DEPARTMENT OF MATHEMATICS AND STATISTICS
620–252 ANALYSIS – Semester 2, 2007
Course Information

Lectures:

This course consists of 36 one hour lectures (three per week) and 11 tutorial/practice class hours (one per week). The lectures are in Theatre 1 - Old Geology on Tuesdays and in the Laby Theatre - David Caro (Physics) Thursdays at 9.00 a.m. and in the Russell Love Theatre - Richard Berry (Mathematics and Statistics) on Fridays at 2.15 p.m.

Subject Description:

• This subject deals with convergence of sequences and series; elementary topology of the real line; the fundamentals of continuity, differentiability of functions of several real variables; analytic functions of a complex variable; complex derivative; power and Laurent series in complex variables; basic topological concepts in the complex plane; and Cauchy's theorem and its applications.

• Students completing this subject develop the ability to determine the convergence or otherwise of sequences and series; differentiate functions of a complex variable; calculate contour integrals; work with analytic functions in the cut plane; and apply Cauchy's integral formula and the residue theorem. The subject demonstrates the differences between functions of a real and a complex variable; and the role of complex analytic methods in solving important problems in science and engineering.

• Sequences and series topics include standard sequences and series, Cauchy convergence, ratio and nth root tests, absolute and conditional convergence, re-arrangements, and power series. Continuity topics include continuity and differentiability of functions of several real variables. Functions of a complex variable topics include elementary functions of a complex variable, branches; differentiation, analytic functions, and Cauchy-Riemann equations. Integration topics include line and contour integrals, Cauchy's integral theorem; Laurent series; singularities, poles, Liouville's theorem; residue theorem, limiting contours, and evaluation of integrals using contour integration.

Assessment/Assignments:

Assessment will be based on a 3-hour end-of-semester written examination [X marks out of 100] and three assignments during semester [A marks out of 15]. The final mark M out of 100 will be $M = \text{Max}(X, 85X/100 + A)$.

Recommended References/Texts:

Almost any book with Complex Analysis/Variables/Functions in the title including:

- A. David Wunsch, *Complex Variables with Applications*, Second Edition (Addison-Wesley).
- E. B. Saff and A. D. Snider, *Fundamentals of Complex Analysis for Mathematics, Science and Engineering* (Prentice Hall).
- Stephen D. Fisher, *Complex Variables*, Second Edition (Wadsworth and Brooks/Cole).
- J. E. Marsden and M. J. Hoffman, *Basic Complex Analysis* (Freeman).
- Murray R. Spiegel, *Theory and Problems of Complex Variables*, Schaum's Outline Series (McGraw-Hill).

Problem Sheets:

There are six problem sheets for this course. These relate to the basic skills to be acquired from the course. It is important to do most of the problems on these problem sheets. The problems marked with an asterisk are more difficult or theoretical.

Subject Web Page:

A web page will be maintained for this subject (http://www.ms.unimelb.edu.au/~s620252/). Lecture notes and supplementary materials will be available from this site.

Acknowledgement:

Paul Pearce has prepared the lecture notes over a number of years. Dave Coulson appreciates being able to use them.

620-252 ANALYSIS – LECTURE OUTLINE

Week 1. Complex Numbers
1. Complex numbers, conjugate, modulus, argument
2. Polar form, principal argument
3. De Moivre, complex exponential, fractional powers

Week 2. Complex Functions
4. Complex plane, argand diagrams, planar sets, topology
5. Functions of a complex variable, limits
6. Extended plane/stereographic projection, continuity

Week 3. Complex Derivatives and Analytic Functions
7. Complex derivative, Cauchy-Riemann equations
8. Analytic functions, entire functions, singularities
9. Harmonic functions

Week 4. Complex Transcendental Functions
10. Complex exponential, trigonometric/hyperbolic functions
11. Complex logarithm, principal branches, branch cuts
12. Riemann sheets, inverse trigonometrics and hyperbolics

Week 5. Line and Contour Integrals
13. Line and contour integrals, paths and curves
14. Properties of line integrals, path dependence
15. Green's theorem, Cauchy-Goursat theorem

Week 6. Cauchy's Integral Formula
16. Fundamental theorem of calculus, path independence
17. Deformation of contours about simple poles
18. General Cauchy integral formula

Week 7. Theorems Related to Cauchy's Integral Formula
19. Maximum modulus principle
20. Liouville's theorem
21. Fundamental theorem of algebra

Week 8. Complex Sequences and Series
22. Complex sequences, Cauchy convergence
23. Complex series, geometric and harmonic series
24. Limit theorems, divergence test

Week 9. Complex Series
25. Absolute convergence, conditional convergence
26. Comparison test, ratio test, nth root test
27. Uniform convergence, Weierstrass M-test

Week 10. Taylor Series and Manipulation of Series
28. Power series, radius of convergence, Taylor-Maclaurin
29. Taylor's theorem, manipulation of series
30. Term-by-term integration and differentiation, analytic continuation

Week 11. Laurent Series and Residue Calculus
31. Laurent series, poles, residues
32. Residue theorem, trigonometric integrals
33. Evaluation of improper integrals and series

