Ch. 1. **Introduction**: Some Basic Notions & Ideas. Binomial Model & Beyond.

1.1 **Intro**

What is this course about?

- Probability Theory and its applications (mostly - to financial mathematics).
- Probability Theory:
  - Provides math'1 models for "random phenomena";

→ Relative frequency interpretation of probability:
  - Suppose a random event can occur (or not occur) in a random experiment;
  - $n$ independent realizations of the experiment;

$$\frac{\text{# trials } A \text{ occurred}}{n} \sim \text{Probability of } A =: P(A).$$

- Modelling: $(\Omega, \mathcal{F}, P)$
  - $\Omega$ = "sample/outcome space"
  - a set of $\omega$'s = outcomes of our experiment ("chance", "state of the world");
\[ \mathcal{F} = \text{collection of "events"} - \text{subsets of } \Omega; \]
\[ P = \text{prob'ly distr on } \mathcal{F}; \]
\[ \forall A \in \mathcal{F}, \ P(A) \in [0,1] \]
\[ (\text{+ something, "3 axioms").} \]

**IMPORTANT:** In our subject, when dealing with options pricing related topics, there will be at least two different prob'ly distributions on the same \((\Omega, \mathcal{F})\)

\[
\text{NB: "Prob'ly distr" = "prob'ly" = "prob'ly measure" = "distr" law.}
\]

1) \(P = \text{"real-world" (a.k.a. "statistical", "actuarial") prob'ly:} \)
\[ \rightarrow \text{long-run relative freq's of events (as above), estimated, say, from mortality tables} \]
\[ \text{and used, say, to compute insurer premiums (to ensure that with a high prob'ly your insurance business will be viable).} \]

2) \(P^* = \text{"risk-neutral" (a.k.a. "arbitrage-free", "fair") prob'ly used for pricing financial derivatives etc:} \)
has little to do with the statistical prob'ly (and has no natural interpretation in terms of relative freq's),

- must have: $P(A) > 0 \Leftrightarrow P^*(A) > 0$, $A \in F$ (some events are possible under $P$ and $P^*$); in this case one says that $P$ and $P^*$ are equivalent;

- "martingale property": loosely speaking, given the past history, the (discounted) asset price remains on the average constant (to be stated in terms of conditional expectations under $P^*$).

The main idea of options pricing is to exclude the role of chance (as we will see soon) - and introducing an "artificial" prob'ly $P^*$ is part of this! Paradox!

The 2nd prob'ly $P^*$ is often called the "equivalent martingale measure" (EMM). It actually appears when solving an optimization problem. And there will be no "with high prob'ability" or "on the average fair" in solution - every will be (at least, theoretically) certain!
4.2 Binomial Asset Pricing Model

(Cox - Ross - Rubinstein (1979), "Option pricing: a simplified approach")

- Introduces important notions & ideas.
- Helps understand risk-neutral ("arbitrage-free") pricing.
- Provides motivation for further studying probability theory (in particular, we'll meet conditional expectations & random processes called "martingales").

Assumptions:

1) Time is discrete: $t = 0, 1, \ldots, T < \infty$
2) There are two assets ("basic underlying securities", "underlying"

- bond (or bank account)
yielding a riskless rate of return $r$ in each time period: the bond price at time $t$ is $B_t = (1 + r)^t$, $t = 0, 1, \ldots, T$;
- a risky stock with price $S_t$ at time $t$; each time period, the price can either move "up" or "down"; there are two numbers $0 < d < u$ s.t.
where \( S_t \) is the stock price at time \( t \), \( S_{t+1} = u S_t \) or \( S_{t+1} = d S_t \), \( 0 \leq t \leq T-1 \).

**Important:** We might say that

(i) "\( u \)" occurs with probability \( p \),
    "\( d \)" - w.r.t. \( q = 1-p \);
(ii) price movements at different times are independent of each other;
(iii) these probs \( p \) can depend on \( t \),

- but all this doesn't matter!!

All what matters now is: the only possible values of \( S_{t+1} \) are \( u S_t \) and \( d S_t \).

