1. Evaluate the following if they exist
   (a) \((3\sqrt{3}i)^{2/3}\)
   (b) \(\lim_{z \to \pi i/2} \log(\cosh z)\)
   (c) \(\sin(3i)\)
   (d) \(\lim_{z \to \pi/2} \frac{(z - \pi/2)e^z}{\cos z}\)
   (e) \(i^{-2i}\) (principal value)

Recall:

**differentiable** If \(f(x + iy) = u(x, y) + iv(x, y)\) has \(u\) and \(v\) \(C^1\) in some open region containing \((x_0, y_0)\) and the Cauchy-Riemann equations hold at \((x_0, y_0)\) then \(f(z)\) is differentiable at \(z_0 = x_0 + iy_0\).

**analytic** If \(f(x + iy) = u(x, y) + iv(x, y)\) has \(u\) and \(v\) \(C^1\) in some open region \(D\) containing \((x_0, y_0)\) and the Cauchy-Riemann equations hold on \(D\) then \(f(z)\) is differentiable on \(D\), that is \(f(z)\) is **analytic** on \(D\).

2. Let \(f(x + iy) = x + iy^2\) define \(f(z)\) where \(z = x + iy\).
   Find where \(f(z)\) is:
   (a) differentiable;
   (b) analytic.

3. Let \(u(x, y) = x^3 - 3xy^2 + e^{-x}\cos y\).
   (a) Show that \(u(x, y)\) is **harmonic** on \(\mathbb{R}^2\).
   (b) Find a **harmonic conjugate** \(v(x, y)\) for \(u(x, y)\).
   (c) Hence find an **entire** function \(f(z)\) such that \(\text{Real}(f(x + iy)) = u(x, y)\).
   Write \(f(z)\) in terms of \(z\).
4. (a) Using the *complex exponential* definitions of sin and cos show that:
   (i) \( \sin(z + w) = \sin z \cos w + \cos z \sin w; \)
   (ii) \( \cos(-z) = \cos z; \)
   (iii) \( \sin(-z) = -\sin z. \)

(b) Hence (or otherwise) show that
   \[ \sin(z - w) = \sin z \cos w - \cos z \sin w. \]

In lectures we have seen that
\[ \sin z = \sin x \cosh y + i \cos x \sinh y \]
where \( z = x + iy. \)

(c) Show that
   \[ |\sin z| = \sqrt{\sin^2 x + \sinh^2 y}. \]

(d) Find all the solutions \( z \in \mathbb{C} \) to
   \[ \sin z = 0. \]

Using the results of parts a) and b) it can be seen that
   \[ \sin z = \sin z' \iff 2 \cos \left( \frac{z + z'}{2} \right) \sin \left( \frac{z - z'}{2} \right) = 0. \]

Additionally, similar techniques to c) and d) can be used to demonstrate that
   \[ \cos z = 0 \iff z = \left( n + \frac{1}{2} \right) \pi \text{ for } n \in \mathbb{Z} \]

(e) Hence (or otherwise) show that \( \sin z \) is *injective*\(^1\) on the domain
   \[ D = \{ z \in \mathbb{C} : |\Re(z)| < \frac{\pi}{2} \}. \]

   *End of work on question four.*

Additionally (and with some difficulty) it can be shown that with domain \( D \) as defined above \( \sin z \) has
- *range* \( R = \mathbb{C} \setminus \{ z \in \mathbb{R} : |z| \geq 1 \}. \)
- *inverse function* \( \text{arcsin} : R \to D \) where \( \text{arcsin} w = -i \log \left( iw + \sqrt{1 - w^2} \right) \).
   where \( \log \) is the principal value of log.

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\(^1\)Recall that \( f(z) \) is injective on \( D \) if and only if \( z, z' \in D \) with \( z \neq z' \Rightarrow f(z) \neq f(z') \) or equivalently \( f(z) = f(z') \) for \( z, z' \in D \Rightarrow z = z' \).