1 Introduction: Some Basic Notions and Ideas. Binomial Model and Beyond

1.1 Introduction (1)

Probabilistic modelling. Real-world probability vs risk-neutral probability. Equivalent probabilities.

1.2 Binomial Asset Pricing Model (7)

Assumptions of the model. Stock and bond dynamics. Sample space, its graphical representation (11). Trading. Contingent claims (derivative securities), options (13). European call/put. The idea of hedging. Perfect hedge and claim price.

1.3 Single Period Binomial Model (18)

Trading strategies. Portfolio hedging. The perfect hedge as the cheapest one (21). The replicating strategy and its value as an expectation under $\mathbf{P}^*$ (23). Arbitrage opportunities. No arbitrage ("arbitrage-free", "fair") price and the replicating portfolio. The expected value of the discounted price process under $\mathbf{P}^*$ (26). Pricing a European call. No arbitrage condition: geometric meaning and existence of $\mathbf{P}^*$ with the property that $S_0 = \mathbf{E}^*(S_1/(1+r))$. Pricing a European put and the put-call parity.

1.4 No Arbitrage Condition for Finite Single Period Markets (37)

The assumptions of the model. Convex sets, the convex hull of a finite set, a representation thereof as a collection of linear combinations of the points from the set. Straight lines and a version of the separation theorem (42). No arbitrage theorem for finite single-period markets (43). EMM property. Theorem on no arbitrage pricing (49). Completeness theorem (52) [the proofs of all three theorems are not examinable].

1.5 Multiperiod Binomial Model (57)

Trading strategies. Replication. Self-financing strategies (61). No arbitrage in the T-period model $\Leftrightarrow$ no arbitrage in each of the time periods. Constructing a replicating portfolio: step by step (65). Diagram method. The value of the replicating portfolio at time $t = 0$ (73). Pricing a European call.

1.6 Black-Scholes Formula as a Limit of the Binomial One (75)

Frequent trading. The Black-Scholes formula.

1.7 Summary of NA Pricing Theory (80)

No arbitrage (NA) $\Leftrightarrow$ under some probability $\mathbf{P}^*$ (EMM), equivalent to the real-world $\mathbf{P}$, the discounted price process is a martingale ("no arbitrage theorem"). An NA market is complete $\Leftrightarrow$ the EMM $\mathbf{P}^*$ is unique. In a NA market, the price of a replicable claim is given by the expectation (under $\mathbf{P}^*$) of the discounted claim value.

2 Solid Foundations: Basic Machinery of Probability Theory

2.1 First Corner Stone: σ-algebras (a.k.a. σ-fields) (82)

Sample (outcome) space Ω. The definition of a σ-algebra (A1–A3, (83)). Measurable space. Examples. Different σ-algebras on the same Ω. Filtration (88). Borel sets.

2.2 Second Corner Stone: Random Variables (92)

Definition. Indicators. The σ-algebra generated by a random variable. Its interpretation as the information contained in the random variable. Filtration as the "history" of a random process (97). Simple random variables. General (positive) random variables as limits of increasing sequences of simple ones.

2.3 Thrid Corner Stone: Probabilities (and Measures) (99)

The definition of a probability (P1–P3, (99)). Probability space. Measures. Independence. The distribution of a random variable.

2.4 Ground Floor: Integrals/Expectations (106)

Integrals of simple random variables. The integral of a general (positive) random variable as the limit of integrals of the approximating simple random variables (108). Lebesgue integral vs conventional (Riemann) integral. Expectations. Expectations under equivalent distributions.

2.5 First Floor: Conditional Expectations (114)

Conditional expectation given an event. Conditional expectation given a simple random variable. The general definition of conditional expectation given a random variable. Properties of conditional expectations (CE1°–CE4°, (117)). The definition of conditional expectation given a σ-algebra (122). Its properties (124). The conditional expectation of a random variable X as the best (in mean quadratic) prediction of its value (from the information contained in the condition). Conditional probability [(127-8), not examinable].

2.6 Transforms (129)

Generating functions, moment generating functions, Laplace transforms, characteristic functions (ch.f.'s). Properties of ch.f.'s (134). Examples (139). The inversion formula (141). "General rules": smooth distribution $\Leftrightarrow$ fast decay of the ch.f. at $\pm\infty$; smooth ch.f. $\Leftrightarrow$ "thin tails" of the distribution.

2.7 Convergence of Distributions: LLN, CLT and More (147)

Definition. Important facts: convergence of distributions $\Leftrightarrow$ (point-wise) convergence of ch.f.'s ["the method of ch.f.'s"](151). The weak LLN (152), Poisson theorem, CLT (156). Stable distributions: definition (157), sums of independent copies of stable RV's, examples (normal, Cauchy). The form of the ch.f. of a general stable distribution (160). Conditions of convergence to stable distributions. The standard CLT vs convergence to stable distributions (162–3). Recognizing stable distributions from data (163). Infinitely divisible distributions (167).

3 Martingales

3.1 Definitions and First Examples (167)

Definition of an adapted process; of a martingale (169). Examples (171).

3.2 Stopping Times (173)

Definition. First hitting times. Examples. American Derivative Securities (177). Theorem on stopped martingales (180). Optional Stopping/Sampling Theorem (OST) (182). Applications thereof.

3.3 Martingales and Claims Pricing (189)

The general framework. Three main facts:

⋄ There exists an EMM $\mathbf{P}^*$ on $(\Omega, \mathcal{F}) \Leftrightarrow$ no arbitrage.

