Recall that the Principal branch of Log is defined on the domain $\mathbb{C}\{ z \in \mathbb{R} - \{0\} \}$ by

$$ \text{Log} z = \log |z| + i \text{Arg} z $$

where $-\pi < \text{Arg} z < \pi$ (for $z \in D$).

If $z = x + iy$ we have

$$ \text{Log}(x + iy) = \log \sqrt{x^2 + y^2} + i \text{Arg}(x + iy) $$

We will use the Cauchy-Riemann Theorem to establish that the principal branch of Log is analytic. But first we must find $v(x, y)$ as an explicit function of $x$ and $y$.

If $z = x + iy = |z|e^{i\theta}$ the following is true:

$$ y > 0 \cos \theta = \frac{x}{\sqrt{x^2 + y^2}} \iff \theta = \arccos \frac{x}{\sqrt{x^2 + y^2}} \text{ as } 0 \leq \arccos \leq \pi. $$

$$ y < 0 \cos \theta = \frac{x}{\sqrt{x^2 + y^2}} \iff \theta = -\arccos \frac{x}{\sqrt{x^2 + y^2}} \text{ as } 0 \geq -\arccos \geq -\pi. $$

$$ x > 0 \tan \theta = \frac{y}{x} \iff \theta = \arctan \frac{y}{x} \text{ as } -\pi/2 < \arctan < \pi/2. $$

Thus we may define

$$ v(x, y) = \text{Arg}(x + iy) = \begin{cases} \arccos \frac{x}{\sqrt{x^2 + y^2}} & \text{for } y > 0 \\ \arctan \frac{y}{x} & \text{for } x > 0 \\ \arccos \frac{x}{\sqrt{x^2 + y^2}} & \text{for } y < 0 \end{cases} $$

Note that if $x > 0$ and $y > 0$ rows 1 and 2 are both applicable but both give the same answer, likewise if $x < 0$ and $y < 0$ rows 2 and 3 are both applicable but both give the same answer.

$$ \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \log \sqrt{x^2 + y^2} = \frac{1}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial x} \sqrt{x^2 + y^2} = \frac{1}{2}x \frac{1}{\sqrt{x^2 + y^2}} = \frac{x}{x^2 + y^2}. $$

By the symmetry of $x, y$ in $u$, $\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$.

The partial derivatives of $u(x, y)$ are continuous (except at $(0, 0) \notin D$) on $D$.

Now we calculate the partial derivatives of $v(x, y)$ (this must be done case wise).

The easy one first $x > 0$:

$$ \frac{\partial v}{\partial x} = \frac{\partial}{\partial x} \arctan(y/x) = \frac{1}{1 + (y/x)^2} \frac{\partial}{\partial x} y/x = \frac{x^2}{x^2 + y^2} \frac{-y}{x^2} = -\frac{y}{x^2 + y^2} $$
\[
\frac{\partial v}{\partial y} = \frac{\partial}{\partial y} \arctan(y/x) = \frac{1}{1 + (y/x)^2} \frac{1}{\partial y} y/x = \frac{x^2}{x^2 + y^2} \frac{1}{x} = \frac{x}{x^2 + y^2}
\]

Now y > 0:

\[
\frac{\partial v}{\partial x} = \frac{\partial}{\partial x} \arccos(x/\sqrt{x^2 + y^2}) = -\frac{1}{\sqrt{1 - x^2/(x^2 + y^2)}} \frac{\partial}{\partial x} x/\sqrt{x^2 + y^2}
\]

\[= -\frac{1}{\sqrt{y^2/(x^2 + y^2)}} [(x^2 + y^2)^{-1/2} - x^2(x^2 + y^2)^{-3/2}] = -\left[\sqrt{x^2 + y^2}/y\right] \frac{y^2/(x^2 + y^2)^{3/2}}{y^2/(x^2 + y^2)^{3/2}} = -\frac{y}{x^2 + y^2}.
\]

\[
\frac{\partial v}{\partial y} = \frac{\partial}{\partial y} \arccos(x/\sqrt{x^2 + y^2}) = -\frac{1}{\sqrt{1 - x^2/(x^2 + y^2)}} \frac{\partial}{\partial y} x/\sqrt{x^2 + y^2}
\]

\[= -\frac{1}{\sqrt{y^2/(x^2 + y^2)}} [-xy(x^2 + y^2)^{-3/2}] = -\left[\sqrt{x^2 + y^2}/y\right] [-xy/(x^2 + y^2)^{3/2}] = \frac{x}{x^2 + y^2}.
\]

Now for y < 0, note that \(\sqrt{y^2} = -y!!:\)

\[
\frac{\partial v}{\partial x} = \frac{\partial}{\partial x} - \arccos(x/\sqrt{x^2 + y^2}) = \frac{1}{\sqrt{1 - x^2/(x^2 + y^2)}} \frac{\partial}{\partial x} x/\sqrt{x^2 + y^2}
\]

\[= \frac{1}{\sqrt{y^2/(x^2 + y^2)}} [(x^2 + y^2)^{-1/2} - x^2(x^2 + y^2)^{-3/2}] = \left[\sqrt{x^2 + y^2}/y\right] \frac{y^2/(x^2 + y^2)^{3/2}}{y^2/(x^2 + y^2)^{3/2}} = -\frac{y}{x^2 + y^2}.
\]

\[
\frac{\partial v}{\partial y} = \frac{\partial}{\partial y} - \arccos(x/\sqrt{x^2 + y^2}) = \frac{1}{\sqrt{1 - x^2/(x^2 + y^2)}} \frac{\partial}{\partial y} x/\sqrt{x^2 + y^2}
\]

\[= \frac{1}{\sqrt{y^2/(x^2 + y^2)}} [-xy(x^2 + y^2)^{-3/2}] = \left[\sqrt{x^2 + y^2}/y\right] [-xy/(x^2 + y^2)^{3/2}] = \frac{x}{x^2 + y^2}.
\]

So in all cases y > 0, x > 0, y < 0

\[
\frac{\partial v}{\partial x} = -\frac{y}{x^2 + y^2}
\]

\[
\frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}
\]

which are continuous on \(D\) and furthermore

\[
\frac{\partial v}{\partial x} = -\frac{y}{x^2 + y^2} = -\frac{\partial u}{\partial y}
\]

\[
\frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2} = \frac{\partial u}{\partial x}
\]

So by the Cauchy-Riemann Theorem

\[u(x, y) + iv(x, y) = \log z \quad \text{Principal branch}\]

is analytic on \(D\).