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620-252 Analysis

Tutorial 11 - Laurent series, Residues and Residue Calculus.

If ζ is a singularity of $f(z)$ and
if there is some non-negative integer m so that

$$\lim_{z\to\zeta}(z-\zeta)^m f(z) \quad \text{exists}$$

then ζ is a POLE of $f(z)$ and it has ORDER m where m has the least value such that the limit exists.
If $m=0$ then we say the singularity is REMOVABLE.
If no m exists then we say the singularity is ESSENTIAL.

1. Find and classify the singularities of the following functions:
   (a) $\dfrac{z^3-z+1}{(z+1)(z-2)^2(z+3)^3}$;
   (b) $\dfrac{z^4-1}{(z+i)^5 z}$;
   (c) $\dfrac{\sin z}{z(z+2)(z-\pi)^2}$.

$$\sum_{n=-\infty}^{\infty} a_n(z-a)^n = \sum_{n=-\infty}^{-1} a_n(z-a)^n + \sum_{n=0}^{\infty} a_n(z-a)^a$$

is a LAURENT SERIES which where it converges (on an open region) will converge to an analytic function $f(z)$.
Suppose $\sum_{n=-\infty}^{-1} a_n(z-a)^n$ converges for $|z-a|>r$ and $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges for $|z-a|<R$ and $r<R$ then the Laurent series converges inside the annulus $r<|z-a|<R$.
(This is the region where BOTH parts of the Laurent series converge.)

Furthermore if a is a pole of order m of $f(z)$, the Laurent series expansion of $f(z)$ in $0<|z-a|<R$ a PUNCTURED DISK around centre a will be of the form

$$\sum_{n=-m}^{\infty} a_n(z-a)^n$$

and the coefficient a_{-1} of $(z-a)^{-1}$ is the RESIDUE of $f(z)$ denoted $\mathrm{Res}\,(f;a)$.

2. Let $f(z)=\dfrac{z+1}{z^2-z}=\dfrac{2}{z-1}-\dfrac{1}{z}$.
   (a) Using geometric series techniques or otherwise find a Laurent series expansion for $f(z)$ valid for:
      i. $0<|z-1|<1$;
      ii. $0<|z|<1$;
      iii. $1<|z-2|<2$.
   (b) By looking at the appropriate Laurent series verify that $\mathrm{Res}\,(f;0)=-1$ and $\mathrm{Res}\,(f;1)=2$.

If $f(z)$ is a function with isolated singularities then provided $f(z)$ is analytic on the simple closed contour Γ

$$\oint_\Gamma f(z)dz = 2\pi i \left(\sum_{\zeta \text{ singularities interior to } \Gamma} \operatorname{Res}(f;\zeta) \right)$$

by the RESIDUE THEOREM calculation of integrals around closed contours by this method is known as RESIDUE CALCULUS.

QUESTION 2 CONTINUED...

(c) Using residue calculus calculate:

i. $\oint_{|z|=1/2} f(z)dz;$

ii. $\oint_{|z-2|=3/2} f(z)dz;$

iii. $\oint_{|z-5|=1} f(z)dz.$

where $f(z) = \dfrac{z+1}{z^2-z} = \dfrac{2}{z-1} - \dfrac{1}{z}$ as before.

Many real integrals (some of which can not be calculated using real techniques) can be calculated using a 'cunning closed contour' and residue calculus.

3. Let $f(z) = \dfrac{1}{(1+z^2)^3}$,

(a) Find the singularities of $f(z)$ and calculate the residues at these singularities.

(b) Let the contour $\Gamma_R = X_R + \gamma_R$ where on $X_R: \ z = -R + 2tR \ t \in [0,1]$ and on $\gamma_R: \ z = Re^{is} \ s \in [0,\pi]$. Calculate

$$\oint_{\Gamma_R} \frac{1}{(1+z^2)^3} dz \qquad \text{where } R > 1$$

(c) Assuming $\lim_{R\to\infty} \int_{\gamma_R} f(z)dz = 0$ find

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^3} dx.$$

(d) Using

$$\left| \int_\gamma f(z)dz \right| \leq |\gamma| M$$

where $|f(z)| \leq M$ for all z on the contour γ and $|\gamma|$ is the length of the contour γ show that

$$\lim_{R\to\infty} \int_{\gamma_R} f(z)dz = 0.$$