

### Tutorial 9 - Solutions

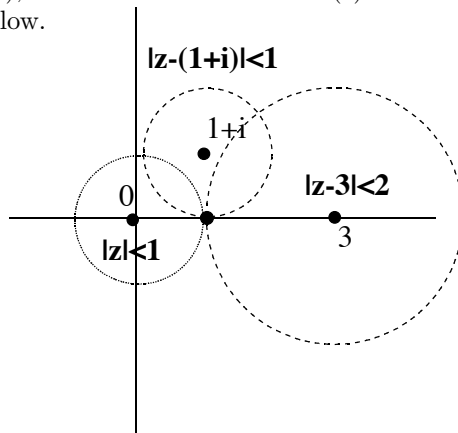
1. For the following geometric series find a maximal region  $R$  on which the series converges and the sum of the series on  $R$  :

(a)  $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \dots$  has  $r = z$  and  $a = 1$  and so converges to  $\frac{1}{1-z}$  provided  $|z| < 1$ .

(b)  $\sum_{n=0}^{\infty} i^{n+1} (z - (1+i))^n = i - (z - (1+i)) - i(z - (1+i))^2 + (z - (1+i))^3 + \dots$   
has  $r = i(z - (1+i))$  and  $a = i$  and so converges to  $\frac{i}{1 - i(z - (1+i))} = \frac{i}{1 - iz + i - 1} = \frac{i}{i - iz} = \frac{1}{1-z}$  provided  $|i(z - (1+i))| < 1 \Leftrightarrow |(z - (1+i))| < 1$ .

(c)  $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{2} \left(\frac{z-3}{2}\right)^n = -\frac{1}{2} + \frac{1}{4}(z-3) - \frac{1}{8}(z-3)^2 + \frac{1}{16}(z-3)^3 + \dots$   
has  $r = -\frac{1}{2}(z-3)$  and  $a = -\frac{1}{2}$  and so converges to  $\frac{-1/2}{1 + 1/2(z-3)} = \frac{-1/2}{-1/2 + z/2} = \frac{1}{1-z}$  provided  $|\frac{1}{2}(z-3)| < 1 \Leftrightarrow |(z-3)| < 2$ .

Notice that all the series have the same sum  $(1-z)^{-1}$ . The series in (b) is an analytic continuation of the series in (a) (they are analytic in domains that intersect and on this intersection they have the same value), and likewise the series in (c) is an analytic continuation of the series in (b). See the diagram below.



2. Let  $a_n = \frac{n^n}{n!}$ .

Remembering that  $(n+1)! = (n+1)n!$  we see that

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{(n+1)}}{(n+1)!} \bigg/ \frac{n^n}{n!} = \frac{(n+1)^{(n+1)}}{(n+1)n!} \bigg/ \frac{n^n}{n!} = (n+1)^n / n^n = \left(1 + \frac{1}{n}\right)^n$$

thus  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$ .

3. On  $R = [0, 1] \subseteq \mathbb{R}$  let  $a_n(x) = x^n$ .

(a) If  $0 \leq x < 1$  then  $\lim_{n \rightarrow \infty} x^n = 0$  and if  $x = 1$  then  $x^n = 1$  for all  $n$  and so  $\lim_{n \rightarrow \infty} x^n = 1$ . Thus

$$a(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

(b)  $a(x)$  is not continuous on  $[0, 1]$  thus the  $a_n(x)$  cannot converge uniformly (else we would contradict the boxed theorem).

4. Using the Ratio test, the Root test or otherwise find the largest open regions on which the following power series converge:

(a) Let  $a_n = \frac{n^n}{n!} \left(\frac{z+2}{3}\right)^n$  then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{(n+1)^{(n+1)}}{(n+1)!} \left(\frac{z+2}{3}\right)^{(n+1)} \bigg/ \frac{n^n}{n!} \left(\frac{z+2}{3}\right)^n \\ &= \left(\frac{z+2}{3}\right) \lim_{n \rightarrow \infty} \frac{(n+1)^{(n+1)}}{(n+1)!} \bigg/ \frac{n^n}{n!} \\ &= e \left(\frac{z+2}{3}\right) \end{aligned}$$

$$\text{thus } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \Leftrightarrow |z+2| < \frac{3}{e}$$

The ratio test also tells us that the series diverges if  $|z+2| > \frac{3}{e}$ . Thus the largest open region of convergence of  $\sum_{n=0}^{\infty} \frac{n^n}{n!} \left(\frac{z+2}{3}\right)^n$  is  $|z+2| < \frac{3}{e}$ .

(b) The 'pesky'  $(-1)^n$  term makes the series  $\sum_{n=0}^{\infty} (3 + (-1)^n) 5^n (z-i)^n$  much more amenable to the Root test.

$$\begin{aligned} \lim_{n \rightarrow \infty} |a_n|^{1/n} &= \lim_{n \rightarrow \infty} |(3 + (-1)^n) 5^n (z-i)^n|^{1/n} \\ &= 5|z-i| \lim_{n \rightarrow \infty} |3 + (-1)^n|^{1/n} \\ &= 5|z-i| \quad (*) \end{aligned}$$

$$(*) \lim_{n \rightarrow \infty} |3 + (-1)^n|^{1/n} = 1 \text{ as } \begin{array}{ccc} (2)^{1/n} & \leq (3 + (-1)^n)^{1/n} \leq & (4)^{1/n} \quad \forall n \in \mathbb{Z} \\ \downarrow & & \downarrow \\ 1 & & 1 \end{array} \text{ as } n \rightarrow \infty.$$

Thus the result (\*) follows by the sandwich rule.

So (using a similar argument as in part (a)) by the root test the largest open set upon which the series  $\sum_{n=0}^{\infty} (3 + (-1)^n) 5^n (z-i)^n$  converges is  $|z-i| < \frac{1}{5}$ .

5. If  $|z| \leq r$  where  $r$  is any real number satisfying  $0 < r < 1$  it follows that  $|z^m| < r^m (= M_n)$  and  $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$  thus invoking the Weierstrass M-test we have uniform convergence on  $|z| \leq r < 1$ .

6. If the power series  $P(z) = \sum_{n=0}^{\infty} a_n(z)$  converges on the open disk  $|z-a| < R$  it will converge uniformly on any closed subdisk of the form  $|z-a| \leq r$  where  $r < R$ . Thus

$$(a) \sum_{n=0}^{\infty} \frac{n^n}{n!} \left(\frac{z+2}{3}\right)^n;$$

$$(b) \sum_{n=0}^{\infty} (3 + (-1)^n) 5^n (z-i)^n.$$

converge uniformly on  $|z+2| \leq r$  where  $r < \frac{3}{e}$  respectively.

$$|z-i| \leq r' \text{ where } r' < \frac{1}{5}$$