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620-252 Analysis

Tutorial 8 - Maximum Modulus Principle, Gauss Mean Value Theorem and more

1. Consider the function $f(z) = \dfrac{z^2}{z+2}$

   (a) The function $f(z) = \dfrac{z^2}{z+2}$ has the single singularity $z = -2$ which is outside $D = \{z \in \mathbb{C} : |z| \leq 1\}$ so by the maximum modulus principle the maximum modulus is attained on the boundary of D namely $|z| = 1$.
   $\left|\dfrac{z^2}{z+2}\right| = \dfrac{1}{|z+2|}$ is maximized when $|z+2|$ is minimized which is when $z = -1$ (the closest the circle $|z| = 1$ is to 2 is at $z = -1$), thus $\left|\dfrac{z^2}{z+2}\right|$ has maximum value 1 on D when $z = -1$

   (b) As $D \subseteq D'$ it follows that the maximum of (any) l attained on D is less than the maximum of l on D' so $M \leq M'$.
   To calculate M', note again $f(z)$ is analytic on D' so we examine $|f(z)|$ on $\partial D'$. We parameterize $\partial D'$ by $\gamma_L :\ z = -1 + it$, where $-1 \leq t \leq 1$, $\gamma_T :\ z = t + i$, where $-1 \leq t \leq 1$, $\gamma_B :\ z = t - i$, where $-1 \leq t \leq 1$, $\gamma_R :\ z = 1 + it$, where $-1 \leq t \leq 1$.

$$\text{On } \gamma_L \quad \begin{aligned} &\left|\frac{z^2}{z+2}\right| \\ =\ & \frac{1+t^2}{\sqrt{(2-1)^2+t^2}} \\ =\ & \sqrt{1+t^2} \\ =\ & \sqrt{2} \text{ at } t = \pm 1 \end{aligned}$$

$$\text{On } \gamma_T \quad \begin{aligned} &\left|\frac{z^2}{z+2}\right| \\ =\ & \frac{t^2+1}{\sqrt{1+(2+t)^2}} \\ \leq\ & \sqrt{1+t^2} \\ \leq\ & \sqrt{2} \end{aligned}$$

   As on the interval $-1 \leq t \leq 1$ it is true that $1+t^2 \leq 1+(2+t)^2 \Rightarrow 1/(1+t^2) \geq 1/(1+(2+t)^2)$.
   By symmetry the same is true on γ_B.
   Comparing the denominators (of γ_R and γ_L respectively) $|z+2| = \sqrt{3+t^2} > \sqrt{3} > \sqrt{2} > |z+2|$ on γ_L where as for the numerators $|-1+it| = |1+it|$. So $|f(z)| < \sqrt{2}$ on γ_R, comparing γ_R with γ_L.
   Thus on D' the maximum modulus of $f(z)$ is $\sqrt{2}$ attained at $z = -1+i$ and $z = -1-i$.

2. Using an appropriate substitution and Cauchy's Integral Formula or Cauchy's General Integral Formula calculate:

   We let $z = e^{i\theta}$ which gives $d\theta = \dfrac{dz}{iz}$ $\cos(\theta) = \dfrac{z+z^{-1}}{2}$ as θ runs from 0 to 2π z does one complete traverse of the unit circle $|z| = 1$.

(a)

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{3+2\cos\theta}d\theta &= \oint_{|z|=1} \frac{1}{3+z+z^{-1}}\frac{dz}{iz} \\
&= \frac{1}{i}\oint_{|z|=1} \frac{1}{z^2+3z+1}dz \\
&= \frac{1}{i}\oint_{|z|=1} \frac{1/(z-(-3-\sqrt{5})/2}{z-(-3+\sqrt{5})/2}dz \\
&\quad \text{the numerator } 1/(z-(-3-\sqrt{5})/2 \text{ is analytic on } |z|<2 \\
&\quad \text{which contains the contour } |z|=1 \text{ So by CIF} \\
&= \frac{1}{i}2\pi i \left.\frac{1}{z-(-3-\sqrt{5})/2}\right|_{z=-3+\sqrt{5})/2} \\
&= 2\pi\frac{1}{\sqrt{5}}
\end{aligned}$$

