DEPARTMENT OF MATHEMATICS AND STATISTICS
620–252 ANALYSIS— Semester 2, 2006

Assignment 1: Limits, Continuity and Derivatives
(Due 4.00 pm Monday 28 August)

1. Evaluate the following if they exist

(a) \((3\sqrt{3}i)^{2/3}\)

\[
(3\sqrt{3}i)^{2/3} = [(3\sqrt{3}i)^2]^{1/3}
\]

\[
|3\sqrt{3}i| = 3^{2/3}
\]

\[
\text{Arg}(3\sqrt{3}) = \frac{\pi}{2}
\]

So \((3\sqrt{3}i)^2 = 3^2 e^{i\pi}\)

Thus one solution is \([3^{2}]^{1/3} e^{i\pi/3} = 3 e^{i\pi/3}\)

\[
= \frac{3}{2} + \frac{3\sqrt{3}}{2} i
\]

\[= r_1 \text{ (say)}\]

Others are \(r_1 e^{2\pi i/3}, r_1 e^{4\pi i/3}\)

As \(e^{2\pi i/3}\) is a primitive cube root of unity.

The set of roots is \(\{3e^{i\pi/3}, 3e^{i\pi}, 3e^{5i\pi/3}\} = \left\{\frac{3}{2} + \frac{3\sqrt{3}}{2} i, -3, \frac{3\sqrt{3}}{2} - \frac{3}{2} i\right\}\)

(b) \(\lim_{z \to \pi i/2} \text{Log}(\cosh z)\)

\(\log z\) (principal branch) is analytic on \(D = \{z \in \mathbb{C} : \text{Im}(z) = 0 \Rightarrow \text{Re} z > 0\}\) and is thus continuous on \(D\) too.

\(\lim_{z \to \pi i/2} \cosh z = \cosh \pi i/2\) as \(\cosh z\) is entire so is continuous on \(\mathbb{C}\).

Thus \(\lim_{z \to \pi i/2} \text{Log}(\cosh z) = \text{Log}(\cosh \pi i/2)\) provided \(\cosh \pi i/2\) is not in the (principal) branch cut of \(\text{Log}\)

\(\cosh \pi i/2 = [e^{\pi i/2} + e^{-\pi i/2}]/2 = (i - i)/2 = 0\)

which is the branch point for \(\text{Log}\) and is an intractable problem (\(\log 0\) does not exist in \(\mathbb{C}\)). Thus the limit DOES NOT EXIST.

(c) \(\sin(3i)\)

\[
\sin(3i) = \frac{e^{i(3i)} - e^{-i(3i)}}{2i} = \frac{e^{-3} - e^{3}}{2i} = i \sinh 3.
\]
(d) \[ \lim_{z \to \pi/2} \frac{(z - \pi/2) e^z}{\cos z} \]

\[ \frac{z - \pi/2}{\cos z} \text{ is of the form } \frac{0}{0} \text{ so by L'hopitals rule } = \lim_{z \to \pi/2} \frac{1}{-\sin z} = -1 \]

So by the algebra of limits

\[ \lim_{z \to \pi/2} \frac{(z - \pi/2) e^z}{\cos z} = \lim_{z \to \pi/2} \frac{(z - \pi/2)}{\cos z} \lim_{z \to \pi/2} e^z = -1 \times e^{\pi/2} = -e^{\pi/2}. \]

(e) \( i^{-2i} \) (principal value)

\[ i^{-2i} \text{ (principal value)} := \exp(-2i \Log i) = \exp(-2i\pi/2i) = e^\pi. \]

2. Let \( f(x + iy) = x + iy^2 \) define \( f(z) \) where \( z = x + iy \).

Find where \( f(z) \) is:

(a) differentiable;

\[ \frac{\partial u}{\partial x} = 1 \quad \frac{\partial v}{\partial x} = 0 \]
\[ \frac{\partial u}{\partial y} = 0 \quad \frac{\partial v}{\partial y} = 2y \]

Note all partial derivatives are continuous

The Cauchy Riemann equations hold

\[ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{everywhere} \]

AND \( \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \Rightarrow \quad 2y = 1 \)

So \( f(z) \) is differentiable on the line \( \text{Im}(z) = 1/2 \).

