Department of Mathematics and Statistics
620-233: Vector Analysis (Advanced) 2008
Assignment 5

This assignment should be returned by 12 noon on Monday May 26.
You should give complete explanations in all questions.

1. Evaluate the following integrals:
   (a) $\int_{\mathbf{c}} f\,ds$ where $f(x,y,z) = xz + y + yz$, $\mathbf{c}(t) = (t, t^2, \frac{2}{3}t^3)$, $0 \le t \le 1$.
   (b) $\int_{\mathbf{c}} xy\,dx + x\,dy$, $\mathbf{c}(t) = (\cos t, \sin t)$, $0 \le t \le \pi$.
   (c) $\int_C (z^3 + 2xy^2)\,dx + 2x^2y\,dy + 3xz^2\,dz$ where C is the rectangle with vertices $(-1,-1,-1)$, $(1,-1,1)$, $(1,1,1)$, and $(-1,1,-1)$ (oriented in that order).

2. Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = a^2$ lying between the planes $z = b$ and $z = c$, where $-a \le b < c \le a$. Verify that this is the same as the surface area of the cylinder $x^2 + y^2 = a^2$ lying between these planes. (You may write down the area of the cylinder without any calculation.)

3. Find the integral $\iint_\Sigma \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z) = (x, y, 2z - x - y)$ and Σ is the surface $x^2 + y^2 = z^2$, $1 \le z \le 4$ with outward pointing normal.

4. Let S be the surface parametrized by $\mathbf{\Phi}(r,\theta) = (r\cos\theta, r\sin\theta, \cos(4\pi r))$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$, and let $\mathbf{F} = (0,0,2)$.
   (a) Find a vector field $\mathbf{G}$ such that $\mathbf{F} = \nabla \times \mathbf{G}$.
   (b) Use Stokes' theorem to compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$. ($S$ has the orientation given by the parametrization above.)

5. Use Green's theorem to compute

$$\iint_D \frac{-2x}{(x^2+y^2)^2}\,dA$$

   where D is the compact region in the first quadrant bounded by the circle $x^2 + y^2 = 1$, the circle $x^2 + y^2 = 2$, the line $x = 0$, and the line $x = 1$.

6. Let D be a compact region in $\mathbb{R}^3$ bounded by a smooth surface S. Show that

$$\text{Volume}(D) = \frac{1}{3}\iint_S \mathbf{r} \cdot \mathbf{n}\,dS$$

   where $\mathbf{r} = (x,y,z)$ and $\mathbf{n}$ is the outward unit normal to S.