DEPARTMENT OF MATHEMATICS AND STATISTICS
620–252 ANALYSIS— Semester 2, 2007

Assignment 5: Week 6 tutorial

1. We are investigating the principal branch of log

$$\operatorname{Log}(z) := \log|z| + i\operatorname{Arg}(z)$$

where $-\pi < \operatorname{Arg} z \le \pi$.

(a) Does $\lim\limits_{z\to -1} \operatorname{Log} z$ exist?

Investigate by

- approaching -1 along the arc of $|z| = 1$ above the real axis - $z(t) = e^{i(\pi - t)}, t > 0, t \to 0 \Rightarrow z \to -1$.
- approaching -1 along the arc of $|z| = 1$ below the real axis - $z(t) = e^{i(\pi + t)}, t > 0, t \to 0 \Rightarrow z \to -1$.

(b) Comment on the continuity of $\operatorname{Log} z$ at -1.

(c) Comment on the analyticity of $\operatorname{Log} z$ at -1.

Note that this ‘Argument’ problem will exist anywhere on the negative real axis.

1. *Solution.*

(a) $(1 + \sqrt{3}i) = 2e^{i\pi/3}$ so $(1+\sqrt{3}i)^3 = 8e^{3\times i\pi/3} = 8e^{i\pi}$ thus $(1+\sqrt{3}i)^{3/4} = 8^{1/4}e^{i\pi/4}, 8^{1/4}e^{3i\pi/4}, 8^{1/4}e^{-i\pi/4}, 8^{1/4}e^{-3i\pi/4}$.

(b) Now $\sinh z$ is entire (hence continuous at $z = \pi i/2$ thus $\lim\limits_{z\to\pi i/2} \sinh z = \sinh(\pi i/2) = \dfrac{e^{i\pi/2} - e^{-i\pi/2}}{2} = \dfrac{i - (-i)}{2} = i$. The principal branch of Log is analytic at $z = i$ so by continuity $\lim\limits_{z\to\pi i/2} \operatorname{Log}(\sinh z) = \operatorname{Log} i = \dfrac{i\pi}{2}$

(c) $\lim\limits_{z\to 0} \operatorname{Log}(z)$ does not exist as $\operatorname{Log} z$ is unbounded as $z \to 0$ as can be seen by $\operatorname{Log}(e^{-n}) = n$ where $e^{-n} \to 0$ as $n \to \infty$.

(d) $\lim\limits_{z\to\pi} \dfrac{(z-\pi)}{\sin z}$ is of the form $\dfrac{0}{0}$ as $\sin\pi = 0$ and $(\pi - \pi) = 0$. So evaluating $\lim\limits_{z\to\pi} \dfrac{(z-\pi)}{\sin z}$ using L’hopitals Rule we obtain $\lim\limits_{z\to\pi} \dfrac{(z-\pi)}{\sin z} = \left.\dfrac{d/dz\ (z-\pi)}{d/dz\ \sin z}\right|_{z=\pi} = \left.\dfrac{1}{\cos z}\right|_{z=\pi} = -1$. Also by continuity $\lim\limits_{z\to\pi} e^{iz} = e^{i\pi} = -1$, thus $\lim\limits_{z\to\pi} e^{iz} \times \lim\limits_{z\to\pi} \dfrac{(z-\pi)}{\sin z} = \lim\limits_{z\to\pi} \dfrac{(z-\pi)\, e^{iz}}{\sin z} = -1 \times -1 = 1$.

(e) $(-2i)^{-2i} = \exp\left(-2i \operatorname{Log}(-2i)\right) = \exp\left(-2i(\log 2 - i\pi/2)\right) = \exp(-\pi - i\log 8) = e^{-\pi}e^{-i\log 8}$.

2. We are using $\epsilon - \delta$ methods to show that $\lim_{z\to 2}\big(z^2-1\big) = 3$.

   (a) Show that $|z-2| < \min\left(\frac{\epsilon}{5}, 1\right) \Rightarrow \big|(z^2-1)-3\big| < \epsilon$.