(b) analytic:

\( f(z) \) is analytic NOWHERE as \( f(z) \) is not differentiable in any open set.

3. Let \( u(x, y) = x^3 - 3xy^2 + e^{-x} \cos y \).
(a) Show that $u(x, y)$ is harmonic on $\mathbb{R}^2$.

\[ \frac{\partial u}{\partial x} = 3x^2 - 3y^2 - e^{-x} \cos y, \quad \frac{\partial^2 u}{\partial x^2} = 6x + e^{-x} \cos y \]

With both the first and second order partial derivatives with respect to $x$ being continuous.

\[ \frac{\partial u}{\partial y} = -6xy - e^{-x} \sin y, \quad \frac{\partial^2 u}{\partial y^2} = -6e^{-x} \cos y \]

With both the first and second order partial derivatives with respect to $y$ being continuous.

Furthermore $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x + e^{-x} \cos y + (-6x - e^{-x} \cos y) = 0$ (Laplace’s equation is satisfied). Thus $u(x, y)$ is harmonic on $\mathbb{R}^2$.

(b) Find a harmonic conjugate $v(x, y)$ for $u(x, y)$.

Reverse engineering the Cauchy Riemann equations.

\[ \frac{\partial u}{\partial y} = -6xy - e^{-x} \sin y \quad \Rightarrow \quad \frac{\partial v}{\partial x} = 6xy + e^{-x} \sin y \]

\[ \Rightarrow \quad v(x, y) = 3x^2y - e^{-x} \sin y + f(y) \]

\[ \frac{\partial u}{\partial x} = 3x^2 - 3y^2 - e^{-x} \cos y \quad \Rightarrow \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2 - e^{-x} \cos y \]

\[ \Rightarrow \quad v(x, y) = 3x^2y - y^3 - e^{-x} \sin y + g(x) \]

So $v(x, y) = 3x^2y - y^3 - e^{-x} \sin y + c$.

(c) Taking $c = 0$,

\[ f(x + iy) = x^3 - 3xy^2 + e^{-x} \cos y + i(3x^2y - y^3 - e^{-x} \sin y) \]

\[ = (x^3 + i3x^2y + i^23xy^2 + i^3y^3) + e^{-x}(\cos y - i \sin y) = (x + iy)^3 + e^{-x-iy} \]

So $f(z) = z^3 + e^{-z}$. 
4. (a) Using the complex exponential definitions of sin and cos show that:

(i) \( \sin(z + w) = \sin z \cos w + \cos z \sin w; \)

\[
\begin{align*}
\text{LHS} &= \sin z \cos w + \cos z \sin w \\
&= \left( \frac{e^{iz} - e^{-iz}}{2i} \right) \left( \frac{e^{iw} + e^{-iw}}{2} \right) + \left( \frac{e^{iz} + e^{-iz}}{2i} \right) \left( \frac{e^{iw} - e^{-iw}}{2} \right) \\
&= \frac{1}{4i} \left( \left[ e^{iz} e^{iw} + e^{iz} e^{-iw} - e^{i(-z)} e^{-iw} - e^{i(-z)} e^{-iw} \right] + \left[ e^{iz} e^{iw} - e^{iz} e^{-iw} + e^{i(-z)} e^{-iw} - e^{i(-z)} e^{-iw} \right] \right) \\
&= \frac{1}{4i} \left( \left[ e^{iz} e^{iw} - e^{-iz} e^{-iw} \right] + \left[ e^{iz} e^{iw} - e^{-iz} e^{-iw} \right] \right) \\
&= \frac{1}{2i} \left( e^{i(z+w)} - e^{-i(z+w)} \right) \\
&= \sin(z + w) \\
&= \text{RHS}
\end{align*}
\]

