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620-252 Analysis

Tutorial 11 - Solutions.

1. Find and classify the singularities of the following functions:

   (a) $\dfrac{z^3-z+1}{(z+1)(z-2)^2(z+3)^3}$ the denominator has zeros at $-1, 2, -3$ of order $1, 2, 3$ respectively thus prima facie the function has poles at $-1, 2, -3$ of order $1, 2, 3$ unless the numerator has canceling zeros at these points. The numerator does not have zeros at $-1, 2, -3$ $\left(z^3-z+1|_{-1}=1, z^3-z+1|_2=7, z^3-\right.$ Thus the function has poles at $-1, 2, -3$ of order $1, 2, 3$ respectively.

   (b) $\dfrac{z^4+1}{(z+i)^5 z}$ at first glance it seems the function has poles of order 5 at $-i$ and order 1 (simple) at 0. however $-i$ is a zero of order 1 of the numerator so $-i$ is a pole of order $5-1=4$ lets check

   If ζ is a pole of $f(z)$ of order m then
   $$\lim_{z\to\zeta}(z-\zeta)^k f(z)=\begin{cases}\infty & k<m\\ w\in\mathbb{C} & k=m\\ 0 & k>m\end{cases}$$

   $\lim_{z\to 0} z^{\mathbf{1}}\dfrac{z^4+1}{(z+i)^5 z}=\lim_{z\to 0}\dfrac{z^4+1}{(z+i)^5}=1/i=-i$ so as the limit is non infinite and nonzero the order of the pole $z=0$ is indeed 1.
   Let's exhibit the property alleged in the box above
   $\lim_{z\to -i}(z+i)^{\mathbf{5}}\dfrac{z^4+1}{(z+i)^5 z}=\lim_{z\to -i}\dfrac{z^4+1}{z}=0/(-i)=0$ so multiplying by $(z+i)^5$ was too much as we got a zero limit.
   $\lim_{z\to -i}(z+i)^{\mathbf{3}}\dfrac{z^4+1}{(z+i)^5 z}=\lim_{z\to -i}\dfrac{z^4+1}{(z+i)^2 z}$ Let's try L'hopitals rule $\infty/\infty=\lim_{z\to -i}\dfrac{4z^3}{2(z+i)z+(z+i)^2}=$ $4i/0$ so multiplying by $(z+i)^3$ was not enough as our limit doesn't exist.
   Thus the order of the pole should be 4 let's double check $\lim_{z\to -i}(z+i)^{\mathbf{4}}\dfrac{z^4+1}{(z+i)^5 z}=\lim_{z\to -i}\dfrac{z^4+1}{(z+i)^1 z}$
   Let's try L'hopitals rule $\infty/\infty=\lim_{z\to -i}\dfrac{4z^3}{z+(z+i)^1}=4i/(-i)=-4$ thus the limit is a non-zero complex number giving the order of the pole as **4**.

   (c) $\dfrac{\sin z}{z(z+2)(z-\pi)^2}$. The sin function has simple zeros at $k\pi$ for $k\in\mathbb{Z}$ thus the order 2 zero of the denominator at π 'cancels' with the order 1 zero of the numerator at π to give a simple pole at $z=\pi$ there is a simple pole at $z=-2$. The 'apparent' singularity at $z=0$ is removable ('loosely' the order 1 zeros of the numerator and denominator cancel) with rigor $\lim_{z\to 0}\dfrac{\sin z}{z(z+2)(z-\pi)^2}=\lim_{z\to 0}\dfrac{\sin z}{z}\times\lim_{z\to 0}\dfrac{1}{(z+2)(z-\pi)^2}=1\times 1/(4\pi^2)$.

2. Let $f(z)=\dfrac{z+1}{z^2-z}=\dfrac{2}{z-1}-\dfrac{1}{z}$.

   (a) Using geometric series techniques or otherwise find a Laurent series expansion for $f(z)$ valid for:

      i. $0<|z-1|<1$;

      We need centre $z=1$ that is a power series involving $\sum a_n(z-1)^n$. Now $\dfrac{2}{z-1}$ is already a Laurent series ($1)^n$) valid for $z\neq 1$ or $0<|z-1|$, so let's look at expanding $\dfrac{1}{z}$ as a Laurent series valid for $0 < |z-1| < 1$ to simplify we use the substitution $w = (z-1)$ giving $\frac{1}{z} = \frac{1}{w+1}$. We can expand $\frac{1}{w+1}$ the 'usual way' to obtain a power series valid for $|w| < 1$ (as $\frac{1}{w+1}$ has singularity $w = -1$) or 'inside out' to obtain a laurent series valid for $|w| > 1$. We want the former option so

$$\begin{aligned}\frac{1}{1+w} &= \frac{1}{1-(-w)} \\ &= 1 + (-w) + (-w)^2 + (-w)^3 + \cdots \text{ provided } |w| < 1 \\ &\quad \text{(using the sum to } \infty \text{ of a G.P.)} \\ &= \sum_{n=0}^{\infty} (-1)^n w^n\end{aligned}$$

