sin $z$ and arcsin $z$

$$\sin : \mathbb{C} \rightarrow \mathbb{C} \text{ via } \sin z = \left(e^{iz} - e^{-iz}\right)/2i.$$ 

We aim to restrict the domain $D$ and codomain $R$ so that $\sin : D \rightarrow R$ is injective (1-1) and surjective (onto). Then arcsin is the inverse function $R \rightarrow D$.

1 $D$ the domain for injectivity.

We claim that if 

$$D = \left\{ z \in \mathbb{C} : |z| < \frac{\pi}{2} \right\}$$

then $\sin : D \rightarrow \mathbb{C}$ is injective.

Suppose that $\sin z = \sin w$ where both $w$ and $z$ are in $D$, then 

$$\left( e^{iz} - e^{-iz} \right)/2i = \left( e^{iw} - e^{-iw} \right)/2i$$

$$\iff \left[ \left( e^{iz} - e^{-iz} \right) - \left( e^{iw} - e^{-iw} \right) \right]/2i = 0$$

$$\iff 2\left( e^{i(z+w)/2} + e^{-i(z+w)/2} \right)/2 \times \left( e^{i(z-w)/2} - e^{-i(z-w)/2} \right)/2i = 0$$

$$\iff e^{i(z+w)/2} = -e^{-i(z+w)/2} \quad \text{or} \quad e^{i(z-w)/2} = e^{-i(z-w)/2}$$

$$\iff e^{i(z+w)} = -1 \quad \text{or} \quad e^{i(z-w)} = 1$$

$$\iff i(z + w) = 2iK\pi + \pi \quad K \in \mathbb{Z} \quad \text{or} \quad i(z - w) = 2iL\pi \quad L \in \mathbb{Z}$$

$$\iff (z + w) = 2K\pi + \pi \quad K \in \mathbb{Z} \quad \text{or} \quad (z - w) = 2L\pi \quad L \in \mathbb{Z}$$

But the LH option is not possible as $|\Re(z+w)| < \pi$ and likewise for the RH option $L = 0$ is the only possibility. Now $L = 0 \Rightarrow z = w$. So 

$$\sin z = \sin w \quad \text{with} \quad z \text{ and } w \in D \Rightarrow z = w$$

that is $\sin$ is injective on $D$.

2 $R = \sin (D)$ the codomain for surjectivity.

We claim that if 

$$R = \left\{ z \in \mathbb{C} : \Im(z) = 0 \Rightarrow |\Re(z)| < 1 \right\}$$

that is $R$ is the complex plane with the two rays $(-\infty, -1]$ and $[1, \infty)$ of the real axis removed then $\sin : D \rightarrow R$ is injective and surjective. (Injectivity was established in section 1). See the diagram overleaf for a sketch of $D$ and $R$.

For all $u + iv$ in $R$ we wish to show that there is $x + iy$ in $D$ so that $\sin (x + iy) = u + iv$.

Now $\sin (x + iy) = \sin x \cosh y + i \cos x \sinh y = sC + iSc$ where $s = \sin x, c = \cos x, S = \sinh y, C = \cosh Y$ so $u = sC$ and $v = Sc$.

Notice that $u^2 = s^2C^2$ and $u^2 + v^2 + 1 = s^2C^2 + c^2S^2 + 1 = s^2C^2 + c^2(C^2 - 1) + s^2 + c^2 = (s^2 + c^2)C^2 - c^2 + c^2 + s^2 = C^2 + s^2$.

So 

$$(X - C^2)(X - s^2) = X^2 - (C^2 + s^2)X + (C^2 + s^2) = X^2 - (u^2 + v^2 + 1)X + u^2.$$
As \( s^2 < 1 \leq C^2 \) on \( D \) \( s^2 \) is the smaller of the roots of the quadratic \( X^2 - (u^2 + v^2 + 1)X + u^2 \) and \( C^2 \) the larger.

So

\[
C^2 = \left( (u^2 + v^2 + 1) + ((u^2 + v^2 + 1)^2 - 4u^2)^{1/2} \right) / 2
\]

and

\[
s^2 = \left( (u^2 + v^2 + 1) - ((u^2 + v^2 + 1)^2 - 4u^2)^{1/2} \right) / 2.
\]

Taking into account the signs of \( s, c, S, C \) on the four “quadrants” of \( D \) we find that \( \text{sign}(u) = \text{sign}(x) \) and \( \text{sign}(v) = \text{sign}(y) \) so,

\[
x = \text{sign}(u) \times \arcsin \left( \left\{ \left( (u^2 + v^2 + 1) - ((u^2 + v^2 + 1)^2 - 4u^2)^{1/2} \right) / 2 \right\}^{1/2} \right)
\]

and

\[
y = \text{sign}(v) \times \text{arccosh} \left( \left\{ \left( (u^2 + v^2 + 1) + ((u^2 + v^2 + 1)^2 - 4u^2)^{1/2} \right) / 2 \right\}^{1/2} \right).
\]

**Checking Analytically**

Recall that if \( X \) and \( Y \) are both non-negative \( \cos(\arcsin X) = \sqrt{1 - X^2}, \sinh(\arccosh(Y)) = \sqrt{Y^2 - 1} \).

