1. (a) Find the largest open region in which the complex series
\[ \sum_{n=0}^{\infty} \left[ \left( -\frac{1}{z} \right)^n + \left( \frac{z}{2} \right)^n \right] \]
converges and find its sum.

(b) Find the circle of convergence for each of the complex power series:

\[ \begin{align*}
(i) & \quad \sum_{n=1}^{\infty} \frac{2 + i^n}{2^n} (z - 1)^n; \\
(ii) & \quad \sum_{n=1}^{\infty} \frac{n^n}{n!} z^n.
\end{align*} \]

For the second series it may be helpful to show \( \lim_{n \to \infty} \frac{(n + 1)^{n+1}}{(n + 1)!} / \frac{n^n}{n!} = e \).

1. Solution.

(a) The series \( S = \sum_{n=0}^{\infty} \left[ \left( -\frac{1}{z} \right)^n + \left( \frac{z}{2} \right)^n \right] \) converges if and only if BOTH \( S_1 = \sum_{n=0}^{\infty} \left( -\frac{1}{z} \right)^n \) and \( S_2 = \sum_{n=0}^{\infty} \left( \frac{z}{2} \right)^n \) converge.

The series \( S_1 \) is a geometric series and converges to \( \frac{1}{1 - \left( -\frac{1}{z} \right)} \) provided \( \left| \frac{1}{z} \right| < 1 \), that is \( |z| > 1 \).

The series \( S_2 \) is also geometric and converges to \( \frac{1}{1 - \left( \frac{z}{2} \right)} \) provided \( \left| \frac{z}{2} \right| < 1 \) that is \( |z| < 2 \).

So the series \( S \) converges when \( |z| > 1 \) and \( |z| < 2 \) (that is \( 1 < |z| < 2 \)) to
\[ \frac{1}{1 - \left( -\frac{1}{z} \right)} + \frac{1}{1 - \left( \frac{z}{2} \right)} = \frac{z}{z+1} + \frac{2}{2 - z} = \frac{z^2 - 4z - 2}{(z+1)(z-2)}. \]

(b) Find the circle of convergence for each of the complex power series:

(i) \( \sum_{n=1}^{\infty} \frac{2 + i^n}{2^n} (z - 1)^n; \)

Now \( |2 + i^n| = \begin{cases} 
1 & \text{if } n = 4m + 2 \\
\sqrt{5} & \text{if } n = 4m + 1 \\
3 & \text{if } n = 4m
\end{cases} \)

Using the root test \( \lim_{n \to \infty} \left| \frac{2 + i^n}{2^n} (z - 1)^n \right|^{1/n} = \left| \frac{z - 1}{2} \right| \lim_{n \to \infty} |2 + i^n|^{1/n} = \)
\[ \left| \frac{z - 1}{2} \right|. \] As

\[ 1^{1/n} \leq |2 + i^n|^{1/n} \leq \sqrt{3}^{1/n} \text{ for all } n \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

hence \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \text{by the sandwich rule.}

From which we see that we have convergence for \(|z - 1| < 2\).

(ii) \[ \sum_{n=1}^{\infty} \frac{n^n}{n!} z^n. \]

We first show that \[ \lim_{n \to \infty} \frac{(n+1)^{n+1}}{(n+1)!} \frac{n^n}{n!} = e. \]

\[ \frac{(n+1)^{n+1}}{(n+1)!} \frac{n^n}{n!} = \frac{n+1}{n} \frac{(n+1)^n}{(n)!} \frac{n^n}{n!} = \left( \frac{n+1}{n} \right)^n. \]

So \[ \lim_{n \to \infty} \left( \frac{n+1}{n} \right)^n = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e. \]

Using the ratio test on the series: \[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} z^{n+1} \frac{n^n}{n!} \right| = |z| \lim_{n \to \infty} \frac{(n+1)^{n+1}}{(n+1)!} \frac{n^n}{n!} = e|z| \]

So the series converges if \(|z| < 1/e\).

2. (a) Using the Weierstrass M-test show that \[ \sum_{n=1}^{\infty} (3n)z^{3n} \] converges uniformly on \(|z| \leq r < 1\).