So substituting $w = z - 1$ and incorperating the $\frac{2}{z-1}$ term

$$f(z) = \frac{z+1}{z^2 - z} = \frac{2}{z-1} - \frac{1}{z} = 2(z-1)^{-1} + \sum_{n=0}^{\infty} (-1)^n (z-1)^n$$

which is valid on $0 < |z-1| < 1$ the intersection of the two domains for each part. (If you really want you can check this using the ratio test)

ii. $0 < |z| < 1$;

This is similar to the above except about centre $z = 0$. Now $\frac{1}{z}$ is a single term Laurent series valid for $|z| > 0$ and we expand $\frac{2}{z-1}$ the 'usual way' (valid for $z < 1$)

$$\begin{aligned}\frac{1}{z-1} &= -\frac{1}{1-z} \\ &= -1 - z - z^2 - z^3 + \cdots \text{ provided } |z| < 1 \\ &\quad \text{(using the sum to } \infty \text{ of a G.P.)} \\ &= \sum_{n=0}^{\infty} -1 z^n\end{aligned}$$

incorperating $-1\times$ the $\frac{1}{z}$ term and doubling to get $2/(z-1)$ rather than $1/(z-1)$

$$f(z) = \frac{z+1}{z^2 - z} = \frac{2}{z-1} - \frac{1}{z} = -1z^{-1} + \sum_{n=0}^{\infty} 2z^n$$

which is valid on $0 < |z| < 1$ the intersection of the two domains for each part.

iii. $1 < |z-2| < 2$;

This is a little trickier but give the technique in general. First we note the centre is $z = 2$. Now $\frac{1}{z-1}$ has a singularity at $z = 1$ (the distance this singularity is away from the centre 2 is $|2-1| = 1$ so we have two choices

- $|z-2| < 1$ if we expand $\frac{1}{z-1}$ 'normally'
- $|z-2| > 1$ if we expand $\frac{1}{z-1}$ 'inside out'

The domain stipulated in the question is $1 < |z-2| < 2$ so we need the second option. Let $w = z - 2$ ($\Leftrightarrow z = w + 2$) $\frac{1}{z-1} = \frac{1}{w+1} = \frac{1/w}{1/w+1}$ the latter step is what Dave calls 'inside out'. Let $u = 1/w$ $\frac{1/w}{1/w+1} = \frac{u}{u+1} = u\left(\frac{1}{1-(-u)}\right) = u - u^2 + u^3 - u^4 \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} u^n$ provided $|u| < 1$. Back-substituting now $\sum_{n=1}^{\infty} (-1)^{n+1} u^n =$

$\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{w}\right)^n$ provided $|u|<1 \Leftrightarrow \left|\frac{1}{w}\right|<1 \Leftrightarrow |w|>1$,
$\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{w}\right)^n = \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{(z-2)}\right)^n = \sum_{n=1}^{\infty}(-1)^{n+1}(z-2)^{-n}$ provided $|(z-2)|>1$.

We now tackle the $\frac{1}{z}$ part
Now $\frac{1}{z}$ has a singularity at $z=0$ (the distance this singularity is away from the centre 2 is $|2-0|=2$ so we have two choices

- $|z-2|<2$ if we expand $\frac{1}{z-1}$ 'normally'
- $|z-2|>2$ if we expand $\frac{1}{z-1}$ 'inside out'

The domain stipulated in the question is $1<|z-2|<2$ so we need the former option.
Let $w=z-2$ ($\Leftrightarrow z=w+2$) $\frac{1}{z}=\frac{1}{w+2}=\frac{1/2}{1+w/2}$ the latter step is what Dave calls 'normal expansion'. $\frac{1/2}{1+w/2}=\frac{1/2}{1-(-w/2)}=1/2-1/2(w/2)+1/2(w/2)^2\cdots=$ $\displaystyle\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^{n+1}w^n$ provided $|w/2|<1 \Leftrightarrow |w|<2$. Back-substituting now $\displaystyle\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^{n+1}w^n=$ $\displaystyle\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^{n+1}(z-2)^n$ provided $|w|<2 \Leftrightarrow |(z-2)|<2$.
Taking the appropriate linear combination of the two series

$$f(z)=\mathbf{2}\frac{1}{z-1}+-\mathbf{1}\frac{1}{z}=\sum_{m=1}^{\infty}2\times(-1)^m(z-2)^{-m}+\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{1}{2}\right)^{n+1}(z-2)^n$$

valid for (the intersection of the two domains above $1<|z-2|<2$.

(b) $\operatorname{Res}(f;0)$ will be the coefficient of z^{-1} for the Laurent series valid in a punctured disk about $z=0$. The series in (ii) is the appropriate one to choose, thus $\operatorname{Res}(f;0)=-1$.
$\operatorname{Res}(f;1)=2$ will be the coefficient of $(z-1)^{-1}$ for the Laurent series valid in a punctured disk about $z=1$. The series in (i) is the appropriate one to choose, thus $\operatorname{Res}(f;1)=-2$.