**NB:** Later on, when extending this simple model to contain time, (i)+(ii) will become important...

**Important:** we assume that \( d < 1+r < u \).

Indeed, otherwise the model would make no sense:

\[ \rightarrow \text{if } 1+r < d < u, \text{ } S_t \text{ is not risky (just borrow money & buy stock)}; \]
\[ \rightarrow \text{if } d < u < 1+r, \text{ } S_t \text{ is not risky again - it's simply hopeless (invest all your money in bonds).} \]

Later on we will review the meaning of this assumption.
Can describe the evolution of our tiny financial market (= outcome of our experiment) using the outcome space

$$\Omega = \{ \omega = (\omega_1, ..., \omega_T) : \omega_t = u \text{ or } d; \ j=1, ..., T \}.$$ 

$$S_0 \xrightarrow{S_t} u^3S_0 \xrightarrow{u} S_t \xrightarrow{d} u^2S_0 \xrightarrow{u} S_0 \xrightarrow{d} uS_0 \xrightarrow{d} S_0 \xrightarrow{d} S_t \xrightarrow{d} S_0 \xrightarrow{d} dS_0 \xrightarrow{u} udS_0 \xrightarrow{u} ud^2S_0 \xrightarrow{u} \cdots$$

3) At any time $t$, both assets can be traded: can sell/buy as many shares/bonds (or parts thereof) as we wish, at their current prices (and this will not influence the prices!), paying no fees/charges (everything is 100% liquid).

"Short selling" of stock/bonds is also allowed:

$$\rightarrow \text{ short selling of a share of stock at time } t : \text{ receive the amount } S_t \text{ now, but deliver the share at a later time } t+k, \ k \geq 1;$$
$$\rightarrow \text{ short selling a bond } = \text{ borrowing money, interest rate } = r$$

$$\text{ "price of bond" } = (1+r)^t$$
Def. A contingent claim (= "derivative security") with maturity date (= "expiring") $T$ is a function

$$X = X(\omega) = g(S_T(\omega)) \geq 0$$

"payoff f" of the underlying asset price $S_T$ at time $T$ (more generally, can have $X(\omega) = g(S_1(\omega), \ldots, S_T(\omega))$).

In financial terms, a claim is a contract that pays the owner the amount $X$ at time $T$.

Ex. An option is a contract giving one the right (but not the obligation) to make a specified transaction at (or by) a specified time (= "maturity"/"expiry") at a specified price (= "strike").

$\rightarrow$ Call option = option to buy.
$\rightarrow$ Put option = option to sell.

- European option: at expiry $T$.
- American option: at any time $\leq T$.

Options can be on:

$\rightarrow$ stock,
$\rightarrow$ currency exchange (Uni Library needs them!),
$\rightarrow$ fuel (airlines need them!),
$\rightarrow$ etc.

Why?

$\rightarrow$ A way to reduce risk.
For a European call with strike $K$ on 1 share of stock $S_t$ and expiry $T$:

$$\text{claim} = \begin{cases} S_T - K & \text{if } S_T > K \\ 0 & \text{if } S_T \leq K \end{cases}$$

Thus the owner of a claim $X$ receives the payment $X > 0$ at time $T$. At that time $T$, we will know the value $X = g(S_T)$.

Q: But how much does this claim (=contract) cost at time $t < T$ (eg $t=0$)?

The key idea: the price (=value) of the claim at time $t=0$ should be the minimum amount of money that allows one to fulfil the claim obligation with certainty. (We will see below what happens if the claim is sold at a different price!)

This is a contingent claim with

$$X = g(S_T) = (S_T - K)^+$$

with $x^+ = \max\{0, x\}$ ("positive part of $x$")

For a European put, $-$

$$\text{claim} = \begin{cases} 0 & S_T > K \\ K - S_T & S_T \leq K \end{cases} = (S_T - K)^-$$

with $x^- = \min\{0, x\}$ ("negative part of $x$")

$N.B.: \quad x = x^+ - x^-$
All the instruments we have for that are: stock & bond, which we can trade. Could try to form a portfolio of them, whose value at time $T$ would be $X(w)$ for all states of the world $(w \in \Omega)$; "hedging". If there is a portfolio with value $X(w)$, $\forall w \in \Omega$, we have a "perfect hedge": (at time $T$,

- at time $t=T$, this portfolio and the claim $X$ generate the same cash flow;
- the value of this portfolio at time $t=0$ will give us the ("risk-neutral," or "arbitrage free," or simply "arbitrage") price of the claim at $t=0$.