(b) Show that \[ \sum_{n=1}^{\infty} (3n)z^{3n-1} = \frac{d}{dz} \left( \frac{1}{1 - z^3} \right) \] on \(|z| \leq r < 1\).

(c) With justification find a function \(f(z)\) that has Taylor series \[ \sum_{n=1}^{\infty} (3n)z^{3n} \] about \(|z| \leq r < 1\).

(d) Hence or otherwise find the sum of the series \[ \sum_{n=1}^{\infty} (3n)(\frac{1}{2})^{3n}. \]

2. Solution.

(a) \( (3n)z^{3n} < (3n)r^{3n} \) for all \(|z| \leq r < 1\). Now \[ \sum_{n=1}^{\infty} (3n)r^{3n} \] converges by the ratio test as \[ \lim_{n \to \infty} \frac{(3n+3)r^{3n+3}}{(3n)r^{3n}} = r^3 < 1. \]

So by the Weierstrass M-test the series converges uniformly for \(|z| \leq r < 1\).
(b) Now \( \frac{1}{1-z^3} = \sum_{n=0}^{\infty} z^{3n} \) for \( z < 1 \) as the series is geometric. Differentiating termwise we obtain \( \frac{d}{dz} \left( \frac{1}{1-z^3} \right) = \sum_{n=1}^{\infty} (3n)z^{3n-1} \) provided the series on the right converges uniformly, which it does for \( z \leq r < 1 \). (Using the Weierstrass M-test gives uniform convergence in any closed subdisk of the disk of convergence.)

(c) From above we have \( \sum_{n=1}^{\infty} (3n)z^{3n-1} = \frac{d}{dz} \left( \frac{1}{1-z^3} \right) = \frac{3z^2}{(1-z^3)^2} \) on \( |z| \leq r < 1 \). So multiplying both sides by \( z \) (strictly speaking on the LHS we are finding the Cauchy product of two series one of which is the single term series \( z \)) we see that \( \sum_{n=1}^{\infty} (3n)z^{3n} = \frac{3z^3}{(1-z^3)^2} \) provided \( |z| \leq r < 1 \).

Now the power series is the same as the Taylor series (provided the centres match) of the function the power series converges to, that is the function is \( \frac{3z^3}{(1-z^3)^2} \).

(d) Thus the sum of the series \( \sum_{n=1}^{\infty} (3n)(\frac{1}{2})^{3n} \) is \( \frac{3z^3}{(1-z^3)^2} \bigg|_{z=1/2} = \frac{24}{49} \) as \( z = 1/2 \) is within the domain \( |z| \leq r < 1 \). (E.g. choose \( r = 3/4 \))

3. Let \( f(z) = \frac{1}{(z-1)(2z-1)} \).

(a) Find Laurent series expansions for \( f(z) \) valid for:

(i) \( 0 < |z - 1| < \frac{1}{2} \); (ii) \( \frac{1}{2} < |z - 1| \).

(b) Write down (with brief justification) or calculate \( \text{Res}(f,1) \)

(c) Hence use residue calculus to calculate

\[ \oint_{|z-1|=1/4} f(z)dz. \]

3. Solution.

(a) (i) For computational simplicity \( w = z - 1 \) (\( \iff z = w + 1 \)),

so \( f(w) = \frac{1}{w(1+2w)} \).

We find the McLaurin series for \( 1/(1+2w) \) using geometric series.

\[ \frac{1}{1+2w} = 1 - 2w + 4w^2 - 8w^3 \ldots = \sum_{n=0}^{\infty} (-2)^n w^n \] valid for \( |w| < 1/2 \). We
now multiply by $1/w$ to get the Mclaurin series for $f(w)$ (strictly speaking we are using the Cauchy product once again).

$$f(w) = 1/w - 2 + 4w - 8w^2 \ldots = \sum_{n=1}^{\infty} (-2)^{n+1} w^n,$$

for $0 < |w| < 1/2$. (Note we now need $w \neq 0$.) Back substitution for $w$ gives

$$f(z) = \sum_{n=1}^{\infty} (-2)^{n+1} (z - 1)^n,$$

for $0 < |z - 1| < 1/2$.