   (b) Find a natural number K so that

$$|z-2| < \min\left(\frac{\epsilon}{K}, 2\right) \Rightarrow \big|(z^2-1)-3\big| < \epsilon.$$

2. *Solution.*

   (a) Let $w = z-2$, so $z = w+2$ substituting gives $\big(z^2-1\big)-3 = (w+2)^2-4 = w^2+4w$ thus if $|z-2| = |w| < \min\left(\frac{\epsilon}{5}, 1\right)$ then $|w| < 1$ and $|w| < \epsilon/5$. So $\big|(z^2-1)-3\big| = \big|w^2+4w\big| = |w||w+4| \le |w|(|w|+4) < \epsilon/5(1+4) = \epsilon$ the desired result.

   (b) Using $|w^2+4w| = |w||w+4| \le |w|\big(|w|+4\big)$ again $|w| < \min\left(\frac{\epsilon}{K}, 2\right)$ means $|w| < 2$ regardless of the value of K so $\big(|w|+4\big) < 2+4 = 6$ so if $|w| < \epsilon/6$ also we have $|w^2+4w| < (\epsilon/6)\times 6 = \epsilon$ so any $K \ge 6$ does the trick.

   Note that no smaller value of K will work. For example if $K \le 5$ then 3.99 satisfies $|2-3.99| < \min(2, 10/K)$ $(\epsilon = 10)$ but $|(3.99^2)-1-3| > 11.84 > 10 = \epsilon$!

3. Let $u(x,y) = -8x^3y + 8xy^3$.

   (a) Show that $u(x,y)$ is *harmonic* on $\mathbb{R}^2$.

   (b) Find a *harmonic conjugate* $v(x,y)$ for $u(x,y)$.

   (c) Hence find an *entire* function $f(z)$ such that $\text{Real}(\,f(x+iy)\,) = u(x,y)$. Write $f(z)$ in terms of z.

3. *Solution.*

   (a)
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(-24x^2y + 8y^3\right) = -48xy.$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(-8x^3 + 24xy^2\right) = 48xy.$$

   Thus $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -48xy + 48xy = 0$, so $u(x, y)$ is harmonic.

   (b) $\frac{\partial u}{\partial x} = -24x^2y + 8y^3$ so $\frac{\partial v}{\partial y} = -24x^2y + 8y^3$ so $v = -12x^2y^2 + 2y^4 + h(x)$. Differentiating partially with respect to x $\frac{\partial v}{\partial x} = -24xy^2 + h'(x) = -\frac{\partial u}{\partial y} = 8x^3 - 24xy^2$ so $h'(x) = 8x^3 \Rightarrow h(x) = 2x^4 + c$. Thus the function
$$v(x, y) = 2x^4 - 12x^2y^2 + 2y^4 + c$$
   is a harmonic conjugate.

   (c) So
$$\begin{aligned} f(z) &= f(x + iy) \\ &= \big(-8x^3y + 8xy^3\big) + i\big(2x^4 - 12x^2y^2 + 2y^4\big) \\ &= 2i\left(x^4 + i4x^3y + 6i^2x^2y^2 + 4i^3xy^3 + i^4y^4\right) \\ &= 2i(x + iy)^4 \\ &= 2iz^4 \end{aligned}$$

4. (a) Using the complex exponential definitions of sin.,cosh, sinh and cos show that if $z = x + iy$ then
$$\cos z = \cos x \cosh y - i \sin x \sinh y.$$