Now with \( X = \left\{ \left( (u^2 + v^2 + 1) - ((u^2 + v^2 + 1)^2 - 4u^2)^{1/2} \right) / 2 \right\}^{1/2} \),

\[
\sqrt{1 - X^2} = \left( 1 - 1/2 \left\{ (u^2 + v^2 + 1) - \sqrt{(u^2 + v^2 + 1)^2 - 4u^2} \right\}^{1/2} \right.
\]

and with \( Y = \left\{ \left( (u^2 + v^2 + 1) + ((u^2 + v^2 + 1)^2 - 4u^2)^{1/2} \right) / 2 \right\}^{1/2} \),

\[
\sqrt{Y^2 - 1} = \left( 1/2 \left\{ (u^2 + v^2 + 1) - \sqrt{(u^2 + v^2 + 1)^2 - 4u^2} \right\} - 1 \right)^{1/2}.
\]

Let’s check \( \sin(x + iy) \) where \( x \) and \( y \) are given as immediately above gives
is

\[
\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y
\]

\[
= \text{sign}(u) \times \left( \arcsin \left[ \left\{ \left( (u^2 + v^2 + 1) - (u^2 + v^2 + 1)^2 - 4u^2 \right)^{1/2} \right\} / 2 \right] \right) \times \text{sign}(v) \times \\
\cosh \left( \text{arccosh} \left[ \left\{ \left( (u^2 + v^2 + 1) + (u^2 + v^2 + 1)^2 - 4u^2 \right)^{1/2} \right\} / 2 \right] \right)
\]

\[
= \text{sign}(u) \left[ \left\{ \left( (u^2 + v^2 + 1) - (u^2 + v^2 + 1)^2 - 4u^2 \right)^{1/2} \right\} / 2 \right] \times \\
\left[ \left\{ \left( (u^2 + v^2 + 1) + (u^2 + v^2 + 1)^2 - 4u^2 \right)^{1/2} \right\} / 2 \right]^{1/2} \times \\
\text{sign}(v) \left[ 1 - 1/2 \left\{ (u^2 + v^2 + 1) - \sqrt{(u^2 + v^2 + 1)^2 - 4u^2} \right\} \right]^{1/2} \times \\
\left[ 1/2 \left\{ (u^2 + v^2 + 1) - \sqrt{(u^2 + v^2 + 1)^2 - 4u^2} \right\} - 1 \right]^{1/2}
\]

Now \((a - b + c)(b - c - a) = -a^2 - b^2 + c^2 + 2ab\) so with \(a = 1, b = (u^2 + v^2 + 1), c = \sqrt{(u^2 + v^2 + 1)^2 - 4u^2}\)

\[
\left[ 1 - 1/2 \left\{ (u^2 + v^2 + 1) - \sqrt{(u^2 + v^2 + 1)^2 - 4u^2} \right\} \right]^{1/2} \left[ \left\{ \left( (u^2 + v^2 + 1) + (u^2 + v^2 + 1)^2 - 4u^2 \right)^{1/2} \right\} / 2 \right]^{1/2}
\]

is \(\sqrt{u^2} = |u|\) by the difference of two squares.

So

\[
\sin(x + iy) = \text{sign}(u)|u| + i\text{sign}(v)|v| = u + iv.
\]

**Checking Numerically**

Let’s use Mathematica to find \(x + iy\) for a value or two of \(u + iv\).

davesarcsin[u_, v_] := Module[{x, y},

    x = Sign[u]*ArcSin[Sqrt[(u^2 + v^2 + 1) - Sqrt[(u^2 + v^2 + 1)^2 - 4 u^2] / 2]];

    y = Sign[v]*

    ArcCosh[Sqrt[(u^2 + v^2 + 1) + Sqrt[(u^2 + v^2 + 1)^2 - 4 u^2] / 2]];

    Return[N[x + I*y]];
]

davesarcsin[Sqrt[3]/2, 0]

1.0472
3 The branch approach to arcsin $z$.

If $\sin z = w$ then $z$ as a function of $w$ will be $\arcsin w$. Recall $\sin z = (e^{iz} - e^{-iz})/(2i)$,

\[
\frac{e^{iz} - e^{-iz}}{2i} = w
\]

\[
\Leftrightarrow e^{iz} - e^{-iz} = (2i)w
\]

\[
\Leftrightarrow (e^{iz})^2 - 1 = (2i)we^{iz}
\]

\[
\Leftrightarrow (e^{iz})^2 - (2iw)e^{iz} - 1 = 0
\]

using the quadratic formula $\Rightarrow e^{iz} = \left(2iw \pm \sqrt{(2iw)^2 + 4}\right)/2$

\[
= iw \pm \sqrt{1 - w^2}
\]

\[
\Rightarrow iz = \log \left(iw \pm \sqrt{1 - w^2}\right)
\]

\[
\Rightarrow z = -i \log \left(iw \pm \sqrt{1 - w^2}\right)
\]

Now the problem we face is that $-i \log \left(iw \pm \sqrt{1 - w^2}\right)$ is NOT a single-valued function (log is multivalued and $\sqrt{}$ is 2-valued) but we can ‘fix’ this with Log the principal value of log on domain $D_{\text{Log}} = \{z \in \mathbb{C} : \text{Im}(z) = 0 \Rightarrow \text{Re}(z) > 0\}$. We can extract one branch of $\sqrt{1 - w^2}$ by $\exp \left[1/2\text{Log}(1 - w^2)\right]$ and use the principal Log in place of the multivalued log to get an analytic function $\arcsin z = -i \log \left(iz + \exp \left[1/2\text{Log}(1 - z^2)\right]\right)$

the catch is we (without work) have no idea of the appropriate domain of the RHS of this definition.

