1, -5, 7)\}$.

(a) Show that $V$ is a set of linearly dependent vectors.

(b) Express one of the vectors in $V$ as a linear combination of the other two.

[6 marks]
10. Consider the linear transformation

\[ T : \mathbb{R}^3 \to \mathbb{R}^3; \quad T(x, y, z) = (y + 3z, x - z, 2x). \]

(a) Show that \( T(\lambda u) = \lambda T(u) \), where \( u = (x, y, z) \in \mathbb{R}^3 \) and \( \lambda \in \mathbb{R} \).

(b) Find a matrix \( A \) such that \( T(u) = Au \).

(c) Does the linear transformation \( T \) have an inverse? Give a reason for your answer.

[6 marks]

11. Consider

\[ R_\theta = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}, \]

the matrix for a reflection in the line through the origin that makes an angle of \( \theta \) radians with the positive \( x \)-axis.

(a) Write down the matrix for a reflection in the line \( y = x \).

(b) Explain why

\[ D = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \]

is the matrix for a dilation in the plane by a factor of 3.

(c) Write down a matrix that represents a reflection in the line \( y = x \) followed by a dilation by a factor of 3.

(d) Write down a matrix that inverts the geometric transformation in part (c) above.

[6 marks]

12. Consider the curve with equation \( y^2 + x^2 \cos y + e^{-x} = 5 \).

(a) Verify that the point \((0, 2)\) lies on the curve.

(b) Show that

\[ \frac{dy}{dx} = \frac{e^{-x} - 2x \cos y}{2y - x^2 \sin y}, \]

(c) Find the equation of the line which is tangent to the curve at the point \((0, 2)\).

[6 marks]
13. (a) Write down the definitions of $\cosh x$ and $\sinh x$ in terms of the exponential function.
(b) Evaluate $\cosh(\log_e 2)$.
(c) Find
\[
\frac{d}{dx} \cosh(\sqrt{x^2 + 1}).
\]
[6 marks]

14. (a) Evaluate $\arcsin \left( \frac{1}{\sqrt{2}} \right)$.
(b) Find an algebraic expression for $\sin(\arccos x)$ in terms of $x$.
(c) Show that
\[
\frac{d}{dx} \arcsinh x = \frac{1}{\sqrt{1 + x^2}}.
\]
[6 marks]

15. Consider the function of two variables, $f(x, y) = x^2 \cos(3y)$.
(a) Write down the first partial derivatives of $f$, namely $f_x$ and $f_y$.
(b) Suppose that $x = 1 + 3t$ and $y = \frac{\pi}{4} - 5t$. Use the chain rule to calculate $\frac{df}{dt}$ at $t = 0$.
[6 marks]

16. Consider the function $f(x, y) = x^3 - y^3 - 3x^2y + 4xy^2$.
(a) Calculate the gradient $\nabla f$ at the point $(1, 2)$.
(b) Use part (a) to find the slope of the tangent line to the curve
\[
x^3 - y^3 - 3x^2y + 4xy^2 = 3
\]
at the point $(1, 2)$.
(c) Calculate the rate of change of $f$, at the point $(1, 2)$, in the direction of the vector $(3, 5)$.
[6 marks]
17. Find the equation of the tangent plane to the surface \( z = x + \frac{1}{4}y^4e^{2x} \) at the point \((0, 2, 4)\). \[6 \text{ marks}\]

18. (a) Express the complex number \((1 + i)(-2 - 3i)\) in cartesian form.
(b) Find the real and imaginary parts of the complex number \(-2i(3 - i)\).
(c) Evaluate \[\left| \frac{2i(1 - 3i)(5 + 2i)}{4(3 + i)(-5 - 2i)} \right| \).
(d) Write the complex number \(2\sqrt{3} + 2i\) in polar form. \[6 \text{ marks}\]

19. (a) Show that \(\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}\).
(b) Use the complex exponential to find \[\int e^t \sin(2t) dt.\] \[6 \text{ marks}\]

20. Find, in cartesian form, all solutions \(z \in \mathbb{C}\) to each of the following equations:
   (a) \(z^2 + 2z + 5 = 0\).
   (b) \(z^4 = -81\). \[6 \text{ marks}\]

END OF THE EXAMINATION PAPER