This paper has 13 pages.

The total number of marks allocated is 120.

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1. Solution.

(a) 
\[ 1033 = 10 \times 100 + 33 \]
\[ 100 = 3 \times 33 + 1 \]

(b) 
\[ 1 = 100 - 3 \times 33 \]
\[ = 100 - 3 \times (1033 - 10 \times 100) \]
\[ = 31 \times 100 - 3 \times 1033 \]

Thus in \( \mathbb{Z}_{1033} \), \( 1 = 31 \times 100 - 3 \times 1033 = 31 \times 100 \) meaning \( 100^{-1} = 31 \) in \( \mathbb{Z}_{1033} \).

(c) 
\[ 100x = -3 \]
\[ \Rightarrow 31 \times 100x = 31 \times -3 \]
\[ \Rightarrow x = -93 \]
\[ = 940 \]

(d) The inverse of 15 in \( \mathbb{Z}_{85} \) does not exist as \( \gcd(15, 85) = 5 \neq 1 \).
2. Solution.

Let $S_n$ be the proposition that $A^n u = \lambda^n u$ where $n \in \mathbb{Z}^+$.

Base case
$S_1$: $A^1 u = \lambda^1 u$ as $u$ is an eigenvector of $A$ with eigenvalue $\lambda$ ($Au = \lambda u$).

Inductive step
Assume $S_k$ is true (where $k \in \mathbb{Z}^+$), that is

\[
A^k u = \lambda^k u \\
\text{thus } A^{k+1} u = A(A^k u) \\
= A(\lambda^k u) \quad (\text{by } S_k) \\
= \lambda^k (Au) \\
= \lambda^k \lambda u \\
= \lambda^{k+1} u
\]

this is $S_{k+1}$.
So $S_k \Rightarrow S_{k+1}$.

So by the principle of mathematical induction $S_n$: $A^n u = \lambda^n u$ for any positive integer $n$. 
3. Solution.

(a) \( n = \phi(m) = \phi(3 \times 29) = (3 - 1) \times (29 - 1) = 2 \times 28 = 56. \)

(b) Encrypting the message ‘16’ is done by \( 16^{11} \) in \( \mathbb{Z}_{87} \).

\[
\begin{array}{ccc}
16 & 11 \\
82 & 5 \\
25 & 2 \\
16 & 1 \\
\end{array}
\]

So (looking at the odd entries in the RH column) \( 16^{11} = 16 \times 82 \times 16 = 16^2 \times 82 = 82 \times 82 = 82^2 = 25 \). Thus ‘16’ is encrypted as ‘25’.

(c) The decrypting key \( d \) needs to be the inverse of \( e \mod n \) (NOT \( \mod m \)! \( 11 \times 51 = 561 = 561 - 10 \times 56 = 1 \) in \( \mathbb{Z}_{56} \). Whereas \( 11 \times 8 = 88 = 32 \neq 1 \) in \( \mathbb{Z}_{56} \). Thus the appropriate decrypting key is \( d = 51 \).

(d) We need \( 62^{51} = (-25)^{51} = -(25^{51}) \).

\[
\begin{array}{ccc}
25 & 51 \\
16 & 25 \\
82 & 12 \\
25 & 6 \\
16 & 3 \\
82 & 1 \\
\end{array}
\]

So (looking at the odd entries in the RH column) \( 25^{51} = 25 \times 16 \times 16 \times 82 = 25 \times 16^2 \times 82 = 25 \times 82 \times 82 \times 82 = 25 \times 82^2 = 25 \times 25 = 25^2 = 16 \), so \( -(25^{51}) = -16 = 71 \) in \( \mathbb{Z}_{87} \). Thus ‘62’ is decrypted as ‘71’.

4. Solution. Current algorithms and computing power mean that prime-factorising a 400 digit \( m \) to find \( p \) and \( q \) may well take more than a lifetime. Note that we need \( p \) and \( q \) to find \( n = (p - 1)(q - 1) \) to in turn find \( d = e^{-1} \) in \( \mathbb{Z}_n \).

5. Solution.

(a) As \( 3^6 = 1 \) the order of 3 divides 6, thus the possibilities are 1, 2, 3 or 6. Now \( 3^1 \neq 1, 3^2 = 9 \neq 1 \) but \( 3^3 = 27 \neq 1 \). So the order of 3 is 3.

(b) Now \( \phi(26) = \phi(2 \times 13) = (2 - 1) \times (13 - 1) = 12 \), so (by Euler’s Theorem) the order of any unit divides 12, giving 1, 2, 3, 4, 6, 12 as the possibilities.

