University of Melbourne
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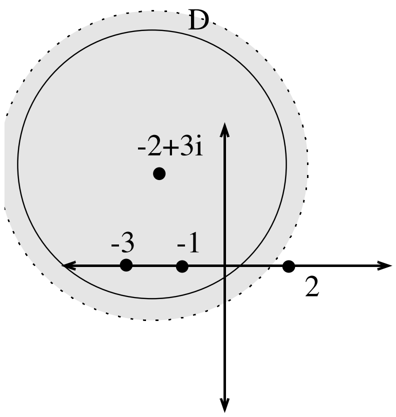
620-252 Analysis

Tutorial 7 - Cauchy Integral Formulae - Solutions

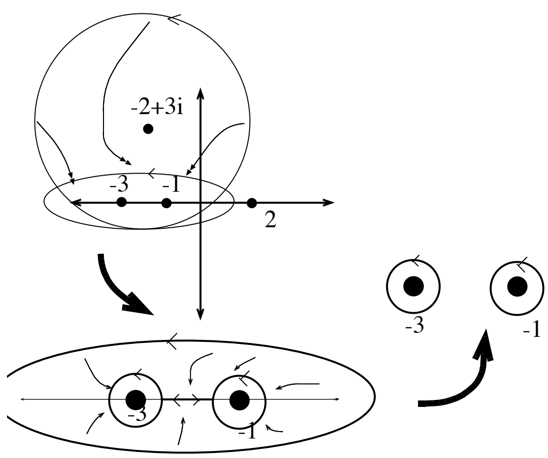
1. Let Γ be the contour $|z-(-2+3i)|=4$.

   (a) Let
$$f(z)=\frac{1}{z-2}+\frac{3}{z+3}+\frac{-2}{z+1}=\frac{2z^2-z+9}{(z+1)(z+3)(z-2)}=\frac{2z^2-z+9}{z^3+2z^2-5z-6}.$$

   i. The singularities of $f(z)$ are (simple poles) at $z=2, z=-1, z=-3$.



   ii. We can deform the contour $|z-(-2+3i)|=4$ to the pair of contours $|z-(-1)|=\epsilon$ and $|z-(-3)|=\epsilon$ (any $0<\epsilon<1$ will work as pictured). See the diagram below.



iii.

$$\begin{aligned}
\oint_\Gamma f(z)dz &= \oint_{|z-(-1)|=\epsilon} \frac{1}{z-2}+\frac{3}{z+3}+\frac{-2}{z+1}dz + \oint_{|z-(-3)|=\epsilon} \frac{1}{z-2}+\frac{3}{z+3}+\frac{-2}{z+1}dz \\
&= \oint_{|z-(-1)|=\epsilon} \frac{-2}{z+1}dz + \oint_{|z-(-1)|=\epsilon} \frac{1}{z-2}+\frac{3}{z+3}dz + \oint_{|z-(-3)|=\epsilon} \frac{3}{z+3}dz + \\
&\quad \oint_{|z-(-3)|=\epsilon} \frac{1}{z-2}+\frac{-2}{z+1}dz \\
&= \oint_{|z-(-1)|=\epsilon} \frac{-2}{z+1}dz + \oint_{|z-(-3)|=\epsilon} \frac{3}{z+3}dz \quad \text{by the Cauchy Goursat Theorem} \\
&= -2\times 2\pi i + 3\times 2\pi i \quad \text{by the Formula } \oint_{|z-a|=1} (z-a)^n\,dz = \begin{cases} 0 & n\in\mathbb{Z}, n\neq -1 \\ 2\pi i & n=-1 \end{cases} \\
&= 2\pi i
\end{aligned}$$

(b) Let $l(z) = \dfrac{18+11z+5z^2}{(z+1)(z+3)(z-2)}$.