⋄⋄ The time t price of an attainable claim X with maturity T is given by the conditional expectation $\mathbf{E}^*(X/(1+r)^{T-t}|\,\mathcal{F}_t)$ (discrete time).

⋄⋄⋄ The EMM $\mathbf{P}^*$ on $(\Omega, \mathcal{F})$ is unique $\Leftrightarrow$ any claim is attainable ("complete market").

Pricing an American Derivative Security (192).

4 Brownian Motion

4.1 Definitions and Basic Properties (196)

Definition. Arithmetic and geometric Brownian motions. Lévy processes and infinitely divisible distributions (200). The covariances of Brownian motion (203). Gaussian processes. A characterization of Brownian motion as a Gaussian process with given mean and covariance functions (205).

4.2 Path Properties of BM (207)

Continuous, but nowhere differentiable. Infinite variation, non-random second variation ($(dW_t)^2 = dt$) (210). What happens when $t \to \infty$ or $t \to 0$.

4.3 Markov Property of BM (212)

Definition of a Markov process. Any process with independent increments is Markovian. Transition density of BM. The joint density of $(W_{t_1}, \ldots, W_{t_k})$ as a product of transition densities (216). Simulating $(W_{t_1}, \ldots, W_{t_k})$.

4.4 Distributions of Some RV's Related to the BM (218)

The distributions of the maximum of a BM on $[0, T]$ and of the hitting time of a level x (219). The distribution of $e^{\mu t + \sigma W_t}$ (224).

4.5 MG's of BM (225)

The three MG's of BM. An application (the distribution of the first hitting time is 1/2-stable) (229).

4.6 The Functional CLT (231)

Convergence of finite-dimensional distributions (233). The space $C[0,1]$, continuous functionals on it (235). Examples. The theorem (239). The multivariate BM.

5 Stochastic calculus

5.1 Introduction. Defining the Itô Integral (241)

A motivating argument. Constructing the Itô Integral: first for step functions and simple processes (248). The properties of the Itô integral of a simple process (250). Extension to the general case; properties remain the same (258). Itô integral of non-random functions (259).

5.2 Itô formula (263)

Itô process. First Itô formula (264). Product rule of Itô calculus (268). Second Itô formula (270). When an Itô process is an MG (272).

5.3 Stochastic Differential Equations (272)

Definitions. Ornstein-Uhlenbeck process (274). Interest rate models (Vasicek, Cox-Ingersol-Ross) (278).

5.4 Black-Scholes model (279)

Black-Scholes SDE. The Black-Scholes market is arbitrage-free and complete (286).

6 Diffusion Processes

6.1 Introduction: from Simple RW's to PDE's (287)

A difference equation for the probabilities in a symmetric simple random walk. Transition to the heat equation.

6.2 Definitions (288)

A definition of diffusion processes in terms of the conditional expectations of X_t and X_t^2: properties A(i)–A(iii) (289). Equivalent definition in terms of an SDE: B (292).

6.3 Kolmogorov Differential Equations and Generators (293)

Reviewing Kolmogorov DE's for jump Markov processes. Backward Kolmogorov DE for $v(s,x) = \mathbf{E}\left(\psi(X_t) \middle| X_s = x\right)$, $0 < s < t$ (295–7). Examples. Forward Kolmogorov DE for $u(t,y) = p(s,x;t,y)$ (the transition density of the diffusion process) (299; the general form of the DE: 305). Generators of diffusion processes are differential operators (305). Stationary densities as solutions to DE's (306, and also 306a-b).

NB: Here ends the common part of 620-302 and 300-332.

6.4 Method of Differential Equations (307)

The probability of hitting a point b before point a given the process starts at x as a solution to the DE $0 = \mu V' + \frac{1}{2}\sigma^2 V''$, $V(a) = 0$, $V(b) = 1$ (309). Examples. The expectation of the hitting time (given the process starts at x) as a solution to the DE $1 = -\mu U' - \frac{1}{2}\sigma^2 U'' = 0$, $U(a) = U(b) = 0$ (316). The Laplace transform of the hitting time as a solution to the DE $\mu W' + \frac{1}{2}\sigma^2 W'' - \lambda W = 0$, $W(a) = W(b) = 1$ (320).

7 Some Applications

7.1 Branching Processes (322)

Assumptions of the model: a large initial number of individuals, the mean offspring number is close to one, finite variance, fast change of generations. Verifying conditions A for a diffusion process (326). Extinction probabilities (329).

7.2 Wright-Fisher Model (329)

Assumptions of the model: a large population, mutation and selectivity rates, new generation formation (331). Case 1: mutation effects only (333). The resulting SDE (335). One-way mutation (336). Case 2: selection only (339).

7.3 Brownian Bridge (340)

The first definition (conditional Brownian motion process). Transition density (342). Second definition (via an SDE). The mean, variance and covariance functions of the Brownian bridge. Third definition (as a Gaussian process with given means and covariances) (346). Fourth definition: $W_t^0 = W_t - tW_1$ (347). Fifth definition: $W_t^0 = (1-t)W_{t/(1-t)}$ (NB: all five are equivalent to each other!). The probability for a Brownian bridge to cross a linear boundary (349). Example: an empirical (sample) distribution function, EDF (349). Statistics as functionals of the EDF. A universal approach to establishing consistency and asymptotic normality of statistics/estimators via the EDF and Brownian bridge approximation (351–355). Goodness-of-fit tests (355).

7.4 Pricing Financial Options (357)

The model. Review of the general pricing formula (360). Revisiting the Black-Scholes formula (363). Barrier options (364). The derivation of the Black-Scholes PDE (366). An alternative probabilistic solution (369).