University of Melbourne
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620-252 Analysis

Tutorial 4 - Harmonic Conjugates, Transcendental Functions

From now on, I will tend to let $z = x + iy$, $x, y \in \mathbb{R}$ without specifically mentioning it.

> In $\mathbb{R}^2$ the function $\phi(x,y)$ is *harmonic* if it is C^2 and satisfies
>
> $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$
>
> - If $f(z) = u(x,y) + iv(x,y)$ is analytic in an open connected set S, then u, v are harmonic on S.
> - If $u(x,y)$ is harmonic on an open disk S, then there exists another harmonic function $v(x,y)$, known as the *harmonic conjugate* of u, such that $f(z) = u(x,y) + iv(x,y)$ is analytic in S.
>
> The Cauchy-Riemann equations are used to find v from u.

1. (a) Show that $u(x,y) = x^3 - 3xy^2 + e^{-x}\cos y$ is harmonic on $\mathbb{R}^2$.
   (b) Find a harmonic conjugate $v(x,y)$ for $u(x,y)$.
   (c) Hence find an entire function $f(z)$ such that $\text{Re}(\, f(z)\,) = u(x,y)$. Write $f(z)$ in terms of z.

> The exponential function.
>
> - Althought probably already understood, we formally define as
>
> $$e^z := e^x(\cos y + i \sin y)$$
>
> - This is not 1-1. Indeed $e^{z_1} = e^{z_2} \Leftrightarrow z_1 = z_2 + 2k\pi i$, $k \in \mathbb{Z}$.
> - $S_0 = \{z \in \mathbb{C} : -\pi < \text{Im}(z) \leq \pi\}$ is known as the principal domain of exp.
> - $\exp : S_0 \to \mathbb{C}\backslash\{0\}$ is 1-1.
>
> Trigonometric and hyperbolic functions are defined via exponentials in the usual manner.

2. (a) Evaluate $\sin(3i)$.
   (b) With $z = x + iy$, use the fact that $\sin z = \sin x \cosh y + i \cos x \sinh y$ to prove that

   $$|\sin z| \leq \cosh y.$$

   (c) WB Qn 25.(b). The above expression for sin may be helpful.

The logarithm function.

- Designed to be an inverse for exp. Due to the fact exp is not 1-1, we don't have a true inverse, but a multivalued function that has similarities to the argument function we looked at earlier in the semester.
- The general defenition is

$$\log(z) := \ln(|z|) + i(\text{Arg}(z) + 2k\pi) , \quad k \in \mathbb{Z}$$

  where ln is the real natural log.
- The *principal value* of log is

$$\text{Log}(z) := \ln(|z|) + i(\text{Arg}(z))$$

  and $\text{Log} : \mathbb{C}\backslash\{0\} \to S_0$ is 1-1, and the true inverse of the restricted version of exp given previously.
- Although defined on $\mathbb{C}\backslash\{0\}$, Log is not continuous on this set. Specifically, it is not continuous on the negative real axis. If we let $P = (-\infty, 0] \subseteq \mathbb{R}$, then Log is continuous and analytic on $\mathbb{C}\backslash P$. This particular domain restriction is known as the *principal branch* of log.

3. (a) WB Qn. 29.(f)
   (b) WB Qn. 28.(d)
   (c) Pick any point x_0 on the negative real axis. Considering different paths approaching this point (informally) prove that Log is not continuous at x_0. If you can't do this reasonably quickly, don't dwell on it too much, I will talk about it when I come around.

For $z \neq 0$, we define complex powers in the following way

$$z^a := e^{a \log z}$$

which is in general multivalued.

- The *principal branch* of z^a is $e^{a\text{Log}(z)}$.
- For future reference, the derivatives of $f(z) = z^a$ and $g(z) = \text{Log}(z)$ are calculated as normal.

4. (a) Calculate all values of i^{-2i}.
   (b) Calculate the principal value of $i^{1/3-i}$. Give answer in cartesian form.
   (c) For what values of a is the value of z^a unique?