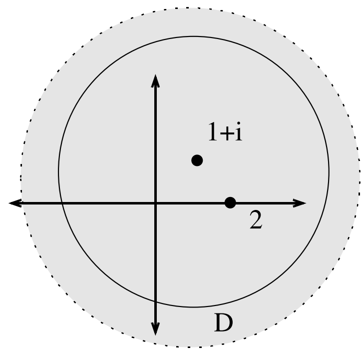
i.

$$\begin{aligned}
\oint_\Gamma l(z)dz &= \oint_{|z-(-1)|=\epsilon} l(z)dz + \oint_{|z-(-3)|=\epsilon} l(z)dz \\
&= \oint_{|z-(-1)|=\epsilon} \frac{h(z)}{z+1}dz \text{ where } h(z) = \left(18+11z+5z^2\right)/((z+3)(z-2)) \\
&\quad + \oint_{|z-(-3)|=\epsilon} \frac{l(z)}{z+3}dz \text{ where } l(z) = \left(18+11z+5z^2\right)/((z+1)(z-2)) \\
&= 2\pi i \;\; h(z)\big|_{z=-1} \text{ by C.I.F. using } D = |z+1| < 2\epsilon \text{ where } \epsilon < 1. \\
&\qquad h(z) \text{ is analytic on } D \text{ and } D \text{ contains the contour } |z-(-1)| = \epsilon \\
&\quad +2\pi i \;\; l(z)\big|_{z=-3} \text{ by C.I.F. using } D' = |z+3| < 2\epsilon \text{ where } \epsilon < 1. \\
&\qquad l(z) \text{ is analytic on } D' \text{ and } D' \text{ contains the contour } |z-(-3)| = \epsilon \\
&= \frac{12}{-6}2\pi i + \frac{30}{10}2\pi i \\
&= 2\pi i
\end{aligned}$$

ii.

$$\begin{aligned}
\oint_\Gamma l(z)dz &= \oint_\Gamma f(z) + \frac{3}{z-2}dz \\
&= \oint_\Gamma f(z)dz + \oint_\Gamma \frac{3}{z-2}dz \\
&\quad \text{as the latter integral is 0 by CG – } z=2 \text{ is outside the contour } \Gamma \\
&= \oint_\Gamma f(z)dz
\end{aligned}$$

2. (a) A suitable domain D is $D : |z-(1+i)| < 6$ as z^4+2z^3-z+1 is analytic in D and D contains the contour $|z-(1+i)| = 5$ – see the following diagram



$$
\begin{aligned}
I &= \oint_{|z-(1+i)|=5} \frac{z^4+2z^3-z+1}{(z-2)^3} dz \\
&= \frac{2\pi i}{2!} \frac{d^2}{dz^2} \left(z^4+2z^3-z+1\right)\Big|_{z=2} \\
& \qquad \text{G.C.I.F. with } n=2 \text{ and } a=2 \\
&= \frac{2\pi i}{2!} \left(12z^2+12z\right)\Big|_{z=2} \\
&= 72\pi i
\end{aligned}
$$

(b) It is true that $(z-2)^4 + 10(z-2)^3 + 36(z-2)^2 + 55(z-2) + 31 = z^4 + 2z^2 - z + 1$.

$$
\begin{aligned}
& \oint_{|z-(1+i)|=5} \frac{z^4+2z^3-z+1}{(z-2)^3} dz \\
= & \oint_{|z-2|=1} \frac{z^4+2z^3-z+1}{(z-2)^3} dz \quad \text{by contour deformation} \\
= & \oint_{|z-2|=1} (z-2)^1 + 10 + 36(z-2)^{-1} + 55(z-2)^{-2} + 31(z-2)^{-3} dz \quad \text{as on } |z=2|=1, \\
& (z-2) \neq 0 \text{ so } \frac{z^4+2z^3-z+1}{(z-2)^3} = (z-2)^1 + 10 + 36(z-2)^{-1} + 55(z-2)^{-2} + 31(z-2)^{-3}
\end{aligned}
$$

(c)

$$
\begin{aligned}
& \oint_{|z-2|=1} (z-2)^1 + 10 + 36(z-2)^{-1} + 55(z-2)^{-2} + 31(z-2)^{-3} dz \\
= & \oint_{|z-2|=1} (z-2)^1 dz + \oint_{|z-2|=1} 10 dz + \oint_{|z-2|=1} 36(z-2)^{-1} dz + \\
& \oint_{|z-2|=1} 55(z-2)^{-2} dz + \oint_{|z-2|=1} 31(z-2)^{-3} dz \\
= & 0 + 0 + 36 \oint_{|z-2|=1} (z-2)^{-1} dz + 0 + 0 \\
= & 36 \times 2\pi i \\
= & 72\pi i
\end{aligned}
$$

You have had a sneak preview of LAURENT SERIES.