(ii) We proceed as in (i) except we generate a series converging to $f(w)$ for $|w| > 1/2$ by expanding $1/(1 + 2w)$ ‘inside out’.

$$\frac{1}{1+2w} = \frac{1/(2w)}{1 + 1/(2w)} = \frac{1}{2w} - \left(\frac{1}{2w}\right)^2 + \left(\frac{1}{2w}\right)^3 \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{2}\right)^n w^{-n},$$

for $|w^{-1}| < 2 \iff |w| > 1/2$. So multiplying by $1/w$ and back substituting for $w$ results in

$$f(z) = \frac{1}{2} (z - 1)^{-2} - \left(\frac{1}{2}\right)^2 (z - 1)^{-3} \ldots = \sum_{n=2}^{\infty} (-1)^n \left(\frac{1}{2}\right)^{n-1} (z - 1)^{-n}$$

for $|z - 1| > 1/2$.

(b) $\text{Res}(f; 1)$ is the coefficient of $(z - 1)^{-1}$ in a Laurent expansion for $f(z)$ valid in a punctured disk about $z = 1$. From (i) $f(z) = \sum_{n=1}^{\infty} (-2)^{n+1} (z - 1)^n,$

for $0 < |z - 1| < 1/2$ in the appropriate Laurent series expansion. By inspection $(n = -1) \text{Res}(f; 1) = 1$

(c) $f(z)$ has two singularities at $z = 1$ and $z = 1/2$, of these only $z = 1$ is interior to the contour $|z - 1| = 1/4$. Thus

$$\oint_{|z-1|=1/4} f(z)dz = 2\pi i \text{Res}(f; 1) = 2\pi i.$$

4. Find and classify singularities, and calculate the residue at isolated singularities, for the following functions:

(a) $\frac{(z - 1)^3}{(z + 1)z}$;  
(b) $\frac{\sin z}{(z - \pi)}$;

(c) $\frac{(z - 2)}{(z - 1)^4}$;  
(d) $\frac{\text{Log}(2z)}{(z - 1)}$ (Principal branch).

4. Solution.

(a) $\frac{(z - 1)^3}{(z + 1)z}$ has singularities at $z = -1$ and $z = 0$. These are both simple poles as the roots of multiplicity 1 of the denominator. If $\eta$ is a simple pole of $f(z)$ $\text{Res}(f; \eta) = \lim_{z \to \eta} (z - \eta)f(z)$. So $\text{Res}(f; 0) = \lim_{z \to 0}(z - 0)f(z) = \frac{(z-1)^3}{(z+1)} \bigg|_{z=0} = -1$ likewise $\text{Res}(f; -1) = \lim_{z \to -1}(z + 1)f(z) = \frac{(z-1)^3}{z} \bigg|_{z=-1} = 8.$
5. (a) Calculate the residues at the poles in the upper half plane of $f(z) = \frac{1}{(z^2 + 1)^2}$.

(b) Using an appropriate contour in the complex plane and appropriate justification calculate the real integral

$$
\int_0^\infty \frac{1}{(1+x^2)^2}dx.
$$

COMMENT ONLY – no work needed here:

A crude upper bound $0 < \int_0^\infty \frac{1}{(1+x^2)^2}dx < \int_0^\infty \frac{1}{1+x^2}dx = \frac{\pi}{2}$.

5. Solution.

(a) $\frac{1}{(z^2 + 1)^2}$ has double poles at $z = i$ and $z = -i$. We are interested in the residue at $z = i$. 

(b) $\frac{\sin z}{(z-\pi)}$ has a singularity at $z = \pi$. This singularity is removable as

$$
\lim_{z\to\pi} \frac{\sin z}{z-\pi} = \lim_{z\to\pi} \frac{\cos z}{1} = -1 \text{ by L'Hopital's rule.}
$$

Alternatively it appears to be a simple pole. If it were: Res$(f; \pi) = \lim_{z\to\pi} (z-\pi)f(z) = \sin z|_{z=\pi} = 0$.

This means the Laurent series of $f(z)$ about $\pi$ has no powers of $(z-\pi)$ beyond $(z-\pi)^{-1}$ in the negative direction. But $(z-\pi)^{-1}$ has coefficient 0, thus the Laurent series is in fact a Taylor series (no -ve powers of $(z-\pi)$) and the singularity is removable.