Selling the claim at time $t=0$ at any other price would create an "arbitrage opportunity" = making something out of nothing without any risk (to be discussed later), and it is a std assumption that (efficient) markets are arbitrage-free.

To understand how this can work, we'll begin with a simple

1.3! SINGLE PERIOD BINOMIAL MODEL ($T=1$)

Def: A trading strategy is a pair $(\Delta, b)$:

$\Delta = \# \text{ shares of stock bought (held)}$

$\Delta = \# \text{ units of bond}$ at time $t=0$. 
\( \Delta < 0 \) means \( |\Delta| \) shares sold short; 
\( b < 0 \) means the amount \( |b|B_0 = |b| \) was borrowed. 
\( E^0 \), the bond price at \( t=0 \).

\( \text{The value of our portfolio is:} \)
\[
V_0 = \Delta S_0 + bB_0 = \Delta S_0 + b \quad \text{at time } t=0,
\]
\[
V_1 = \Delta S_1 + bB_1 = \Delta S_1 + b(1+r) \quad \text{at time } t=1.
\]

\( \text{Hedging: } V_1(w) \geq X(w), \forall w \in \Omega. \)

We have only two \( w \)'s: \( u \) and \( d \), 
and if
\[
X(w) = \begin{cases} 
X_u & \text{if } w = u, \\
X_d & \text{if } w = d,
\end{cases}
\]
we get:
\[
\begin{cases} 
\Delta u S_0 + b(1+r) \geq X_u, \\
\Delta d S_0 + b(1+r) \geq X_d.
\end{cases}
\]
Equivalently, \[
\begin{cases}
\dot{b} = -\frac{uS_0}{1+r} \Delta + \frac{X_u}{1+r} \quad \text{\(\text{Q}\)}
\end{cases}
\]
\[
\begin{cases}
\dot{b} = -\frac{dS_0}{1+r} \Delta + \frac{X_d}{1+r} \quad \text{\(\text{D}\)}
\end{cases}
\]

Solving for this point:
\[
\begin{cases}
\Delta uS_0 + \dot{b}(1+r) = X_u \\
\Delta dS_0 + \dot{b}(1+r) = X_d
\end{cases} \implies \Delta S_0(u-d) = X_u - X_d,
\]
so
\[
\Delta = \frac{X_u - X_d}{(u-d)S_0} = \frac{g(uS_0) - g(dS_0)}{uS_0 - dS_0}
\]

- a discrete-time version of the "delta-hedging rule" for derivative securities (the # of shares in portfolio is derivative of the option value w.r.t. the price of the underlying), and

\[
b = \frac{X_u - \Delta uS_0}{1+r} = \frac{1}{1+r} \left[ X_u - u \frac{X_u - X_d}{u-d} \right] = \frac{uX_d - dX_u}{(1+r)(u-d)}.
\]

\[\implies \text{Now the replicating portfolio \((\Delta, b)\) generates the same cash flow at time } t=1 \text{ as the claim } X, \text{ and the value of the portfolio at time } t=0 \text{ is}\]