Week 12 Singularities and Meromorphic Functions
34. Isolated zeros and poles
35. Singularities: isolated, removable, essential
36. Partial fraction expansions

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Sheet 1: Complex Numbers

1. *Complex Arithmetic.* Simplify the following complex expressions:

(a) $\dfrac{1}{1-i}$ (b) $\left(\dfrac{1+i}{1-i}\right)^3$

(c) $\dfrac{2+i}{1-i}+\dfrac{2-i}{1+i}+i$ (d) $(2+i)(-1-i)(3-2i)$

(e) $\dfrac{3}{i}+\dfrac{i}{3}$ (f) $\left(\dfrac{2+i}{6i-(1-2i)}\right)^2$

2. *Powers of* i. Show that for integer k

$$i^{4k}=1, \qquad i^{4k+1}=i, \qquad i^{4k+2}=-1, \qquad i^{4k+3}=-i$$

Hence evaluate

(a) i^7 (b) i^{62}

(c) i^{-202} (d) i^{-4321}

3. *Modulus, Conjugate and Argument.* Evaluate the following expresions involving complex modulus, conjugate and argument:

(a) $\overline{(3-4i)(1+i\sqrt{5})}$ (b) $\overline{\left(\dfrac{1}{(1-i)(3-2i)}\right)}$

(c) $\left|\dfrac{1}{1-i}\right|$ (d) $\left|i\overline{(3+4i)}(2-i)\right|$

(e) $\left|\left(\dfrac{(3+4i)(1+i)}{\overline{(3-4i)}}\right)^4\right|$ (f) $\left|i+\dfrac{1+i\sqrt{3}}{2+i\sqrt{2}}\right|$

(g) $\big||1+i|-|2-i|\big|$ (h) $\big||1+i|+i|2-i|\big|$

(i) $\operatorname{Arg}(-1+i\sqrt{3})$ (j) $\operatorname{Arg}\big((-1+i\sqrt{3})+(-1+i)\big)$

4. *Properties of Conjugation, Real and Imaginary Parts.* For $z_1, z_2 \in \mathbb{C}$, prove the following properties:

(a) $\overline{z_1-z_2}=\overline{z}_1-\overline{z}_2$ (b) $\overline{z_1 z_2}=\overline{z}_1\,\overline{z}_2$

(c) $\overline{\left(\dfrac{1}{z_1}\right)}=\dfrac{1}{\overline{z}_1}$ (d) $\overline{\left(\dfrac{z_1}{z_2}\right)}=\dfrac{\overline{z}_1}{\overline{z}_2}$

(e) $\operatorname{Re}(z_1 z_2)=\operatorname{Re}(\overline{z}_1\,\overline{z}_2)$ (f) $\operatorname{Im}(z_1 z_2)=-\operatorname{Im}(\overline{z}_1\,\overline{z}_2)$

5. *Principal Argument.*

   (a) Show, by finding a counter-example, that in general the principal argument does not satisfy

   $$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2), \qquad z_1, z_2 \in \mathbb{C}$$

   (b) How does this relation need to be modified so that it is true in general?

6. **Complex Algebra in Real Matrices.* Consider the field of 2×2 real matrices of the form:

   $$Z = xI + yJ, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad x, y \in \mathbb{R}$$

   (a) Show that this number field is algebraically equivalent (isomorphic) to the complex number field in the sense that they have the same algebraic laws (commutative, associative, distributive) under addition and multiplication.

   (b) Define a conjugate matrix $\overline{Z}$ and verify the usual properties of conjugation.

   (c) Show that the matrix

   $$R(\theta) = e^{\theta J} = \cos\theta I + \sin\theta J$$

   rotates vectors in the plane clockwise through θ radians.

   (d) Show that the inverse matrix of $R(\theta)$ is $R(-\theta)$.

7. *Complex Equations.* Find all solutions of the following equations for $z = x + iy$:

   (a) $z^2 = -5$
   (b) $z^2 + z + 1 = 0$
   (c) $2z^2 + 2z + 5 = 0$
   (d) $5z^2 + 4z + 1 = 0$
   (e) $z^3 = 1$
   (f) $z^4 = -z^{-1}$
   (g) $z^2 = i\,\overline{z}$
   (h) $z^4 - 2\sqrt{3}z^2 + 4 = 0$

8. *Polar Form.* Express the following complex numbers in polar form $r(\cos\theta + i\sin\theta)$ and exponential polar form $re^{i\theta}$:

   (a) $z_1 = 2 + 2\sqrt{3}\,i$
   (b) $z_2 = -5 + 5i$
   (c) $z_3 = -\sqrt{6} - \sqrt{2}\,i$
   (d) $z_4 = -3i$

   (e) Hence obtain the products $z_1 z_2$, $z_1 z_3$ and $z_3 z_4$ in exponential polar form.

9. *Polar to Cartesian Form.* Give the complex number $z = x + iy$ whose polar coordinates (r, θ) are:

   (a) $(\sqrt{3}, \pi/4)$
   (b) $(1/\sqrt{2}, \pi)$
   (c) $(4, -\pi/2)$
   (d) $(2, -\pi/4)$
   (e) $(1, 4\pi)$
   (f) $(\sqrt{2}, 9\pi/4)$