1. Consider the function \( f(z) = \frac{z^2}{z+2} \)

(a) The function \( f(z) = \frac{z^2}{z+2} \) has the single singularity \( z = -2 \) which is outside \( D = \{ z \in \mathbb{C} : |z| \leq 1 \} \) so by the maximum modulus principle the maximum modulus is attained on the boundary of \( D \) namely \( |z| = 1 \).

\[ \left| \frac{z^2}{z+2} \right| = \frac{1}{|z+2|} \text{ is maximized when } |z+2| \text{ is minimized which is when } z = -1 \text{ (the closest the circle } |z| = 1 \text{ is to } 2 \text{ is at } z = -1), \text{ thus } \left| \frac{z^2}{z+2} \right| \text{ has maximum value } 1 \text{ on } D \text{ when } z = -1 \]

(b) As \( D \subseteq D' \) it follows that the maximum of (any) \( l \) attained on \( D \) is less than the maximum of \( l \) on \( D' \) so \( M \leq M' \).

To calculate \( M' \), note again \( f(z) \) is analytic on \( D' \) so we examine \( |f(z)| \) on \( \partial D' \). We parameterize \( \partial D' \) by \( \gamma_L : z = -1 + it \), where \(-1 \leq t \leq 1\), \( \gamma_T : z = t + i, \) where \(-1 \leq t \leq 1\), \( \gamma_B : z = t - i, \) where \(-1 \leq t \leq 1\), \( \gamma_R : z = 1 + it, \) where \(-1 \leq t \leq 1\).

On \( \gamma_L \)
\[
\left| \frac{z^2}{z+2} \right| = \frac{1 + t^2}{\sqrt{(2-1)^2 + t^2}} = \frac{1 + t^2}{\sqrt{1 + t^2}} = \sqrt{2} \text{ at } t = \pm 1
\]

On \( \gamma_T \)
\[
\left| \frac{z^2}{z+2} \right| = \frac{t^2 + 1}{\sqrt{1 + (2+t)^2}} \leq \frac{t^2 + 1}{\sqrt{1 + t^2}} \leq \frac{\sqrt{2}}{\sqrt{2}}
\]

As on the interval \(-1 \leq t \leq 1\) it is true that \(1 + t^2 \leq 1 + (2+t)^2 \Rightarrow 1/(1+t^2) \geq 1/(1+(2+t)^2)\). By symmetry the same is true on \( \gamma_B \).

Comparing the denominators (of \( \gamma_R \) and \( \gamma_L \) respectively) \(|z+2| = \sqrt{3 + t^2} > \sqrt{3} > \sqrt{2} > |z+2|\) on \( \gamma_L \) where as for the numerators \(|-1 + it| = |1 + it|\). So \(|f(z)| < \sqrt{2}\) on \( \gamma_R \), comparing \( \gamma_R \) with \( \gamma_L \).

Thus on \( D' \) the maximum modulus of \( f(z) \) is \( \sqrt{2} \) attained at \( z = -1 + i \) and \( z = -1 - i \).

2. Using an appropriate substitution and Cauchy’s Integral Formula or Cauchy’s General Integral Formula calculate:

We let \( z = e^{i\theta} \) which gives \( d\theta = \frac{dz}{iz} \cos(\theta) = \frac{z + z^{-1}}{2} \) as \( \theta \) runs from 0 to \( 2\pi \) \( z \) does one complete traverse of the unit circle \( |z| = 1 \).
\[
\int_0^{2\pi} \frac{1}{3 + 2 \cos \theta} d\theta = \oint_{|z|=1} \frac{1}{3 + z + z^{-1}} \frac{dz}{iz}
\]
\[
= \frac{1}{i} \oint_{|z|=1} \frac{1}{z^2 + 3z + 1} dz
\]
\[
= \frac{1}{i} \oint_{|z|=1} \frac{1/(z - (-3 - \sqrt{5})/2)}{z - (-3 + \sqrt{5})/2} dz
\]
the numerator \(1/(z - (-3 - \sqrt{5})/2)\) is analytic on \(|z| < 2\) which contains the contour \(|z| = 1\) so by CIF
\[
= \frac{1}{i} 2\pi i \frac{1}{z - (-3 - \sqrt{5})/2} \Big|_{z=-3+\sqrt{5}/2}
\]
\[
= 2\pi \frac{1}{\sqrt{5}}
\]

