Exam duration — Three hours
Reading time — 15 minutes
This paper consists of 3 pages

Identical Examination Papers: None.

Authorized Materials: No materials are authorized.

Mathematical tables and calculators are not permitted. Candidates are reminded that no written or printed material related to the subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators: Script books only are required. The students may remove the exam paper at the conclusion of the examination. No written or printed material related to the subject may be brought into the examination.

Instructions to Students: This examination consists of six questions. All questions may be answered.

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1. (a) Find all complex solutions of the equation

\[ z^2 + \overline{z} + 1 = 0 \]

in cartesian form. Indicate these on an Argand diagram and explain why the fundamental theorem of algebra does or does not apply.

(b) Show that

\[ \text{Arctan } z = \frac{1}{2} i \text{Log} \left( \frac{1 - iz}{1 + iz} \right) \]

and hence evaluate \( \text{Arctan}(i/2) \).

2. Find the following limits if they exist, otherwise, explain why they do not exist:

(a) \( \lim_{z \to 1} \frac{z^3 + 2z^2 - 3}{z^2 + z - 2} \)  
(b) \( \lim_{z \to \infty} \frac{3z^2 - z - 3}{2z^2 + z - 2} \)  
(c) \( \lim_{z \to 0} \frac{\text{Re}(z)}{z} \)  
(d) \( \lim_{z \to 0} \frac{\cos z - 1}{z^2 \cosh z} \)

3. (a) Clearly state Cauchy’s theorem and the general Cauchy integral formula. Hence or otherwise, evaluate the following closed contour integrals:

(i) \( \oint_{|z|=1} \frac{z \, dz}{z - 2} \)  
(ii) \( \oint_{|z|=2} \frac{z \, dz}{z - 1} \)  
(iii) \( \oint_{|z|=2} \frac{z \, dz}{(z - 1)^3} \)  
(iv) \( \oint_{|z|=2} \frac{z \, dz}{(z - 1)(z - 3)} \)

(b) Let \( \Gamma \) be the arc of the unit circle \( |z| = 1 \) going anti-clockwise from \( z = 1 \) to \( z = i \). Fully simplifying your answer, evaluate the contour integral

\[ \int_{\Gamma} z^{1/3} \, dz \]

where \( z^{1/3} \) denotes the principal branch cube root (i) by using the fundamental theorem of calculus and (ii) by parametrizing the curve \( \Gamma \).

4. (a) Use the ratio or root tests as appropriate to find the largest disk of absolute convergence of the following complex power series and find their sums:

(i) \( \sum_{n=0}^{\infty} (2 - 3(-1)^n) z^{2n} \)  
(ii) \( \sum_{n=1}^{\infty} \frac{(3z)^{2n}}{(2n)!} \)

(b) Use the Weierstrass \( M \)-test to show that the following complex series converges absolutely and uniformly on an arbitrarily large disk

\[ \sum_{n=1}^{\infty} \frac{1}{n^2 + |z|^2}, \quad |z| \leq R \]
5. Write down the Taylor or Laurent series for the following complex functions valid in the indicated domain:

(a) \( \log(1 + z^2) \), \( |z| < 1 \)
(b) \( \sin \left( \frac{1}{z} \right) \), \( |z| > 0 \)
(c) \( \frac{\sinh z - z}{z^4} \), \( 0 < |z| < \infty \)
(d) \( \frac{z^2}{(1 + z)(3 - z)} \), \( 1 < |z| < 3 \)

6. By converting to a complex contour integral and using residue calculus, evaluate the trigonometric integral involving the real angle \( \theta \)

\[
\int_0^{2\pi} \frac{\cos 2\theta}{(5 - 4\cos \theta)^2} \, d\theta
\]