1. Evaluate the following if they exist
   (a) \((1 + \sqrt{3}i)^{3/4}\)
   (b) \(\lim_{z \to \pi i/2} \text{Log}(\sinh(z))\)
   (c) \(\lim_{z \to 0} \text{Log}(z)\)
   (d) \(\lim_{z \to \pi} \frac{(z - \pi) e^{iz}}{\sin z}\)
   (e) \((-2i)^{-2i}\) (principal value)

1. Solution.
   (a) \((1 + \sqrt{3}i) = 2e^{i\pi/3}\) so \((1 + \sqrt{3}i)^3 = 8e^{3i\pi/3} = 8e^{i\pi}\) thus \((1 + \sqrt{3}i)^{3/4} = \sqrt[3]{8} e^{i\pi/4}, \sqrt[3]{8} e^{3i\pi/4}, \sqrt[3]{8} e^{-i\pi/4}, \sqrt[3]{8} e^{-3i\pi/4}\).
   (b) Now \(\sinh z\) is entire (hence continuous at \(z = \pi i/2\) thus \(\lim_{z \to \pi i/2} \sinh(z) = \sinh(\pi i/2) = e^{i\pi/2} - e^{-i\pi/2} = i\). The principal branch of \(\text{Log}\) is analytic at \(z = i\) so by continuity \(\lim_{z \to \pi i/2} \text{Log}(\sinh(z)) = \text{Log}(i) = \frac{i\pi}{2}\)
   (c) \(\lim_{z \to 0} \text{Log}(z)\) does not exist as \(\text{Log} z\) is unbounded as \(z \to 0\) as can be seen by \(|\text{Log}(e^{-n})| = n\) where \(e^{-n} \to 0\) as \(n \to \infty\).
   (d) \(\lim_{z \to \pi} \frac{(z - \pi)}{\sin z}\) is of the form \(0/0\) as \(\sin \pi = 0\) and \((\pi - \pi) = 0\). So evaluating \(\lim_{z \to \pi} \frac{(z - \pi)}{\sin z}\) using L’hopital’s Rule we obtain \(\lim_{z \to \pi} \frac{(z - \pi) e^{iz}}{\sin z} = \frac{d}{dz} \frac{(z - \pi)}{\sin z} |_{z = \pi} = -1\). Also by continuity \(\lim_{z \to \pi} e^{iz} = e^{i\pi} = -1\), thus \(\lim_{z \to \pi} e^{iz} \times -1 = -1 \times -1 = 1\).
   (e) \((-2i)^{-2i} = \exp(-2i \text{Log}(-2i)) = \exp(-2i(\log 2 - i\pi/2)) = \exp(-\pi - i \log 4) = e^{-\pi} e^{-i \log 4}\).

2. We are using \(\epsilon - \delta\) methods to show that \(\lim_{z \to 2} (z^2 - 1) = 3\).
   (a) Show that \(|z - 2| < \min(\frac{\epsilon}{3}, 1) \Rightarrow |(z^2 - 1) - 3| < \epsilon\).
   (b) Find a natural number \(K\) so that \(|z - 2| < \min(\frac{\epsilon}{K}, 2) \Rightarrow |(z^2 - 1) - 3| < \epsilon\).
2. **Solution.**

(a) Let \( w = z - 2 \), so \( z = w + 2 \) substituting gives \( z^2 - 1 - 3 = (w + 2)^2 - 4 = w^2 + 4w \) thus if \( |z - 2| = |w| < \min \left( \frac{\epsilon}{5}, 1 \right) \) then \( |w| < 1 \) and \( |w| < \epsilon/5 \). So \( |(z^2 - 1) - 3| = |w^2 + 4w| = |w||w + 4| \leq |w|(|w| + 4) < \epsilon/5(1 + 4) = \epsilon \) the desired result.