3. Using residue calculus calculate:

   (a) $\displaystyle\oint_{|z|=1/2}f(z)dz=2\pi i\left(\operatorname{Res}(f;0)\right)=2\pi i(-1)=-2\pi i$ as the contour $|z|=1/2$ contains the singularity $z=0$ and no others.

   (b) $\displaystyle\oint_{|z-2|=3/2}f(z)dz=2\pi i\left(\operatorname{Res}(f;1)\right)=2\pi i(2)=-4\pi i$ as the contour $|z-2|=3/2$ contains the singularity $z=1$ and no others.

   (c) $\displaystyle\oint_{|z-5|=1}f(z)dz=0$ as the contour $|z-5|=1$ contains no singularities.

4. $f(z)=\dfrac{1}{(1+z^2)^3}=\dfrac{1}{\left((z-i)(z+i)\right)^3}=\dfrac{1}{(z-i)^3(z+i)^3}$,

   (a) Thus the singularities of $f(z)$ are i and $-i$ both poles of order 3. We calculate the residues using two techniques (of course you may use your favorite).
   Let's find the Laurent series for $f(z)$ about centre $z=i$. Now $\frac{1}{z+i}=\frac{1}{(z-i)+2i}=$ $\frac{-i/2}{1-i(z-i)/2}=-i/2+1/4(z-i)+i/8(z-i)^2+\cdots$ So using the Cauchy Product

$$\frac{1}{(z+i)^3}=\left(-i/2+1/4(z-i)+i/8(z-i)^2+\cdots\right)^3=(-i/2)^3+3((-i/2)^2 1/4)(z-i)+\left(3(-i/2)^2 i/8+3(1/4)^2(\cdots\right.$$

Thus the Laurent series for $f(z)$ obtained by dividing by $(z-i)^3$ is $\cdots - \frac{3i}{16}(z-i)^{-1} + \cdots$ which means $\text{Res}(f;i) = -\frac{3i}{16}$.

Lets calculate the residue at $z=-i$ using limits. As $z=-i$ is a pole of order 3 $\text{Res}(f;-i) = \frac{1}{2!}\lim_{z\to -i}\left(\frac{d^2}{dz^2}(z-(-i))^3 f(z)\right)$ this is $\frac{1}{2!}\frac{d^2}{dz^2}(z-i)^{-3}\Big|_{z=-i} = \frac{12}{2}(z-i)^{-5}\Big|_{z=-i} = \frac{12}{2\times 32}i^5 = \frac{3}{16}i$.

Note that as $f(z)$ is a function with real coefficients the singularities occur in conjugate pairs and the residues at a conjugate pair are mutually conjugate, that is if ζ is a singularity of $f(z)$ so is $\overline{\zeta}$ and moreover

$$\text{Res}\left(f;\overline{\zeta}\right) = \overline{\text{Res}\left(f;\zeta\right)}.$$

(b) The contour Γ_R is a semicircle of radius R with 'diameter' along the real axis, centre at the origin and arc in the upper half plane.

The only singularity inside Γ_R is i so by residue calculus

$$\oint_{\Gamma_R}\frac{1}{(1+z^2)^3}dz = 2\pi i\text{Res}\left(\frac{1}{(1+z^2)^3};i\right) = 2\pi i\frac{-3i}{16} = \frac{3}{8}\pi$$

(c) Assuming $\lim_{R\to\infty}\int_{\gamma_R} f(z)dz = 0$

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{1}{(1+x^2)^3}dx &= \lim_{R\to\infty}\int_{X_R}\frac{1}{(1+z^2)^3}dz \\
&= \lim_{R\to\infty}\left(\oint_{\Gamma_R}\frac{1}{(1+z^2)^3}dz - \int_{\gamma_R}\frac{1}{(1+z^2)^3}dz\right) \\
&= \lim_{R\to\infty}\oint_{\Gamma_R}\frac{1}{(1+z^2)^3}dz - \lim_{R\to\infty}\int_{\gamma_R}\frac{1}{(1+z^2)^3}dz \\
&= \frac{3}{8}\pi - 0 \\
&= \frac{3}{8}\pi
\end{aligned}$$

(d) On γ_R $|z| = R$ so $(z^2+1)^3 > (R^2-1)^3$ if $R>1$ so

$$|f(z)| = \left|\frac{1}{(1+z^2)^3}\right| < \frac{1}{(R^2-1)^3}$$

now the length of the contour γ_R is $|\gamma_R| = \pi R$ thus

$$\left|\int_{\gamma_R}\frac{1}{(1+z^2)^3}dz\right| \le \pi R \times \frac{1}{(R^2-1)^3} \to 0 \quad \text{as} \quad R\to\infty$$