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Tutorial 7 - Cauchy Integral Formulae

Recall:

$$\oint_{|z-a|=1} (z-a)^n \, dz = \begin{cases} 0 & n \in \mathbb{Z}, n \neq -1 \\ 2\pi i & n = -1 \end{cases}$$

where the circle $|z-a| = 1$ has positive orientation.

DEFORMATION THEOREM:
If the contours Γ and Γ' are both positively oriented and the set of singularities of $f(z)$ interior to Γ is exactly the same as the set interior to Γ' then

$$\oint_{\Gamma} f(z)dz = \oint_{\Gamma'} f(z)dz.$$

1. Let Γ be the contour $|z-(-2+3i)| = 4$.

   (a) Let

   $$f(z) = \frac{1}{z-2} + \frac{3}{z+3} + \frac{-2}{z+1} = \frac{2z^2 - z + 9}{(z+1)(z+3)(z-2)} = \frac{2z^2 - z + 9}{z^3 + 2z^2 - 5z - 6}.$$

   i. List the singularities of $f(z)$.

   ii. With the aid of a diagram deform the contour Γ to a set of contours each containing just one singularity.

   iii. Use this set of contours and the partial fraction decomposition of $f(z)$ to calculate $\oint_{\Gamma} f(z)dz$.

CAUCHY'S INTEGRAL FORMULA:

If $f(z)$ is analytic on D a simply connected open domain that contains the simple closed (positively oriented) contour Γ and a is interior to Γ, then

$$\oint_{\Gamma} \frac{f(z)}{z-a} dz = 2\pi i f(a).$$

   (b) Let $l(z) = \dfrac{18 + 11z + 5z^2}{(z+1)(z+3)(z-2)}$.

   i. Use contour deformation (the one from **??** will be perfect) and the Cauchy Integral Formula to calculate

   $$\oint_{\Gamma} l(z)dz.$$

   Hint you will need to express $l(z)$ as $\dfrac{h(z)}{z+1}, \dfrac{g(z)}{z+3}, \ldots$ as is appropriate to each part of your contour.

   ii. As $l(z) = f(z) + \dfrac{3}{z-2}$ you should get the same answer for $\oint_{\Gamma} f(z)dz$ as for $\oint_{\Gamma} l(z)dz$. Demonstrate this using the Cauchy Goursat theorem.

CAUCHY'S GENERAL INTEGRAL FORMULA:

If $f(z)$ is analytic on D a simply connected open domain that contains the simple closed (positively oriented) contour Γ and a is interior to Γ, then

$$\oint_{\Gamma} \frac{f(z)}{(z-a)^{n+1}} dz = 2\pi i \frac{f^{(n)}(a)}{n!}.$$

2. (a) By defining a suitable domain D use C.G.I.F. to evaluate

$$I = \oint_{|z-(1+i)|=5} \frac{z^4 + 2z^3 - z + 1}{(z-2)^3} dz.$$

(b) It is true that $(z-2)^4 + 10(z-2)^3 + 36(z-2)^2 + 55(z-2) + 31 = z^4 + 2z^2 - z + 1$. Demonstrate that

$$I = \oint_{|z-2|=1} (z-2)^1 + 10 + 36(z-2)^{-1} + 55(z-2)^{-2} + 31(z-2)^{-3} dz.$$

(c) Re-evaluate I using the RHS of the above and the formula at the top of the first page.

You have had a sneak preview of LAURENT SERIES.