   (b) Hence (or otherwise) show that
$$|\cos z| = \sqrt{\cos^2 x + \sinh^2 y}.$$

   (c) Find all the solutions $z \in \mathbb{C}$ to
$$\cos z = 0.$$

4. *Solution.*

(a)

$$\begin{aligned}
\cos x\cosh y - i\sin x\sinh y &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)\left(\frac{e^{y}+e^{-y}}{2}\right) - i\left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{y}-e^{-y}}{2}\right)\\
&= \frac{1}{4}\left(\left[e^{ix+y}+e^{ix-y}+e^{-ix+y}+e^{-ix-y}\right]\right.\\
&\qquad \left. + \left[e^{ix+y}-e^{ix-y}-e^{-ix+y}+e^{-ix-y}\right]\right)\\
&= \frac{1}{4}\left(\left[e^{ix+y}+ \cancel{e}^{ix-y}+ \cancel{e}^{-ix+y}+e^{-ix-y}\right]\right.\\
&\qquad \left. + \left[e^{ix+y}- \cancel{e}^{ix-y}- \cancel{e}^{-ix+y}+e^{-ix-y}\right]\right)\\
&= \frac{1}{2}\left(e^{ix+y}+e^{-ix-y}\right)\\
&= \left(e^{iz}+e^{-iz}\right)/2\\
&= \cos z
\end{aligned}$$

(b) As $|u(x,y)+iv(x,y)| = \left(u^2(x,y)+v^2(x,y)\right)^{1/2}$ we see that

$$\begin{aligned}
|\cos z| &= \sqrt{(\cos x\cosh y)^2+(\sin x\sinh y)^2}\\
&= \sqrt{\cos^2 x\cosh^2 y+\sin^2 x\sinh^2 y}\\
&= \sqrt{\cos^2 x\left(\sinh^2 y+1\right)+\sin^2 x\sinh^2 y}\\
&= \sqrt{\left(\cos^2 x+\sin^2 x\right)\sinh^2 y+\cos^2 x}\\
&= \sqrt{\cos^2 x+\sinh^2 y}.
\end{aligned}$$

(c) Now $\cos z = 0 \Rightarrow |\cos z| = 0 \Rightarrow \cos^2 x+\sinh^2 y = 0 \Rightarrow \cos x = 0$ & $\sinh y = 0$. Thus $y = 0$ and $x = \frac{2n+1}{2}\pi \quad n\in\mathbb{Z}$ giving $z = \frac{2n+1}{2}\pi \quad n\in\mathbb{Z}$. Thus extending the domain of cos from $\mathbb{R}$ to $\mathbb{C}$ introduces no more zeros!

Please turn over ...

5. Our aim in this question is to show that $\operatorname{Log} z$ (the principal branch) is analytic on the domain

$$D = \{z \in \mathbb{C} : \operatorname{Im}(z) = 0 \Rightarrow \operatorname{Re}(z) > 0\} = \{x+iy : x \in \mathbb{R}, y \in \mathbb{R} \text{ with } y = 0 \text{ only if } x > 0\}.$$

If $z = re^{i\theta}$ $\quad -\pi < \theta < \pi$ then

$$\operatorname{Log}(x+iy) = \operatorname{Log} z = \log r + i\theta = u(x,y) + iv(x,y)$$

where

$$u(x,y) = \frac{1}{2}\log(x^2+y^2)$$

and

$$v(x,y) = \begin{cases} \arccos\left(x/(\sqrt{x^2+y^2})\right) & y > 0 \\ 0 & y = 0 \\ -\arccos\left(x/(\sqrt{x^2+y^2})\right) & y < 0 \end{cases}$$

(a) Using (without verification) $\dfrac{d}{dt}\arccos t = -\dfrac{1}{\sqrt{1-t^2}}$, show that:

(i) $\dfrac{\partial}{\partial x}\arccos\left(\dfrac{x}{\sqrt{x^2+y^2}}\right) = -\dfrac{|y|}{(x^2+y^2)}$ note $|y| = \sqrt{y^2}$;

(ii) $\dfrac{\partial}{\partial y}\arccos\left(\dfrac{x}{\sqrt{x^2+y^2}}\right) = \dfrac{xy}{|y|(x^2+y^2)}$.

(b) Note that if $x+iy$ **in** $\boldsymbol{D}$ then $\lim\limits_{y\to 0}\arccos\left(\dfrac{x}{\sqrt{x^2+y^2}}\right) = 0$. So we can have the region $y > 0$ 'creep' to $y \geq 0$ and $y < 0$ 'creep' to $y \leq 0$.
In domain D, carefully evaluate:

(i) $\dfrac{\partial v}{\partial x}$;

(ii) $\dfrac{\partial v}{\partial y}$.