**Recall:** \(d < 1 + r < u\)

**N.B.:** The perfect hedge is the cheapest one:
\[
\min_{\Delta S_0 + b} \text{ for all hedges}
\]
is attained at the point where \(=\) holds in \(\text{Q} + \text{D}\), i.e. when the value of the portfolio at time \(t=1\) is \(V_1(\omega) = X(\omega), \omega \in \Omega\), exact replication!
\[ V_0 = \Delta S_0 + b = \frac{X_u X_d - d X_u}{u - d} + \frac{u X_d - d X_u}{u - d} \]

\[ = \frac{(1+r)(X_u - X_d) + u X_d - d X_u}{(1+r)(u-d)} \]

\[ = \frac{1}{1+r} \left[ \frac{1+r-d}{u-d} X_u + \frac{u-(1+r)}{u-d} X_d \right] \]

\[ = \rho^* = P^*(au) = 1 - \rho^* = P^*(a\bar{d}) \]

\[ \text{NB: } \rho^* \in (0,1) \]

\[ E^* \left[ \frac{X}{1+r} \right] = \frac{1}{1+r} X^* \]

\[ \text{discounted claim value} \]

\[ \text{expt wrt. distr } P^* \]

\[ \Rightarrow \text{Now this } X^* \text{ is the fair price of the claim } X \text{ at time } t=0. \text{ Indeed, if the claim is sold at another price } X_0 \neq X^*, \text{ there will be an arbitrage opportunity: making a net investment of } \$0 \text{ at time } t=0, \text{ we would have at time } t=1 \text{ an amount } \geq 0 \text{, where some } w \in \Omega. \]

\[ \text{Cover the claim: } -X \]

\[ \text{Buy bonds: } -(X_0-X^*) \]

\[ \text{Balance: } \$0 \]

\[ (4+r)(X_0-X^*) > 0 \]

\[ \text{Sell the claim: } X_0 \]

\[ \text{Buy bonds: } -(X_0-X^*) \]

\[ \text{Balance: } \$0 \]

\[ (4+r)(X_0-X^*) > 0 \]

\[ \text{(i) If } X_0 > X^*: \]

\[ \text{Sell short 4 shares: } -4S_0 \]

\[ \text{Sell short 6 bonds: } -6 \]

\[ \text{Buy the claim: } -X_0 \]

\[ \text{Buy bonds: } -(x^*-x_0) \]

\[ \text{Balance: } \$0 \]

\[ (4+r)(x^*-x_0) > 0 \]

\[ \text{In both cases: riskless profit = arbitrage!} \]

\[ \Rightarrow \text{Only if the claim price at time } t=0 \text{ is } X^*, \text{ there is no such opportunity. This is the main principle of derivative pricing: the } \]
price should exclude arbitrage opportunities — they are not present in efficient markets. Replicating portfolios are means to make all this work — and their construction is perhaps even more important than just pricing options/claims.

→ Back to the claim price $X^*$: (23).
The answer came out in the form of the expectation (under some "artificial" probability) of the discounted claim value.

→ There is something special about this probability distribution. Namely,

$$E^*_* \left( \frac{S_t}{1+r} \right) = S_0$$

→ i.e. the expected (under $P^*$) discounted stock price at time $t=1$ coincides with its price at $t=0$ (as we will see later, this means that the discounted price process $S_t (1+r)^t$ forms what's called a martingale — if we adopt $P^*$);

→ note that there is a unique $P^*$ ($\Leftrightarrow P^*$) for which this holds;

→ verifying: this is a special case of the pricing formula for $X=S_t$; direct computation:

$$E^*_* \left( \frac{S_t}{1+r} \right) = \frac{1}{1+r} \left[ \frac{1+r-d}{u-d} u S_0 + \frac{u-(1+r)}{u-d} d S_0 \right]$$

$$= \frac{u(1+r-d) + d(u-1-r)}{(1+r)(u-d)} S_0 = S_0.$$

→ This is not just a nice coincidence. This actually is the main result of the arbitrage-free pricing theory.
Ex. Pricing a European call. As we saw (15), the payoff of an EC is 
\[ g(S_1) = (S_1 - K)^+ . \]
So the price \( C \) of the call is:
\[ C = \frac{1}{1+r} \left[ p^*(uS_0-K)^+ + (1-p^*) (dS_0-K)^+ \right] , \]
where 
\[ p^* = \frac{1+r-d}{u-d} \in (0,1). \]

Need to know: \( r, u, d, S_0, K. \)
Suppose: \( r=0.25 \quad S_0=1 \quad u=1.75 \quad d=0.5 \)
\( \Rightarrow \) Price & replicate! NB: \( d < 1+r < u \), OK!