Now \( 15^6 = 3^6 \times 5^6 = (3^3)^2 \times 5^4 \times 5^2 = 1 \times 1 \times -1 \neq 1 \). Thus the order of 15 is not 6 (or anything dividing 6). Also \( 15^4 = 3^4 \times 5^4 = 3^3 \times 3^1 \times 1 = 1 \times 3 = 3 \neq 1 \) so the order of 15 is not 4 (or anything dividing 4). This leaves \( 1, 2, 3, 6, 12 \) as the only possibility. As the order of 15 in \( \mathbb{Z}_{26} \) is \( \phi(26) \) it is a primitive unit.

(c) \( 13 \equiv 1 \mod \phi(26) \) so by the extended Eulers Theorem \( a^{13} = a \) in \( \mathbb{Z}_{26} \) for any \( a \).
7. Solution.

(a) The rank of $A$ is 3.

(b) The first, second and fourth columns of the original matrix,

$$\begin{bmatrix}
1 & -2 & 1 \\
2 & 4 & 2 \\
-7 & 3 & 1 \\
4 & -2 & 0
\end{bmatrix}.$$ 

(c) The dimension of the row space of $A$ is the rank of $A$ which is 3.

(d) The rows of $A$ are not linearly independent – it is impossible for 4 vectors to be linearly independent in a space of dimension $< 4$.

(e) The non-trivial rows of $B$ form a basis for the row space of $A$,

$$\{(1, 0, 1, 0, 1), (0, 1, -1, 0, 0), (0, 0, 0, 1, 4)\}.$$ 

(f) No the vectors $(1, -2, 1, 2), (-2, 4, 3, -7), (3, -6, -2, 9), (1, -2, 1, 0), (5, -10, 5, 2)$ do not span $\mathbb{R}^4$ – these vectors are the columns of $A$ and their span (the column space) is only of dimension 3. Give a reason.

(g) $(3, -6, -2, 9) = 1(1, -2, 1, 2) - 1(-2, 4, 3, -7) -$ see column 3 of $B$.

(h) Let $x_3 = s$ and $x_5 = t$ (the third and fifth columns of $B$ have no row leaders). Row 3 $\Rightarrow x_4 = -4t$. Row 2 $\Rightarrow x_2 = s$. Row 1 $\Rightarrow x_1 + s + t = 0 \Rightarrow x_1 = -s - t$. Thus to be in the solution space $(x_1, x_2, x_3, x_4, x_5) = (-s - t, s, -4t, t) = s(-1, 1, 1, 0, 0) + t(-1, 0, 0, -4, 1)$. So a basis for the solution space is

$$\begin{bmatrix}
-1 \\
1 \\
0 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
-1 \\
0 \\
1 \\
0 \\
-4
\end{bmatrix}.$$ 

(i) The kernel of $T$ is the solution space of $A$ thus a basis is the one given to the previous question part.

(a) Now the vector $w = (-1, -1, -1)$ is in $P$ as $(-1) + (-1) + (-1) \leq 0$. But $-1w = (1, 1, 1)$ is not in $P$ as $1 + 1 + 1 \not\leq 0$! So $P$ isn’t closed under scalar multiplication and is not a subspace of $\mathbb{R}^3$.

(b) Let

$$T : \mathbb{R}^5 \rightarrow \mathbb{R}^4$$

be a linear transformation.

Suppose that $T(u) = u', T(v) = v'$.

(i) $T(u + v) = T(u) + T(v) = u' + v'$ and $T(\alpha u) = \alpha T(u) = \alpha u'$.

(ii) $\text{Ker}(T) = \{w \in \mathbb{R}^5 : T(w) = 0 \ (\in \mathbb{R}^4)\}$.

(iii) $\text{Ker}(T)$ contains $0 \in \mathbb{R}^5$ as $T(0) = T(0w) = 0T(w) = 0 \in \mathbb{R}^4$. So $\text{Ker}(T)$ is non-empty.

Suppose both $u$ and $v$ are in $\text{Ker}(T)$ then $T(u) = 0$ and $T(v) = 0$ so (with $u' = 0$ and $v' = 0$) we see $T(u + v) = 0 + 0 = 0$. Thus $u + v$ is $\text{Ker}(T)$.

Thus $\text{Ker}(T)$ is closed under addition.

$T(\alpha u) = \alpha T(u) = \alpha 0 = 0$ so $\alpha u$ is in $\text{Ker}(T)$ and $\text{Ker}(T)$ is closed under scalar multiplication.