(b) Using \( |w^2 + 4w| = |w||w + 4| \leq |w|(|w| + 4) \) again \( |w| < \min \left( \frac{\epsilon}{K}, 2 \right) \) means \( |w| < 2 \) regardless of the value of \( K \) so \( (|w| + 4) < 2 + 4 = 6 \) so if \( |w| < \epsilon/6 \) also we have \( |w^2 + 4w| < (\epsilon/6) \times 6 = \epsilon \) so any \( K \geq 6 \) does the trick.

Note that no smaller value of \( K \) will work. For example if \( K \leq 5 \) then 3.99 satisfies \( |2 - 3.99| < \min(2, 10/K) \) \( (\epsilon = 10) \) but \( |(3.99^2) - 1 - 3| > 11.84 > 10 = \epsilon \)!

3. **Let** \( u(x, y) = -8x^3y + 8xy^3 \).

   (a) **Show that** \( u(x, y) \) **is harmonic on** \( \mathbb{R}^2 \).

   (b) **Find a harmonic conjugate** \( v(x, y) \) **for** \( u(x, y) \).

   (c) **Hence find an entire function** \( f(z) \) **such that** \( \text{Real}(f(x + iy)) = u(x, y) \). **Write** \( f(z) \) **in terms of** \( z \).
3. Solution.

(a) Note that \( u(x, y) \) is a polynomial in \( x \) and \( y \) and is thus \( C^\infty \) and \( C^2 \) in particular.

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} (-24x^2y + 8y^3) = -48xy.
\]

\[
\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} (-8x^3 + 24xy^2) = 48xy.
\]

Thus \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -48xy + 48xy = 0 \), so \( u(x, y) \) is harmonic.

(b) \( \frac{\partial u}{\partial x} = -24x^2y + 8y^3 \) so \( \frac{\partial v}{\partial y} = -24x^2y + 8y^3 \) so \( v = -12x^2y^2 + 2y^4 + h(x) \).

Differentiating partially with respect to \( x \) \( \frac{\partial v}{\partial x} = -24xy^2 + h'(x) = -\frac{\partial u}{\partial y} = 8x^3 - 24xy^2 \) so \( h'(x) = 8x^3 \Rightarrow h(x) = 2x^4 + c \). Thus the function

\[
v(x, y) = 2x^4 - 12x^2y^2 + 2y^4 + c
\]

is a harmonic conjugate.

(c) So

\[
f(z) = f(x + iy) = (-8x^3y + 8xy^3) + i(2x^4 - 12x^2y^2 + 2y^4)
\]

\[
= 2i(x^4 + 4x^3y + 6i^2x^2y^2 + 4i^3xy^3 + i^4y^4)
\]

\[
= 2i(x + iy)^4
\]

\[
= 2iz^4
\]

4. (a) Using the complex exponential definitions of \( \sin, \cosh, \sinh \) and \( \cos \) show that if \( z = x + iy \) then

\[
\cos z = \cos x \cosh y - i \sin x \sinh y.
\]

(b) Hence (or otherwise) show that

\[
|\cos z| = \sqrt{\cos^2 x + \sinh^2 y}.
\]

(c) Find all the solutions \( z \in \mathbb{C} \) to

\[
\cos z = 0.
\]
4. Solution.

(a) 
\[
\cos x \cosh y - i \sin x \sinh y = \left(\frac{e^{ix} + e^{-ix}}{2}\right) \left(\frac{e^{iy} + e^{-iy}}{2}\right) - i \left(\frac{e^{ix} - e^{-ix}}{2i}\right) \left(\frac{e^{iy} - e^{-iy}}{2}\right)
\]
\[
= \frac{1}{4} \left(\left[ e^{ix+y} + e^{ix-y} + e^{-ix+y} + e^{-ix-y}\right]
- \left[ e^{ix+y} - e^{ix-y} - e^{-ix+y} + e^{-ix-y}\right]\right)
\]
\[
= \frac{1}{4} \left(\left[ e^{ix+y} + e^{ix-y} + e^{-ix+y} + e^{-ix-y}\right]
- \left[ e^{ix+y} - e^{ix-y} - e^{-ix+y} + e^{-ix-y}\right]\right)
\]
\[
= \frac{1}{2} \left( e^{ix-y} + e^{-ix+y}\right)
= \left( e^{iz} + e^{-iz}\right)/2
= \cos z
\]