- \( p^* = \frac{1.25-0.5}{1.75-0.5} = \frac{0.75}{1.25} = 0.6 \), so that
\[ C = \frac{1}{1.25} \left[ 0.6 \times (1.75-1)^+ + 0.4 \times (0.5-1)^+ \right] = \frac{0.75 \times 0.6 - 0.36}{1.25} = 0.36 \text{, mm} \]

- Replicating portfolio:
\[ \Delta = \frac{(uS_0-K)^+ - (dS_0-K)^+}{uS_0 - dS_0} = \frac{0.75 - 0}{1.75 - 0.5} = 0.6, \]
\[ b = \frac{u(uS_0-K)^+ - d(dS_0-K)^+}{(1+r)(u-d)} = \frac{-0.5 \times 0.75}{1.25 \times 1.25} = -0.24. \]

- Checking: the value of the portfolio at \( t=0 \) is:
\[ V_0 = \Delta S_0 + b = 0.6 \times 1 + (-0.24) \times 1 = 0.36 \text{ (=C, ok!)} \]
The value at \( t=1 \) is:
\[ V_1(u) = \Delta uS_0 + b (1+r) = 0.6 \times 1.75 + (-0.24) \times 1.25 \]
\[ = 1.05 - 0.3 = 0.75; \]
\( \Rightarrow \) if the stock goes up,
\[ V_1(d) = \Delta dS_0 + b (1+r) = 0.6 \times 0.5 + (-0.24) \times 1.25 \]
\[ = 0.3 - 0.3 = 0, \]
in both cases \( (S_1-K)^+ \), replication!
- **Graphical representation** (can be used for multiperiod models as well).

  **Stock price:**

  \[ \begin{array}{c}
  1.75 \\
  0.5 \\
  \end{array} \]

  **Claim value:**

  \[ \begin{array}{c}
  \text{Claim value} \\
  \frac{p*=0.6}{q*=0.4} \\
  0.36 \\
  0.75 \\
  \end{array} \]

  - **NB:** If we buy one share at \( t=0 \), then at \( t=1 \): +0.75 or −0.5.
    - If we buy one call at \( t=0 \) (−0.36), then at \( t=1 \): +0.39 or −0.36 (less risky).

  **END OF EX.**

- **The absence of such opportunity is central for any market model.**

  - When are there no arbitrage opportunities in our simple market?

  \[ \begin{array}{c}
  t=0 \\
  \text{either} \\
  \text{or} \\
  \end{array} \]

  \[ \begin{array}{c}
  \frac{dS_0}{1+r} \\
  S_0 \\
  \frac{dS_0}{1+r} \\
  S_0 \\
  \end{array} \]

  \[ \begin{array}{c}
  t=1 \\
  \text{if} \\
  \text{if} \\
  \text{if} \\
  \end{array} \]

  \[ \begin{array}{c}
  uS_0 \text{ or } S_0 \\
  S_0 \text{ or } S_0 \\
  S_0 \text{ or } S_0 \\
  \end{array} \]

  \[ \begin{array}{c}
  d<1+r<u \\
  d<1+r<u \\
  1+r<u \\
  \end{array} \]

  **No arbitrage:**

  - Sell stock short, buy bonds

  **Arbitrage:**

  - Borrow money, buy stock

  \[ \text{ Indeed, } \]

  \[ \text{ if } V_0 = \Delta S_0 + b = 0, \text{ then } b = -\Delta S_0 \]
so that
\[ V_1 = \Delta S_1 + b(1+r) = \Delta S_1 - \Delta S_0 (1+r) \]
\[ = \Delta (1+r) \left[ \frac{S_1}{1+r} - S_0 \right] \]
either \( u \) \( S_0 > S_0 \) or \( d \) \( S_0 < S_0 \)

where \( \ldots \) can be \( > 0 \) and \( < 0 \) — hence regardless of the sign of \( \Delta \), \( V_1 < 0 \) is possible. \( \text{No arbitrage!} \)

\textbf{IMPORTANT:} In the \textit{NA} case,
\[ S_0 \in \left( \frac{S_1(d)}{1+r}, \frac{S_1(u)}{1+r} \right) \]
i.e. can be represented as
\[ S_0 = \rho^* \frac{S_1(u)}{1+r} + (1-\rho^*) \frac{S_1(d)}{1+r}, \quad \rho^* \in (0,1) \]
\[ = E^* \left( \frac{S_1}{1+r} \right) \]
(\text{giving values of the RHS is interval, etc.)