(b) With the same substitution and approach as above
\[
\int_0^{2\pi} \frac{1}{(3 + 2 \cos \theta)^2} d\theta = \oint_{|z|=1} \frac{1}{(3 + z + z^{-1})^2} \frac{dz}{iz}
\]
\[
= \frac{1}{i} \oint_{|z|=1} \frac{z}{(z^2 + 3z + 1)^2} dz
\]
\[
= \frac{1}{i} \oint_{|z|=1} \frac{z/(z - (-3 - \sqrt{5})/2)^2}{(z - (-3 + \sqrt{5})/2)^2} dz
\]
the numerator \(n(z) = z/(z - (-3 - \sqrt{5})/2)^2\) is analytic on \(|z| < 2\) which contains the contour \(|z| = 1\) so by GCIF
\[
= \frac{1}{i} 2\pi i \frac{d}{dz} n(z) \Big|_{z=3+\sqrt{5}/2}
\]
\[
= 2\pi \left( (z - (-3 - \sqrt{5})/2)^{-2} - 2z(z - (-3 - \sqrt{5})/2)^{-3} \right) \Big|_{z=-3+\sqrt{5}/2}
\]
\[
= 2\pi \left( \frac{1}{5} \left( 1 + (3 - \sqrt{5})/(\sqrt{5}) \right) \right)
\]
\[
= 2\pi \left( \frac{1}{5} \left( 3/\sqrt{5} \right) \right)
\]
\[
= \frac{6\pi}{5\sqrt{5}}
\]

3. Consider the series
\[
S = \sum_{n=0}^{\infty} (-1)^n (2z)^n = 1 - (2z) + (2z)^2 - (2z)^3 + \ldots
\]

(a) The series \(S\) is geometric with common ratio \(r = (-2z)\) and so will converge provided \(|r| = | -2z| < 1\) that is if \(|z| < \frac{1}{2}\).

(b) Using \(\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}\) (for \(|z| < \frac{1}{2}\)) we obtain \(S = \frac{1}{1+2z}\)

(c) We cannot include \(z = -\frac{1}{2}\) as this is a singularity of \(\frac{1}{1+2z}\) so the largest disk open centred at the origin is \(|z| < \frac{1}{2}\).

Note that this is the same open disk as for the convergence of \(S\).
4. (a) Let \( z_0 = 0 \), \( r = 1 \) and \( f(z) = e^z \) then

\[
\int_0^{2\pi} e^{i\theta} d\theta = 2\pi \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta = 2\pi f(z_0) = 2\pi e^0 = 2\pi
\]

(b) i. Expanding out

\[
\left( \int_0^{2\pi} e^{i\theta} d\theta \right) = (2\pi)
\]

\[
\Leftrightarrow \int_0^{2\pi} (e^{\cos \theta + i \sin \theta} d\theta) = 2\pi
\]

\[
\Leftrightarrow \int_0^{2\pi} (e^{\cos \theta} e^{i \sin \theta} d\theta) = 2\pi
\]

\[
\Leftrightarrow \int_0^{2\pi} (e^{\cos \theta} (\cos(\sin \theta) + i \sin(\sin \theta))) d\theta = 2\pi
\]

\[
\Leftrightarrow \int_0^{2\pi} (e^{\cos \theta} \cos(\sin \theta)) d\theta + i \int_0^{2\pi} (e^{\cos \theta} \sin(\sin \theta)) d\theta = 2\pi + i0
\]

So taking real parts we obtain \( \int_0^{2\pi} e^{\cos(\theta)} \cos(\sin \theta) d\theta = 2\pi \)

ii. Where as \( \int_0^{2\pi} e^{\cos(\theta)} \sin(\sin \theta) d\theta = 0 \) is obtained by taking imaginary parts.