(c) In domain D, verify that:

(i) the Cauchy-Riemann equations hold;

(ii) the functions u and v are C^1.

(d) Justify the analyticity of Log on domain D.

End of assignment 1.

5. *Solution.*

(a) (i)

$$\begin{aligned}
&\frac{\partial}{\partial x}\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\\
= \;& \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\times -1/\sqrt{1-x^2/(x^2+y^2)}\\
= \;& \left(\frac{(x^2+y^2)^{1/2}-1/2\times 2x\times(x^2+y^2)^{-1/2}}{x^2+y^2}\right)\times -\frac{1}{|y|}(x^2+y^2)^{1/2}\\
= \;& \left(\frac{(x^2+y^2)-x^2}{(x^2+y^2)^{3/2}}\right)\times -\frac{1}{|y|}(x^2+y^2)^{1/2}\\
= \;& -\frac{|y|^2}{|y|}\frac{1}{(x^2+y^2)}\\
= \;& -\frac{|y|}{(x^2+y^2)}
\end{aligned}$$

(ii)

$$\begin{aligned}
&\frac{\partial}{\partial y}\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\\
= \;& \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\times -1/\sqrt{1-x^2/(x^2+y^2)}\\
= \;& \left(-1/2\times 2y\times x\times(x^2+y^2)^{-3/2}\right)\times -\frac{1}{|y|}(x^2+y^2)^{1/2}\\
= \;& \frac{xy}{|y|(x^2+y^2)}
\end{aligned}$$

(b) (i) Suppose $y \ge 0$ then $v(x,y) = \arccos\left(x/(\sqrt{x^2+y^2})\right)$ so $\frac{\partial v}{\partial x} = -\frac{|y|}{(x^2+y^2)} = -\frac{y}{(x^2+y^2)}$.

Suppose $y \le 0$ then $v(x,y) = -\arccos\left(x/(\sqrt{x^2+y^2})\right)$ so $\frac{\partial v}{\partial x} = -1\times -\frac{|y|}{(x^2+y^2)} = -1\times\frac{y}{(x^2+y^2)}$. So in either case

$$\frac{\partial v}{\partial x} = -\frac{y}{(x^2+y^2)}.$$

(ii) Suppose $y \geq 0$ then $\dfrac{\partial v}{\partial y} = \dfrac{xy}{|y|(x^2+y^2)} = \dfrac{xy}{y(x^2+y^2)} = \dfrac{x}{(x^2+y^2)}$.
Suppose $y \leq 0$ then $\dfrac{\partial v}{\partial y} = -1 \times \dfrac{xy}{|y|(x^2+y^2)} = -1 \times \dfrac{xy}{-y(x^2+y^2)} = \dfrac{x}{(x^2+y^2)}$. So in either case

$$\frac{\partial v}{\partial y} = \frac{x}{(x^2+y^2)}.$$

(c) In domain D:

(i) $\dfrac{\partial u}{\partial x} = \dfrac{\partial 1/2 \log(x^2+y^2)}{\partial x} = \dfrac{1}{2}\dfrac{2x}{x^2+y^2} = \dfrac{x}{x^2+y^2}$. By the symmetry of $u(x,y)$ in x and y we see that $\dfrac{\partial u}{\partial y} = \dfrac{y}{x^2+y^2}$. Thus on D

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} = -\frac{\partial v}{\partial x},$$

so the Cauchy Riemann equations hold.

(ii) The partial derivatives calculated above are continuous (on D) as the partial derivatives are continuous on each half and (by the 'creep') correspond on their join (the positive real axis), thus u and v are C^1 on D.

(d) Hence by the Cauchy Riemann Theorem Log in differentiable everywhere on the open domain D (u and v are C^1 and satisfy the Cauchy Riemann equations) and hence is analytic on D.