\textbf{\( \rightarrow \)} Thus our \textquotedblleft natural	extquotedblright{} assumption that \( d < 1+r < u \) is in fact the \textit{NA} condition for our binomial market.

\textbf{Ex: Pricing a European Put.}

1) As we saw \( \text{ii) \(, the payoff of an EP is} \)
\[ g(S_1) = (S_1 - K)^-. \]

Using the general arbitrage-free pricing formula \( \text{ii}, the put price} \)
\[ P = \frac{1}{1+r} \left[ \rho^*(uS_0 - K)^- + (1-\rho^*) (dS_0 - K)^- \right]. \]

For the numerical values from the \textit{EC} example \( \text{ii),} \)
\[ P = \frac{1}{1.25} \left[ 0.6 \times (1.75 - 1)^- + 0.4 \times (0.5 - 1)^- \right] = 0.16. \]
2) Using the "put-call parity": a fundamental relation derived using an arbitrage-based argument.

→ Let $C$ and $P$ be the EC and EP prices, resp, on one share of a common stock, with a common strike $K$.

→ Form a portfolio consisting of:
  - 1 share;
  - 1 put on 1 share;
  - "short position" in 1 call on 1 share (write a call).

The value of the portfolio at $t=0$:

$$V_0 = S_0 + P - C.$$  

The value at $t=1$:

$$V_1 = S_1 + (S_1-K)^- - (S_1-K)^+.$$  

$$= S_1 - [(S_1-K)^+ - (S_1-K)^-]$$  

$$= S_1 - (S_1-K) = K = \text{const} \quad \text{(as } x^+ - x^- = x)$$  

→ How much is it worth at $t=0$?

→ Must have $V_0 = \frac{K}{1+r}$, for otherwise arbitrage opp'ty! Say, if $V_0 < \frac{K}{1+r}$, borrow $V_0$ from the bank, buy the portfolio $\Rightarrow$ at time $t=1$ you'll get $K - (1+r) V_0 > K - (1+r) \frac{K}{1+r} > 0$, ar'ge!

If $V_0 > \frac{K}{1+r}$, sell the portfolio; again!!

→ Thus

$$V_0 = S_0 + P - C = \frac{K}{1+r} \quad \text{"put-call parity"},$$

→ knowing one of $C/P$ is eq't to knowing them both.
For our numerical values,
\[ P = \frac{K}{1+r} - S_0 + C = \frac{1}{1.25} - 1 + 0.36 = 0.16 \] (\textbf{same})

\[ \text{Ex Real-life figures (Bingham & Kiesel)} \]
For Deutsche Bank stock, on 23/06/97 (t=0), the resp. prices (in DM)
were:
\[ S_0 = 97.70 \]
\[ C = 23.20 \]
\[ P = 4.16 \] \hspace{1cm} (t=1).

The relevant (simple) interest rate
for the period was \( r = 3.15\% \).
We have:
\[ S_0 + P - C = 97.70 + 4.16 - 23.20 = 78.66, \]
\[ \frac{K}{1+r} = \frac{80}{1.0315} \approx 77.56 \], pretty close - and
note that:

- these were in fact not European, but American options;
- DB may pay a dividend before maturity;
- the price data may be influenced by "bid-ask"/non-synch. trading
  effects,

So can hardly expect exact equality.

\[ \text{NB: If time is cont's, } T \text{ is the expiring, } r \text{ the (const. compounding)
  interest rate, the put-call parity takes the form:} \]
\[ S_t + P_t - C_t = K e^{-r(T-t)}, \text{ } \forall t \in [0,T] \]
(\text{using the same derivation argument}).