Thus by the subspace Theorem $\text{Ker}(T)$ is a subspace of $\mathbb{R}^5$. 
(a) The word 0001111 is a codeword as
\[ B[0 0 0 0 1 1 1]^T = [0 0 0 0]^T \]
whereas the other two words are not codewords as they have non-zero syndrome.

(b) The dimension of $C$ is 3 so there are $|\mathbb{Z}_2|^3 = 2^3 = 8$ codewords.

(c) The minimum distance of the code is 4 so the code can:

(i) correct 1 error (but no more than 1);
(ii) detect 3 errors.

(d) 1000000 $\rightarrow$ 0000000 (the zero word is always a codeword), 0001011 has syndrome 0111 which is the 5th column of the check matrix (so the single error is in the 5th bit) thus 0001011 $\rightarrow$ 0001111. (0001111 is a codeword).

(e) The rank of the matrix $B$ is 4 as it is already in RREF form. Thus by the rank/nullity theorem the dimension of the solution space is (the number of columns - the rank) $7-4 = 3$. The solution space of $B$ is exactly the code $C$ thus $C$ has dimension 3.

(f) The word 1100000 has at least 2 errors as the syndrome 1100\(^T\) is Not 0000\(^T\) (hence at least one error is made) and is NOT a column of the check matrix – if only one error is made in the $i$th bit then the syndrome is $B_i$ the $i$th column of the check matrix $B$. 

(a) (i) With the data points $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & -2 & 1 & 2 & 3 \\ 5 & 5 & 2 & 0 & -2 \end{pmatrix}$ we have $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -4 & -2 & 1 & 2 & 3 \end{pmatrix}^T$.

So

$$A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -4 & -2 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -4 \\ 1 & -2 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 \\ 0 & 34 \end{pmatrix},$$

$$A^T y = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -4 & -2 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 5 \\ 5 \\ 2 \\ 0 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 10 \\ -34 \end{pmatrix}$$

Solving

$$A^T A \bar{u} = A^T y \quad \text{for} \quad \bar{u} = \begin{pmatrix} a \\ b \end{pmatrix}$$

gives

$$5a = 10$$

$$34b = -34$$

So $a = 2, b = -1$.

So the line of (least squares) best fit is $y = 2 - x$.

(ii) The points on the line are $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 & 4 & 1 & 0 & -1 \end{pmatrix}$ giving a least squares error of $(6 - 5)^2 + (4 - 5)^2 + (1 - 2)^2 + (0 - 0)^2 + (-1 - -2)^2 = 4$.

(b) $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -4 & -2 & 1 & 2 & 3 \\ 16 & 4 & 1 & 4 & 9 \end{pmatrix}^T$

10. Solution.

(a) (i) The properties $\langle \ , \ \rangle$ must satisfy (to be an inner product on $\mathbb{R}^2$) are:

1 $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v$ in $\mathbb{R}^2$.

2 $\langle u, \alpha v \rangle = \alpha \langle u, v \rangle$ for all $\alpha \in \mathbb{R}$ and for all $u, v$ in $\mathbb{R}^2$.

3 $\langle u, (v + w) \rangle = \langle u, v \rangle + \langle u, w \rangle$ for all $u, v, w$ in $\mathbb{R}^2$.

4a $\langle u, u \rangle \geq 0$ for all $u$ in $\mathbb{R}^2$.

4b $\langle u, u \rangle = 0$ if and only if $u = 0$ in $\mathbb{R}^2$. 
(ii) $\| (3, 2) \| = \left( \langle (3, 2), (3, 2) \rangle \right)^{1/2} = \left( 3 \times 3 - 3 \times 2 - 2 \times 3 + 3 \times 2 \times 2 \right)^{1/2} = \sqrt{9} = 3$

and

$\langle (3, 2), (-3, 1) \rangle = (3 \times -3 - 2 \times -3 - 3 \times 1 + 3 \times 2 \times 1) = (-9 + 6 - 3 + 6) = 0$.

(b) Show (by exhibiting an inner product property that fails) that the following formula DOES NOT define an inner product on $\mathbb{R}^2$:

$\langle (1, 0), (1, 0) \rangle = 1 \times 0 + 0 \times 1 = 0$ but $(1, 0) \neq 0$ so property 4b fails – positive definiteness. $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1y_2 + x_2y_1$. 
11. Solution.