(c) $\frac{(z-2)}{(z-1)^4}$ has a pole of order 4 at $z = 1$.

Res$(f; 1) = \frac{1}{3!} \lim_{z\to1} \frac{d^3}{dz^3} (z-2)^4 (z-1)^4 = \frac{1}{3!} \lim_{z\to1} \frac{d^3}{dz^3} (z-2) = 0$

Alternatively for 'series zealots' $z-2 = (z-1) - 1$ thus $\frac{(z-2)}{(z-1)^4} = -1((z-1)^{-4} + (z-1)^{-3}$ is a Laurent series expansion (valid for $0 < |z-1|$) giving Res$(f; 1) = 0$ as there is no $(z-1)^{-1}$ term in the expansion.

(d) $\frac{\log(2z)}{(z-1)}$ has a simple pole at $z = 1$ a branch point at $z = 0$ and for the principal branch a branch cut along the negative real axis. Note the branch point and points on the branch cut are NOT ISOLATED singularities.

Res$(f; 1) = \log(2z)|_{z=1} = \log 2$ (the real log of 2).
This is \((m = 2)\)
\[
\frac{1}{1!} \lim_{z \to -i} \frac{d}{dz} \frac{(z - i)^2}{(z^2 + 1)^2} = \lim_{z \to -i} \frac{d}{dz} \frac{1}{(z^2 + 1)^2} = \lim_{z \to -i} -2 \frac{1}{(z + i)^3} = \frac{-2}{-8i} = \frac{1}{4i}.
\]

(b) Now \(\int_0^\infty \frac{1}{(1 + x^2)^2} \, dx = \lim_{R \to \infty} \int_0^R \frac{1}{(1 + x^2)^2} \, dx = \frac{1}{2} \lim_{R \to \infty} \int_{-R}^R \frac{1}{(1 + x^2)^2} \, dx\)
as the integrand is an even function of \(x\).
Let \(\Gamma_R = \gamma_R + C_R\) where \(\gamma_R\) is from \(-R\) to \(R\) along the real axis and \(C_R\) is the positively oriented semi circular arc in the upper half plane running from \(R\) to \(-R\). or more explicitly
\[
\gamma_R : \quad z(t) = -R + 2Rt \quad \text{for} \quad t \in [0, 1]
\]
\[
C_R : \quad z(t) = Re^{it} \quad \text{for} \quad t \in [0, \pi].
\]
We note/show

- \(\int_{\gamma_R} \frac{1}{(z^2 + 1)^2} \, dz = \int_{-R}^{R} \frac{1}{(1 + x^2)^2} \, dx.\)
- \(\lim_{R \to \infty} \int_{C_R} \frac{1}{(z^2 + 1)^2} \, dz = 0.\)

(If \(R > 1\)) and \(z\) on \(C_R (\Rightarrow |z| = R)\) then \(|\int_{\Gamma} f(z) \, dz| \leq M|\Gamma|\) for any \(f(z)\). So in our case
\[
|\int_{C_R} \frac{1}{(z^2 + 1)^2} \, dz| \leq \frac{\pi R}{(R^2 - 1)^2} \to 0 \quad \text{as} \quad R \to \infty.
\]
Thus \(2 \int_0^\infty \frac{1}{(1 + x^2)^2} \, dx = \lim_{R \to \infty} \int_{\gamma_R} \frac{1}{(z^2 + 1)^2} \, dz.\)
As \(R\) goes to \(\infty\) the contour \(\Gamma_R\) includes all singularities in the upper half plane.
So by residue calculus \(\lim_{R \to \infty} \int_{\Gamma_R} \frac{1}{(z^2 + 1)^2} \, dz = 2\pi i \text{Res} \left( \frac{1}{(z^2 + 1)^2}, i \right) = \frac{2\pi i}{4i} = \frac{\pi}{2}.\)
Giving \(\int_0^\infty \frac{1}{(1 + x^2)^2} \, dx = \frac{\pi}{4}.\)
Which lies between 0 and \(\pi/2\) as it must by our crude bound.