(b) As \(|u(x, y) + iv(x, y)| = (u^2(x, y) + v^2(x, y))^{1/2}\) we see that
\[
|\cos z| = \sqrt{(\cos x \cosh y)^2 + (\sin x \sinh y)^2}
= \sqrt{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}
= \sqrt{\cos^2 x (\sinh^2 y + 1) + \sin^2 x \sinh^2 y}
= \sqrt{(\cos^2 x + \sin^2 x) \sinh^2 y + \cos^2 x}
= \sqrt{\cos^2 x + \sinh^2 y}.
\]

(c) Now \(\cos z = 0 \Rightarrow |\cos z| = 0 \Rightarrow \cos^2 x + \sinh^2 y = 0 \Rightarrow \cos x = 0 \& \sinh y = 0\). Thus \(y = 0\) and \(x = \frac{2n+1}{2} \pi \ n \in \mathbb{Z}\) giving \(z = \frac{2n+1}{2} \pi \ n \in \mathbb{Z}\). Thus extending the domain of \(\cos\) from \(\mathbb{R}\) to \(\mathbb{C}\) introduces no more zeros!
5. Our aim in this question is to show that Log $z$ (the principal branch) is analytic on the domain

$$D = \{ z \in \mathbb{C} : \text{Im}(z) = 0 \Rightarrow \text{Re}(z) > 0 \} = \{ x+iy : x \in \mathbb{R}, y \in \mathbb{R} \text{ with } y = 0 \text{ only if } x > 0 \}.$$

If $z = re^{i\theta} - \pi < \theta < \pi$ then

$$\text{Log}(x + iy) = \text{Log} z = \log r + i\theta = u(x, y) + iv(x, y)$$

where

$$u(x, y) = \frac{1}{2} \log(x^2 + y^2)$$

and

$$v(x, y) = \begin{cases} 
\arccos \left( \frac{x}{\sqrt{x^2 + y^2}} \right) & y > 0 \\
0 & y = 0 \\
-\arccos \left( \frac{x}{\sqrt{x^2 + y^2}} \right) & y < 0 
\end{cases}$$

(a) Using (without verification) $\frac{d}{dt} \arccos t = -\frac{1}{\sqrt{1-t^2}}$, show that:

(i) $\frac{\partial}{\partial x} \arccos \left( \frac{x}{\sqrt{x^2 + y^2}} \right) = -\frac{|y|}{(x^2 + y^2)}$ note $|y| = \sqrt{y^2}$;

(ii) $\frac{\partial}{\partial y} \arccos \left( \frac{x}{\sqrt{x^2 + y^2}} \right) = \frac{xy}{|y|(x^2 + y^2)}.$

(b) Note that if $x + iy$ in $D$ then $\lim_{y \to 0} \arccos \left( \frac{x}{\sqrt{x^2 + y^2}} \right) = 0$. So we can have the region $y > 0$ ‘creep’ to $y \geq 0$ and $y < 0$ ‘creep’ to $y \leq 0$.

In domain $D$, carefully evaluate:

(i) $\frac{\partial v}{\partial x}$;

(ii) $\frac{\partial v}{\partial y}$.

(c) In domain $D$, verify that:

(i) the Cauchy-Riemann equations hold;

(ii) the functions $u$ and $v$ are $C^1$.

(d) Justify the analyticity of Log on domain $D$. 
5. Solution.

