Assignment 2: Line and Contour Integrals
(Due 4.00 pm Monday 2 October)

Note: All closed contours are positively oriented unless otherwise stated.

1. (a) By parametrizing the contours, evaluate the closed contour integrals

\[ I_1 = \oint_{\Gamma} z \, dz, \quad I_2 = \oint_{\Gamma} \overline{z} \, dz \]

around the contour \( \Gamma = \gamma_1 + \gamma_2 \) where

\( \gamma_1 \) \( \gamma_1 \) is the section of the parabola \( \text{Im } z = (\text{Re } z)^2 \) from \( z = 0 \) to \( z = 1 + i \)

and

\( \gamma_2 \) \( \gamma_2 \) is the straight line segment from \( z = 1 + i \) to \( z = 0 \).

(b) Explain why Cauchy’s theorem does or does not apply to these integrals.

(c) Without evaluating any more integrals, what is the area enclosed by the contour \( \Gamma \).

2. Let \( f(z) = \frac{5}{2} z^{3/2} \) (a multivalued function).

(a) Define/find a branch \( F(z) \) of \( f(z) \), stating a simply connected domain \( D \) on which \( F(z) \) is analytic.

(b) Verify that \( G(z) = \exp(\frac{5}{2} \log z) \) is a primitive of \( F(z) \) on \( D \).

(c) Hence, using the fundamental theorem of calculus, calculate the contour integral

\[ I_1 = \oint_{|z|=1} F(z) \, dz \]

In this integral you should integrate between \( z_1 = e^{-i(\pi-\epsilon)} \) and \( z_2 = e^{i(\pi-\epsilon)} \) and take the limit \( \epsilon \to 0 \).

(d) Show that \( |\frac{5}{2} z^{3/2}| = \frac{5}{2} \) for \( |z| = 1 \).

(e) Using \( |\int_{\Gamma} g(z) \, dz| \leq |\Gamma|M \) where \( |g(z)| \leq M \) for \( z \) on \( \Gamma \) show that

\[ \lim_{\epsilon \to 0} \int_{e^{i(\pi-\epsilon)}}^{e^{i(\pi+\epsilon)}} F(z) \, dz = 0 \] where \( z = e^{it}, \ t \in [\pi - \epsilon, \pi + \epsilon] \).
3. (a) Show that \( \frac{1}{2} \log\left( \frac{1-z}{1+z} \right) \) (the Principal branch of log) is a primitive for \( \frac{1}{z^2-1} \).

(b) Show that the domain \( D \) for which this primitive is valid is \( \{ z \in \mathbb{C} : \text{Im}(z) = 0 \Rightarrow |\text{Re}(z)| < 1 \} \).

It may be useful to show that \( \frac{1-z}{1+z} \in \mathbb{R} \Leftrightarrow z - \overline{z} = 0 \).

(c) Hence evaluate \( A = \int_{\Gamma} \frac{1}{z^2-1} \, dz \) where \( \Gamma \) is the ‘crazy contour’ consisting of line segments starting from \(-i\) to \(1+3i\), then \(1+3i\) to \(3+4i\) and finally \(3+4i\) to \(i\).

- Explain and justify the steps you used in your evaluation. (A sketch may be useful.)

(d) For which contours \( \Gamma \) starting at \(-i\) and ending at \(i\) will \( A = \int_{\Gamma} \frac{1}{z^2-1} \, dz \)?

4. (a) Use the Cauchy integral formula to evaluate the trigonometric integral

\[
I = 2 \int_{0}^{2\pi} \frac{1}{(5 + \cos \phi)} \, d\phi.
\]

(b) (Just 1 mark)\(^1\) Using the double angle formula and periodicity (real techniques) show that

\[
\int_{0}^{2\pi} \frac{1}{(2 + \cos^2 \theta)} \, d\theta = 2 \int_{0}^{2\pi} \frac{1}{(5 + \cos \phi)} \, d\phi.
\]

(c) Use the generalized Cauchy integral formula with appropriate contour deformations to evaluate the integral

\[
I = \oint_{|z-i|=2} \frac{dz}{z^3(z-1)^2(z-2)}.
\]

\(^1\)The LHS integral may rekindle memories of the 5/9 lecture.