University of Melbourne
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620-252 Analysis

Tutorial 8 - Maximum Modulus Principle, Gauss Mean Value Theorem and more

> MAXIMUM MODULUS PRINCIPLE:
>
> If $f(z)$ is analytic on a closed bonded region R
> then $\max\limits_{z\in R}|f(z)| = f(w)$ for $w \in \partial R$
> that is the maximum modulus of $f(z)$ is attained at the boundary or R.

1. Consider the function $f(z) = \dfrac{z^2}{z+2}$

   (a) Let $D = \{z \in \mathbb{C} : |z| \leq 1\}$. Find M the maximum modulus of $f(z)$ on D and $w \in D$ satisfying $|f(w)| = M$.

   (b) Let $D' = \{z \in \mathbb{C} : |\mathrm{Im}(z)| \leq 1 \text{ and } |\mathrm{Re}(z)| \leq 1\}$.
   If M' is the maximum modulus of $f(z)$ on D' (see the solutions for the actual value of M') then how do we know $M \leq M'$ without calculation?

2. Using an appropriate substitution and CAUCHY'S INTEGRAL FORMULA or CAUCHY'S GENERAL INTEGRAL FORMULA calculate:

   (a)
   $$\int_0^{2\pi} \frac{1}{3 + 2\cos\theta} d\theta$$

   (b)
   $$\int_0^{2\pi} \frac{1}{(3 + 2\cos\theta)^2} d\theta.$$

> RATIO TEST:
> If $\lim\limits_{n\to\infty} \dfrac{|a_{n+1}(z)|}{|a_n(z)|} = r$ and $r < 1$
> then
> the series $\sum\limits_{n=0}^{\infty} a_n(z)$ converges absolutely.
>
> SUM TO INFINITY OF A G.P.:
> $\sum\limits_{n=0}^{\infty} ar^n$ converges if and only if $|r| < 1$ and furthermore it converges to the limit $\dfrac{a}{1-r}$.

3. Consider the series

   $$S = \sum_{n=0}^{\infty} (-1)^n (2z)^n = 1 - (2z) + (2z)^2 - (2z)^3 + \dots$$

   (a) Using the ratio test or otherwise find the largest open disk for which S converges.

   (b) Find the sum of the series on this disk – it will be $f(z)$ a function of z.

   (c) What is the largest open disk (centred at the origin) on which $f(z)$ is analytic?

Gauss' Mean Value Theorem:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

provided $f(z)$ is analytic in a domain D which contains the circle of radius r centred at z_0.

4. (a) By defining appropriate $f(z), r, z_0$ and using the Gauss Mean Value Theorem show that

$$\int_0^{2\pi} e^{e^{i\theta}}\, d\theta = 2\pi.$$

(b) Hence show that:

i. $\displaystyle\int_0^{2\pi} e^{\cos(\theta)} \cos(\sin\theta)\ d\theta = 2\pi;$

ii. $\displaystyle\int_0^{2\pi} e^{\cos(\theta)} \sin(\sin\theta)\ d\theta = 0.$