(a) (i)

\[
\frac{\partial}{\partial x} \arccos \left( \frac{x}{\sqrt{x^2 + y^2}} \right) \\
= \frac{\partial}{\partial x} \left( \frac{x}{\sqrt{x^2 + y^2}} \right) \times -1/\sqrt{1 - x^2/(x^2 + y^2)} \\
= \left( \frac{(x^2 + y^2)^{1/2} - 1/2 \times 2x \times (x^2 + y^2)^{-1/2}}{x^2 + y^2} \right) \times -\frac{1}{|y|} (x^2 + y^2)^{1/2} \\
= \left( \frac{(x^2 + y^2) - x^2}{(x^2 + y^2)^{3/2}} \right) \times -\frac{1}{|y|} (x^2 + y^2)^{1/2} \\
= -\frac{|y|^2}{|y|} \frac{1}{x^2 + y^2} \\
= -\frac{|y|}{(x^2 + y^2)}
\]

(ii)

\[
\frac{\partial}{\partial y} \arccos \left( \frac{x}{\sqrt{x^2 + y^2}} \right) \\
= \frac{\partial}{\partial y} \left( \frac{x}{\sqrt{x^2 + y^2}} \right) \times -1/\sqrt{1 - x^2/(x^2 + y^2)} \\
= \left( -1/2 \times 2y \times x \times (x^2 + y^2)^{-3/2} \right) \times -\frac{1}{|y|} (x^2 + y^2)^{1/2} \\
= \frac{xy}{|y|(x^2 + y^2)}
\]

(b) (i) Suppose \( y \geq 0 \) then \( v(x, y) = \arccos \left( x/(\sqrt{x^2 + y^2}) \right) \) so \( \frac{\partial v}{\partial x} = -\frac{|y|}{(x^2 + y^2)} = -\frac{y}{(x^2 + y^2)}. \)

Suppose \( y \leq 0 \) then \( v(x, y) = -\arccos \left( x/(\sqrt{x^2 + y^2}) \right) \) so \( \frac{\partial v}{\partial x} = -1 \times -\frac{|y|}{(x^2 + y^2)} = -1 \times \frac{y}{(x^2 + y^2)}. \) So in either case

\[
\frac{\partial v}{\partial x} = -\frac{y}{(x^2 + y^2)}.
\]
(ii) Suppose \( y \geq 0 \) then
\[
\frac{\partial v}{\partial y} = \frac{xy}{|y|(x^2 + y^2)} = \frac{xy}{y(x^2 + y^2)} = \frac{x}{(x^2 + y^2)}.
\]
Suppose \( y \leq 0 \) then
\[
\frac{\partial v}{\partial y} = -1 \times \frac{xy}{|y|(x^2 + y^2)} = -1 \times \frac{xy}{-y(x^2 + y^2)} = \frac{x}{(x^2 + y^2)}.
\]
So in either case
\[
\frac{\partial v}{\partial y} = \frac{x}{(x^2 + y^2)}.
\]

(c) In domain \( D \):

(i) \( \frac{\partial u}{\partial x} = \frac{\partial 1/2 \log(x^2 + y^2)}{\partial x} = \frac{1}{2} \frac{2x}{x^2 + y^2} = \frac{x}{x^2 + y^2} \). By the symmetry of \( u(x, y) \) in \( x \) and \( y \) we see that \( \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2} \). Thus on \( D \)
\[
\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} = \frac{\partial v}{\partial y}
\]
and
\[
\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2} = -\frac{\partial v}{\partial x},
\]
so the Cauchy Riemann equations hold.

(ii) The partial derivatives calculated above are continuous (on \( D \)) as the partial derivatives are continuous on each half and (by the 'creep') correspond on their join (the positive real axis), thus \( u \) and \( v \) are \( C^1 \) on \( D \).

(d) Hence by the Cauchy Riemann Theorem \( \log \) is differentiable everywhere on the open domain \( D \) (\( u \) and \( v \) are \( C^1 \) and satisfy the Cauchy Riemann equations) and hence is analytic on \( D \).