(a) 
\[
\begin{align*}
\mathbf{u}_1 \cdot \mathbf{u}_1 &= \frac{1}{9}((-2)^2 + 2^2 + 1^2 + 0^2) = \frac{9}{9} = 1 \\
\mathbf{u}_1 \cdot \mathbf{u}_2 &= \frac{1}{9}((-2) \times 2 + 2 \times 2 + 1 \times 0 + 0 \times 1) = 0 \\
\mathbf{u}_2 \cdot \mathbf{u}_2 &= \frac{1}{9}(2^2 + 2^2 + 0^2 + 1^2) = \frac{9}{9} = 1
\end{align*}
\]

Thus the set of vectors is orthogonal and of unit length meaning the set is orthonormal.

(b) 
\[
\begin{align*}
\mathbf{v}_1 &= \frac{1}{3}(-2, 2, 1, 0) \quad \rightarrow \quad \mathbf{u}_1 = \frac{1}{3}(-2, 2, 1, 0) \\
\mathbf{v}_2 &= \frac{1}{3}(2, 2, 0, 1) \quad \rightarrow \quad \mathbf{u}_2 = \frac{1}{3}(2, 2, 0, 1)
\end{align*}
\]
the first two vectors form an orthonormal set from (a)
\[
\begin{align*}
\mathbf{v}_3 &= (-2, 5, 4, 3) \quad \rightarrow \quad \mathbf{w}_3 &= \mathbf{v}_3 - (\mathbf{u}_1 \cdot \mathbf{v}_3)\mathbf{u}_1 - (\mathbf{u}_2 \cdot \mathbf{v}_3)\mathbf{u}_2 \\
&= (-2, 5, 4, 3) - ((-2, 5, 4, 3) \cdot \frac{1}{3}(-2, 2, 1, 0))\frac{1}{3}(-2, 2, 1, 0) \\
&\quad - ((-2, 5, 4, 3) \cdot \frac{1}{3}(2, 2, 0, 1))\frac{1}{3}(2, 2, 0, 1) \\
&= (-2, 5, 4, 3) - 2(-2, 2, 1, 0) - (2, 2, 0, 1) = (0, -1, 2, 2) \\
&\quad \rightarrow \quad \mathbf{u}_3 = \frac{1}{3}(0, -1, 2, 2)
\end{align*}
\]

(c) We use the properties of our orthonormal basis
\[
\mathbf{w} = \sum (\mathbf{w} \cdot \mathbf{u}_i) \mathbf{u}_i
\]
Now \((0, 3, 3, 3) \cdot \frac{1}{3}(-2, 2, 1, 0) = 3, (0, 3, 3, 3) \cdot \frac{1}{3}(2, 2, 0, 1) = 3, (0, 3, 3, 3) \cdot \frac{1}{3}(0, -1, 2, 2) = 3 \)
so
\[
(0, 3, 3, 3) = 3\mathbf{u}_1 + 3\mathbf{u}_2 + 3\mathbf{u}_3.
\]
12. Solution. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be shears along the $y$-axis and $x$-axis respectively given by

$$
S \left[ \begin{array}{c} x \\ y \end{array} \right] = \left[ \begin{array}{cc} 1 & -4 \\ 0 & 1 \end{array} \right] \left[ \begin{array}{c} x \\ y \end{array} \right] \quad \text{and} \quad T \left[ \begin{array}{c} x \\ y \end{array} \right] = \left[ \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right] \left[ \begin{array}{c} x \\ y \end{array} \right].
$$

(a) $S \circ T$ has the standard matrix representation

$$
\left[ \begin{array}{cc} 1 & -4 \\ 0 & 1 \end{array} \right] \left[ \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right] = \left[ \begin{array}{cc} -3 & -4 \\ 1 & 1 \end{array} \right].
$$

(b) The image with respect to $S \circ T$ of the vector $\left[ \begin{array}{c} 1 \\ 1 \end{array} \right]$ is

$$
\left[ \begin{array}{cc} -3 & -4 \\ 1 & 1 \end{array} \right] \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] = \left[ \begin{array}{c} -7 \\ 2 \end{array} \right].
$$

(c) As $S \circ T : [1 \ 1]^T \to [-7 \ 2]^T$ the line $y = x$ maps to $2x + 7y = 0$. 
13. Solution.

(a) \[
\begin{bmatrix}
3 & 4 \\
-2 & -3
\end{bmatrix}
\].

(b) The transition matrix \( P_{B\rightarrow S} \) from \( B \) to \( S \) is \[
\begin{bmatrix}
2 & -1 \\
-1 & 1
\end{bmatrix}
\].

(c) The transition matrix \( P_{S\rightarrow B} \) from \( S \) to \( B \) is \((P_{B\rightarrow S})^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \).