Let $F = \{z \in \mathbb{C} : \text{Im}(z) = 0 \Rightarrow \text{Re}(z) \leq 0\}$ note that $D_{\text{Log}} = \mathbb{C} \setminus F$ the set complement in $\mathbb{C}$ of $F$.

The domain $R$ for $\arcsin$ needs to ensure neither $1 - z^2$ nor $(iz + \exp \left[1/2\text{Log}(1 - z^2)\right])$ lies in $F$ for any $z \in R$.

Let’s tackle $1 - z^2$ first. Suppose $1 - z^2 \in \mathbb{R}$ then $1 - z^2 = 1 - z^2 \Leftrightarrow z^2 - z^2 = (z - \overline{z})(z + \overline{z}) = 0$ so either $z = \overline{z}$ in which case $z$ is real OR $z = -\overline{z}$ in which case $z$ is purely imaginary, $z = iy, y \in \mathbb{R}$ but $z = iy \Rightarrow 1 - z^2 = 1 + y^2 \in \mathbb{R}^+$ so the latter option of $z = iy$ DOES NOT give $1 - z^2 \in F$. If $z = x$ is real $1 - z^2 = 1 - x^2 \leq 0 \Leftrightarrow |x| \geq 1$. We remove these ‘bad’ $z$ to get
$R = \{ z \in \mathbb{C} : \text{Im}(z) = 0 \Rightarrow |\text{Re}(z)| < 1 \}$ as the appropriate domain for $\exp \left[ \frac{1}{2} \log(1 - z^2) \right]$.

We now need to look at the possibility of $(iz + \exp \left[ \frac{1}{2} \log(1 - z^2) \right])$ lying in $F$.

\[
(iz + \exp \left[ \frac{1}{2} \log(1 - z^2) \right]) = r \in \mathbb{R} \\
\Rightarrow \exp \left[ \frac{1}{2} \log(1 - z^2) \right] = r - iz \\
\Rightarrow (\exp \left[ \frac{1}{2} \log(1 - z^2) \right])^2 = (r - iz)^2 \\
\Rightarrow 1 - z^2 = r^2 - 2riz - z^2 \\
\Rightarrow 1 = r^2 - 2riz \\
\Rightarrow z = \frac{r^2 - 1}{2ri}
\]

So $z$ is thus imaginary. If $z = iy$ where $y \in \mathbb{R}$

\[
(iz + \exp \left[ \frac{1}{2} \log(1 - z^2) \right]) = iy + \sqrt{1 + y^2} \text{ the usual real } \sqrt{1 + y^2} = \sqrt{1 + y^2 - y} > 0 \ \forall y \ (\text{as } y \leq |y| = \sqrt{y^2} < \sqrt{1 + y^2})
\]

Thus no $z \in \mathbb{C}$ gives $(iz + \exp \left[ \frac{1}{2} \log(1 - z^2) \right])$ lying in $F$.

So $R$ is the domain of analyticity for

\[
\arcsin z = -i \log (iz + \exp \left[ \frac{1}{2} \log(1 - z^2) \right]) .
\]

4 Some Notes

• This complex domain $D$ for arcsin contains the domain we are used to using for real arcsin (except the end points $-1$ and $1$ are missing to ensure analyticity).

• A different choice of branch of log gives rise to a different domain. For example if we choose $\log^{*}$ defined by insisting $0 < \text{Arg} < 2\pi$ (so for example $\log^{*}(-i) = 3\pi/2$) we would get domain $R^{*} = \{ z \in \mathbb{C} : \text{Im}(z) = 0 \Rightarrow |\text{Re}(z)| > 1 \}$. That is the excised region (the branch cut) is along the real axis from $-1$ to $1$.

• Yet another choice of branch, $\log^{\#}$ defined by insisting $-\pi/2 < \text{Arg} < 3\pi/2$ has the domain $R^{\#} = \mathbb{C} \setminus \{ \text{pair of curves} \}$ where (pair of curves) = $\{ \cosh t + i \sinh t : t \geq 0 \} \cup \{ -\cosh t + i \sinh t : t \leq 0 \}$.

• All these domains are topologically equivalent on the Riemann sphere: the single branch cut in each case joins $-1$ to $1$. In the case of $R$ and $R^{\#}$ the cut goes through the point at $\infty$, whereas in the case of $R^{*}$ the cut goes through $0$. 