(b) With the same substitution and approach as above

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{(3+2\cos\theta)^2}d\theta &= \oint_{|z|=1} \frac{1}{(3+z+z^{-1})^2}\frac{dz}{iz} \\
&= \frac{1}{i}\oint_{|z|=1} \frac{z}{(z^2+3z+1)^2}dz \\
&= \frac{1}{i}\oint_{|z|=1} \frac{z/(z-(-3-\sqrt{5})/2)^2}{(z-(-3+\sqrt{5})/2)^2}dz \\
&\quad \text{the numerator } n(z) = z/(z-(-3-\sqrt{5})/2)^2 \text{ is analytic on } |z|<2 \\
&\quad \text{which contains the contour } |z|=1 \text{ So by GCIF} \\
&= \frac{1}{i}2\pi i \left.\frac{d}{dz}n(z)\right|_{z=(-3+\sqrt{5})/2} \\
&= 2\pi \left.\left((z-(-3-\sqrt{5})/2)^{-2} - 2z(z-(-3-\sqrt{5})/2)^{-3}\right)\right|_{z=(-3+\sqrt{5})/2} \\
&= 2\pi\left(\sqrt{5}^{-2} - 2((-3-\sqrt{5})/2)(\sqrt{5})^{-3}\right) \\
&= 2\pi\left(\frac{1}{5}\left(1+(3-\sqrt{5})/\sqrt{5})\right)\right) \\
&= 2\pi\left(\frac{1}{5}\left(3/\sqrt{5}\right)\right) \\
&= \frac{6\pi}{5\sqrt{5}}
\end{aligned}$$

3. Consider the series

$$S = \sum_{n=0}^{\infty}(-1)^n(2z)^n = 1-(2z)+(2z)^2-(2z)^3+\ldots$$

(a) The series S is geometric with common ratio $r=(-2z)$ and so will converge provided $|r| = |-2z| < 1$ that is if $|z| < \frac{1}{2}$.

(b) Using $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ (for $|z| < \frac{1}{2}$) we obtain $S = \frac{1}{1+2z}$

(c) We cannot include $z = -\frac{1}{2}$ as this is a singularity of $\frac{1}{1+2z}$ so the largest disk open centred at the origin is $|z| < \frac{1}{2}$.

Note that this is the same open disk as for the convergence of S

4. (a) Let $z_0 = 0$ $r = 1$ and $f(z) = e^z$ then

$$\begin{aligned} \int_0^{2\pi} e^{e^{i\theta}}\, d\theta &= 2\pi \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta \\ &= 2\pi f(z_0) \\ &= 2\pi e^0 \\ &= 2\pi \end{aligned}$$

(b) i. Expanding out

$$\begin{aligned} \left(\int_0^{2\pi} e^{e^{i\theta}}\, d\theta \right) &= (2\pi) \\ \Leftrightarrow \int_0^{2\pi} \left(e^{\cos\theta + i\sin\theta}\, d\theta \right) &= 2\pi \\ \Leftrightarrow \int_0^{2\pi} \left(e^{\cos\theta} e^{i\sin\theta}\, d\theta \right) &= 2\pi \\ \Leftrightarrow \int_0^{2\pi} \left(e^{\cos\theta} \left(\cos(\sin\theta) + i\sin(\sin\theta) \right) d\theta \right) &= 2\pi \\ \Leftrightarrow \int_0^{2\pi} \left(e^{\cos\theta} \cos(\sin\theta) \right)\, d\theta + i \int_0^{2\pi} \left(e^{\cos\theta} \sin(\sin\theta) \right)\, d\theta &= 2\pi + i0 \end{aligned}$$

So taking real parts we obtain $\int_0^{2\pi} e^{\cos(\theta)}\ \cos(\sin\theta)\ d\theta = 2\pi$

ii. Where as $\int_0^{2\pi} e^{\cos(\theta)}\ \sin(\sin\theta)\ d\theta = 0$ is obtained by taking imaginary parts.