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620-252 Analysis

Tutorial 10 - Taylor series and Series Manipulation.

> IF a power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges uniformly (on $|z-a| \le r$ say) to $f(z)$.
> THEN this sum function $f(z)$ must be analytic and moreover the TAYLOR SERIES of $f(z)$ about centre $z=a$ is the original power series $\sum_{n=0}^{\infty} a_n(z-a)^n$.

1. Using techniques other than evaluating $\frac{f^{(n)}(a)}{n!}$ (in particular geometric series are useful) calculate the Taylor series of the following functions around the given centres and give the disk of convergence.
   (a) $\frac{1}{1-2z}$ around $z = 0$;
   (b) $\frac{1}{z}$ around $z = 1$;
   (c) $\frac{1}{2-z}$ around $z = 0$;
   (d) $\frac{1}{3+z}$ around $z = 0$;
   (e) $\frac{5}{6-z-z^2}$ around $z = 0$;
   (f) $\frac{2}{1-z^2}$ around $z = 2$.

2. Using known McLaurin Series and simple series manipulation find McLaurin series for the following functions
   (a) $(1-\cos z)$;
   (b) $\left(\frac{1-\cos z}{z^2}\right)$;
   (c) $\frac{\sin z}{z}$.
   (d) For the functions $f(z)$ in (b) and (c) use your Mclaurin Series to find $\lim_{x\to 0} f(z)$.
   (e) What kind of singularity is $z = 0$ for both these functions?

The McLaurin Series for Log$(1+z)$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} \ldots$

3. Using the McLaurin Series for Log:
   (a) Calculate $\sum_{n=1}^{\infty} \frac{1}{n2^n}$.
   (b) Give a series that converges to Log(5).

THE CAUCHY PRODUCT:
IF
the power series about centre $z = a$ $\sum_{n=0}^{\infty} a_n (z-a)^n$ and $\sum_{n=0}^{\infty} b_n (z-a)^n$ converge to $f(z)$ and $g(z)$ on the disks $|z-a| < R_f$ and $|z-a| < R_g$ respectively
THEN
the power series (Taylor series) of $f(z)g(z)$ about centre $z = a$ is

$$\sum_{n=0}^{\infty} c_n (z-a)^n$$

where $c_n = \sum_{k=0}^{n} a_k \times b_{n-k}$ which converges on $|z-a| < \min(R_f, R_g)$.

This isn't as scary as it looks – it means we multiply series like polynomials (that go on for ever).

4. Using the CAUCHY PRODUCT;
   (a) Find the Mclaurin series for $\frac{1+z^2}{1-z}$.
   (b) We are investigating the Mclaurin series for $z \cot z$.
      i. Write down the Mclaurin series for $\cos z$.
      ii. Write down the Mclaurin series for $\sin z$.
      iii. Write down the Mclaurin series for $z \cos z$.
      iv. Suppose that $T(z) = t_0 + t_1 z + t_2 z^2 + \ldots$ is the Mclaurin series for $z \cot z = \frac{z \cos z}{\sin z}$. It must be the case that
      $\left(t_0 + t_1 z + t_2 z^2 + \ldots\right) \sin z = z \cos z$
      equate coefficients (using the Cauchy product to calculate the coefficient of z^1 on the left) to find t_0.
      equate coefficients (using the Cauchy product to calculate the coefficient of z^2 on the left) to find t_1.
      Iterate this process to find t_2, t_3 and t_4.

Using essentially this series and process to calculate coefficients and residue calculus around a cunning contour it can be shown (we will do one or two cases) that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} \cdots \quad \sum_{n=1}^{\infty} \frac{1}{n^{2m}} = \frac{\pi^{2m}}{\text{rational}} \cdots$$