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620-252 Analysis

Tutorial 2 Answers

1. $\sqrt{3} - i = 2e^{-i(\pi/6+2\pi k)}$, $k \in \mathbb{Z}$

   So to solve we consider, $z^{2/3} = (\sqrt{3} - i) = 2e^{-i(\pi/6+2\pi k)}$, $k = 0, 1$

$$\begin{aligned} \Rightarrow \quad z &= 2^{3/2}e^{-i(3/2)(\pi/6+2\pi k)} \\ &= 2\sqrt{2}e^{-i(\pi/4+3\pi k)} \\ &= 2\sqrt{2}e^{-i\pi/4},\ 2\sqrt{2}e^{-i13\pi/4} \\ &= 2\sqrt{2}(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}),\ 2\sqrt{2}(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) \\ &= 2(1-i),\ 2(-1+i) \end{aligned}$$

2. (c) i.

   ii. S is not open, as $3i \in S$, but there is no open disk around $3i$ contained within S. Every open disk around $3i$ will contain both points of S and points not in S.

   iii. S is not closed, because $\mathbb{C}\backslash S$ is not open. This is because $-2i \in \mathbb{C}\backslash S$, but every open disk around $-2i$ will contain points of S.

3. (a) $|z - a| = r$ is the set of points that are distance r from the point a.

   $|z - a| + |z - b| = d$ is the set of points with the property have the sum of their distance from point a and their distance from point b is equal to d.

   (b)

iii. For this one we need some calculation to help see the picture. Let $z = x + iy$. Then

$$\begin{aligned} & \sqrt{(x-2)^2 + y^2} = 2\sqrt{(x+1)^2 + y^2} \\ \Rightarrow \quad & (x-2)^2 + y^2 = 4(x+1)^2 + 4y^2 \\ \Rightarrow \quad & x^2 - 4x + 4 + y^2 = 4x^2 + 8x + 4 + y^2 \\ \Rightarrow \quad & 3x^2 + 12x + 3y^2 = 0 \\ \Rightarrow \quad & (x+2)^2 + y^2 - 4 = 0 \\ \Rightarrow \quad & (x+2)^2 + y^2 = 4 \end{aligned}$$

So the set is a circle of radius 2 centred at $z = -2$.

Sets (i) and (iii) are closed, (ii) is open and (iv) is neither (think about trying to form an open disk around $z = 0$, which belongs to the complement).

4. (a) Let $z = x + iy$. Then

$$f(z) = \frac{2z^2 + 3}{|z-1|} = \frac{2(x+iy)^2 + 3}{\sqrt{(x-1)^2 + y^2}} = \frac{2x^2 - 2y^2 + 3 + i(4xy)}{\sqrt{(x-1)^2 + y^2}}.$$

So

$$u(x,y) = \frac{2x^2 - 2y^2 + 3}{\sqrt{(x-1)^2 + y^2}}, \quad v(x,y) = \frac{4xy}{\sqrt{(x-1)^2 + y^2}}$$

$$\text{and } u, v \text{ undefined} \Leftrightarrow (x-1)^2 = 0 \ \& \ y^2 = 0 \Leftrightarrow x = 1 \ \& \ y = 0$$

Thus $\text{dom}(f) = \mathbb{C} \backslash \{1\}$.

(b) $\text{Re}(z) > 5 \ \Rightarrow \ \text{Re}(f(z)) > 10$. So $\text{Ran}(f) = \{w \in \mathbb{C} : \text{Re}(w) > 10\}$.

(c) Let $z = re^{i\theta}$. Then

$$f(z) = -2re^{i3\theta} = 2re^{i(3\theta + \pi)}. \quad \text{So } |f(z)| = 2r, \ \arg(f(z)) = 3\theta + \pi.$$

Now enforce the restrictions on the domain:

$$|z| < 1 \Rightarrow |f(z)| < 2$$

and as $\theta = \arg(z)$

$$0 < \theta < \pi/2 \ \Rightarrow \ 0 < 3\theta < 3\pi/2 \ \Rightarrow \ \pi < 3\theta + \pi < 5\pi/2 \ \Rightarrow \ -\pi < \text{Arg}(f(z)) < \pi/2.$$

Thus $\text{Ran}(f) = \{w \in \mathbb{C} : |w| < 2, \ -\pi < \text{Arg}(w) < \pi/2\}$

(d) Let $z = re^{i\theta}$, and as $|z| = 1$ then $r = 1$. So

$$f(z) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \cos(\theta).$$

But $\theta \in \mathbb{R}$, so we in fact have $\text{Ran}(f) = [-1, 1] \subseteq \mathbb{R}$.