(ii) \( \cos(-z) = \cos z; \)

\[
\begin{align*}
\text{LHS} &= \cos(-z) \\
&= \left( \frac{e^{i(-z)} + e^{-i(-z)}}{2} \right) \\
&= \left( \frac{e^{-iz} + e^{iz}}{2} \right) \\
&= \cos z \\
&= \text{RHS}
\end{align*}
\]

(iii) \( \sin(-z) = -\sin z. \)

\[
\begin{align*}
\text{LHS} &= \sin(-z) \\
&= \left( \frac{e^{i(-z)} - e^{-i(-z)}}{2i} \right) \\
&= - \left( \frac{e^{iz} + e^{-iz}}{2i} \right) \\
&= -\sin z \\
&= \text{RHS}
\end{align*}
\]

(b) Hence (or otherwise) show that

\[ \sin(z - w) = \sin z \cos w - \cos z \sin w. \]

\[ \sin(z - w) = \sin(z + (-w)) = \sin z \cos(-w) + \cos z \sin(-w) = \sin z \cos w + \cos z \times - \sin w = \sin z \cos w - \cos z \sin w. \]
In lectures we have seen that

\[ \sin z = \sin x \cosh y + i \cos x \sinh y \]

where \( z = x + iy \).

(c) Show that

\[ |\sin z| = \sqrt{\sin^2 x + \sinh^2 y}. \]

\[
|\sin z| = |\sin x \cosh y + i \cos x \sinh y| \\
= \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y} \\
= \sqrt{\sin^2 x \cosh^2 y + (1 - \sin^2 x) \sinh^2 y} \\
= \sqrt{\sin^2 x (\cosh^2 y - \sinh^2 y) + \sinh^2 y} \\
= \sqrt{\sin^2 x + \sinh^2 y}
\]

(d) Find all the solutions \( z \in \mathbb{C} \) to

\[ \sin z = 0. \]

\[ \sin z = 0 \iff |\sin z| = 0 \iff \sin^2 x + \sinh^2 y = 0 \iff \sin x = 0 \& \& \sinh y = 0 \]

Now \( \sinh y = 0 \iff y = 0 \) and \( \sin x = 0 \iff x = n\pi \) for \( n \in \mathbb{Z} \). Thus \( \sin z = 0 \)

for \( z \in \mathbb{C} \) if and only if \( z = n\pi \) for \( n \in \mathbb{Z} \).

(Amazingly expanding the domain of \( \sin \) from \( \mathbb{R} \) to \( \mathbb{C} \) obtains no more zeros.)

Using the results of parts a) and b) it can be seen that

\[ \sin z = \sin z' \iff 2 \cos \left( \frac{z + z'}{2} \right) \sin \left( \frac{z - z'}{2} \right) = 0. \]

Additionally, similar techniques to c) and d) can be used to demonstrate that

\[ \cos z = 0 \iff z = \left( n + \frac{1}{2} \right) \pi \text{ for } n \in \mathbb{Z} \]

(e) Hence (or otherwise) show that \( \sin z \) is injective on the domain

\[ D = \left\{ z \in \mathbb{C} : |\text{Re}(z)| < \frac{\pi}{2} \right\}. \]
\[
\sin z = \sin z' \\
\iff 0 = \sin z - \sin z' \\
\iff 0 = 2 \cos \left( \frac{z + z'}{2} \right) \sin \left( \frac{z - z'}{2} \right) \\
\iff \cos \left( \frac{z + z'}{2} \right) = 0 \text{ OR } \sin \left( \frac{z - z'}{2} \right) = 0
\]

Now for \( z, z' \) in \( D \) \( \left| \Re \left( \frac{z + z'}{2} \right) \right| < \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} \) so \( \cos \left( \frac{z + z'}{2} \right) \neq 0 \) for \( z, z' \in D \).

Also for \( z, z' \) in \( D \) \( \left| \Re \left( \frac{z - z'}{2} \right) \right| \leq \left| \Re \left( \frac{z}{2} \right) \right| + \left| \Re \left( \frac{-z'}{2} \right) \right| < \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} \)

so \( \sin \left( \frac{z - z'}{2} \right) = 0 \iff \left( \frac{z - z'}{2} \right) = 0 \iff z = z' \).