(d) We want \([w]_B, [w]_S = P_{S\rightarrow B}[w]_S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -7 \\ 11 \end{bmatrix} = \begin{bmatrix} 4 \\ 15 \end{bmatrix} \), so
\[
\begin{bmatrix} -7 \\ 11 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 15 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.
\]

(e) \[
[T]_{B\rightarrow B} = P_{S\rightarrow B}[T]_{S\rightarrow S}P_{B\rightarrow S} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]

(f) \([T(w)]_B = [T]_{B\rightarrow B}[w]_B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 15 \end{bmatrix} = \begin{bmatrix} 4 \\ -15 \end{bmatrix} \).


\[
\det (A - \lambda I) = \det \left( \begin{bmatrix} 5 - \lambda & 6 \\ -3 & -4 - \lambda \end{bmatrix} \right)
\]

\[
= (5 - \lambda)(-4 - \lambda) - (-3 \times 6)
\]

\[
= \lambda^2 - \lambda - 2
\]

if and only if \( \lambda \in \{-1, 2\} \). So the eigenvalues are 2 and -1.

\( \lambda = 2 \)
\[
\begin{bmatrix}
5 - \lambda & 6 \\
-3 & -4 - \lambda
\end{bmatrix} = \begin{bmatrix} 3 & 6 \\
-3 & -6
\end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\
0 & 0
\end{bmatrix}
\]

which has non-zero vector \([2 & -1]^T\) in its solution space. Thus \([2 & -1]^T\) is an eigenvector for \( \lambda = 2 \).

\( \lambda = -1 \)
\[
\begin{bmatrix}
5 - \lambda & 6 \\
-3 & -4 - \lambda
\end{bmatrix} = \begin{bmatrix} 6 & 6 \\
-3 & -3
\end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\
0 & 0
\end{bmatrix}
\]

which has non-zero vector \([1 & -1]^T\) in its solution space. Thus \([1 & -1]^T\) is an eigenvector for \( \lambda = -1 \).
15. Solution.

(a) \[
\begin{bmatrix}
7 & 10 & -2 \\
10 & 4 & -8 \\
-2 & -8 & -2
\end{bmatrix}
\begin{bmatrix}
2 \\
-1 \\
2
\end{bmatrix}
= \begin{bmatrix}
36 \\
-18 \\
0
\end{bmatrix}
= 18 \begin{bmatrix}
2 \\
-1 \\
2
\end{bmatrix}
\]
so \(v_1\) is an eigenvector with eigenvalue 18.

\[
\begin{bmatrix}
7 & 10 & -2 \\
10 & 4 & -8 \\
-2 & -8 & -2
\end{bmatrix}
\begin{bmatrix}
2 \\
-1 \\
2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]
so \(v_2\) is an eigenvector with eigenvalue 0.

\[
\begin{bmatrix}
7 & 10 & -2 \\
10 & 4 & -8 \\
-2 & -8 & -2
\end{bmatrix}
\begin{bmatrix}
-1 \\
2 \\
2
\end{bmatrix}
= \begin{bmatrix}
9 \\
-18 \\
9
\end{bmatrix}
= -9 \begin{bmatrix}
2 \\
-2 \\
2
\end{bmatrix}
\]
so \(v_3\) is an eigenvector with eigenvalue -9.

(b) \[D = P^{-1}AP\]
where
\[
P = \begin{bmatrix}
2 & 2 & -1 \\
2 & -1 & 2 \\
-1 & 2 & 2
\end{bmatrix}
\]
and
\[
D = \begin{bmatrix}
18 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -9
\end{bmatrix}
\]

Note that \(P^{-1}\) exists as eigenvectors for different eigenvalues are linearly independent.

(c) We merely need to normalize the orthogonal set of eigenvectors that formed the columns of \(P\) thus
\[
Q = \frac{1}{3} \begin{bmatrix}
2 & 2 & -1 \\
2 & -1 & 2 \\
-1 & 2 & 2
\end{bmatrix}
\]

(d) The matrix \(A\) is symmetric with distinct eigenvalues thus the eigenvectors \(v_1, v_2, v_3\) are orthogonal.

(e) \(\langle u, v \rangle = u^T A w\) does not define an inner product on \(\mathbb{R}^3\) as (choosing an eigenvector for a negative eigenvalue)
\[
\langle v_3, v_3 \rangle = v_3^T A v_3 = -9v_3^T v_3 = -81.
\]

So the alleged inner product is not positive definite.