Hecke Algebras
from
Group Actions on Trees

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Abstract

The Hecke algebra associated to a group $G$ and a subgroup $H$ of $G$ is related to the representations of $G$ generated by its $H$-fixed vectors. Many groups have an associated tree on which they act upon, and Baugmartner, Laca, Willis and Ramagge [BLRW09] give a characterisation of the Hecke algebra for three separate actions and multiplication tables for two of the actions.

We give an explicit example for each of the three actions calculating the explicit multiplication tables and prove the results in the paper associated to each example.
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Introduction

A Hecke Algebra of a group $G$ and subgroup $H$ of $G$ is an algebra with basis double cosets of $H$ in $G$ and has relations to the representation theory of $G$.

Groups acting on trees are examples (with an appropriate natural topology) of totally disconnected locally compact groups, an area of much interest after George Willis’ paper on their structure in 1994 [Wil94].

Baugmartner, Laca, Willis and Ramagge in [BLRW09] give a characterisation of the Hecke algebra for three separate actions of groups on trees roughly:

- A group $G$ acting transitively on vertices with the stabilizer $H$ of a vertex $v_0$ acting transitively on spheres of vertices around $v_0$.
- A group $G$ acting transitively on edges with the stabilizer $H$ of an edge $e_0$ acting transitively on infinite geodesics through $e_0$.
- The action of an HNN-extension on its associated tree.

This document presents an explicit example for each of the three actions and calculates the Hecke Algebras.

Chapter 1 introduces the group theory and graph theory used throughout the document. Chapter 1 also introduces the $p$-adics and Laurent series over finite fields, which will be related to two of the examples.

Chapter 2 introduces Bass-Serre theory, a theory developed by Bass and Serre which associates a tree and an action on the tree to a group. This is used in the construction of the examples of groups acting on trees.

Chapter 3 defines Hecke Algebras.

Chapter 4 covers some of the results of [BLRW09].

Chapter 5 is the first example and corresponds to the vertex stabilizer case. It describes the Bruhat-Tits tree of $\text{GL}_2(\mathbb{Q}_p)$, an example which dates back to at least Serre’s book on trees [Ser80].

Chapter 6 gives another example from [Ser80] of the decomposition of $\text{SL}_2(\mathbb{F}_p((t)))$ as an amalgamated free product. This corresponds to the edge stabilizer case.

Chapter 7 gives the example of the Baumslag-Solitar groups which have been much studied in group theory. This corresponds to the HNN-extension case.

Both Chapter 5 and Chapter 6 also both contain alternate proofs to some of the relevant results from [BLRW09].
Chapter 1

Groups and Graphs

1.1 Graphs and Trees

Definition 1.1.1 (Graphs and directed graphs). A graph \( G \) consists of

- A set \( V(G) \) called the vertex set of \( G \). Elements of \( V(G) \) are called vertices.
- A set \( E(G) \) called the edge set of \( G \) whose elements are two element subsets of \( V(G) \). Elements of \( E(G) \) are called edges. If \( \{e_1, e_2\} \) is an edge we will say that \( e_1 \) is adjacent to \( e_2 \).

A directed graph \( G \) consists of

- A set \( V(G) \) called the vertex set of \( G \). Elements of \( V(G) \) are called vertices.
- A set \( E(G) \) called the edge set of \( G \) whose elements are ordered pairs of distinct elements from \( V(G) \). Elements of \( E(G) \) are called edges. If \( (e_1, e_2) \) is an edge we will say that \( e_1 \) is adjacent to \( e_2 \).

Remark. This definition of a graph excludes the possibility of loops or parallel edges.

Definition 1.1.2 (Path, length of path). Let \( G \) be a graph.

A path in \( G \) is a sequence \( (v_0, v_1, \ldots, v_n) \) such that \( v_i \in V(G) \) for all \( 0 \leq i \leq n \) and \( \{v_i, v_{i+1}\} \in E(G) \) for \( 0 \leq i \leq n-1 \).

If \( p = (v_0, v_1, \ldots, v_n) \) we call \( p \) a path from \( v_0 \) to \( v_n \).

A similar definition for path exists in directed graphs by replacing \( \{v_i, v_{i+1}\} \) by \( (v_i, v_{i+1}) \) in the above definition.

The remaining definitions in this section will only be given for undirected graphs (and, once defined, only undirected trees), but there exist standard generalisations.

Definition 1.1.3. Let \( p = (v_0, v_1, \ldots, v_n) \) be a path in a graph \( G \).

The length of \( p \), denoted \( l(p) \), is \( n \).

Definition 1.1.4 (Connected). A graph \( G \) is connected if given \( v_1, v_2 \in V(G) \), there exists a path in \( G \) from \( v_1 \) to \( v_2 \).

Definition 1.1.5 (Backtracks). Let \( p = (v_0, v_1, \ldots, v_n) \) be a path in a graph \( G \).

Then \( p \) contains a backtrack if \( v_i = v_{i+2} \) for some \( 0 \leq i \leq n-2 \).

Definition 1.1.6 (Cycle). Let \( G \) be a graph.

A cycle in \( G \) is a path \( p = (v_0, v_1, \ldots, v_n) \) such that \( n \geq 1 \) and \( v_0 = v_n \).

Definition 1.1.7 (Tree). A tree is a connected graph such that if a a path is a cycle then it must contain a backtrack.
Definition 1.1.8 (distance). Let $G$ be a graph. Let $v_1, v_2 \in V(G)$.
If $v_1 \neq v_2$ then the distance from $v_1$ to $v_2$, denoted $d(v_1, v_2)$, is
\[
\min\{l(p) \mid p \text{ is a path from } v_1 \text{ to } v_2\}
\]
if $v_1 = v_2$ define $d(v_1, v_2) = 0$.

Definition 1.1.9 (Degree). Let $G$ be a graph and let $v \in V(G)$.
The degree of $v$ is the cardinality of the set $\{e \in E(G) \mid v \in e\}$.

Remark. More specifically, if the degree of a vertex $v$ is finite then the degree of $v$ is the number of edges connected to $v$.

Definition 1.1.10 (Locally finite). A graph $G$ is locally finite if the degree of every vertex in $V(G)$ is finite.

Remark. Equivalently, a graph is locally finite if every vertex in $G$ has only a finite number of edges connected to it.

Definition 1.1.11 (Geodesics, doubly infinite/singly infinite). Let $G$ be a tree.
A doubly infinite geodesic in $G$ is an equivalence class of functions $F$ such that for all $f \in F$,
\begin{itemize}
  \item $f : \mathbb{Z} \to V(G)$
  \item $\{f(i), f(i+1)\} \in E(G)$ for all $i \in \mathbb{Z}$
  \item $f(i) = f(j)$ if and only if $i = j$
\end{itemize}
and if $f, g \in F$ there exists $\alpha \in \mathbb{Z}$ such that $f(x + \alpha) = g(x)$ for all $x \in \mathbb{Z}$.

A singly infinite geodesic or ray in $G$ is a function $f : \mathbb{Z}_{\geq 0} \to V(G)$ such that:
\begin{itemize}
  \item $\{f(i), f(i+1)\} \in E(G)$ for all $i \in \mathbb{Z}_{\geq 0}$
  \item $f(i) = f(j)$ if and only if $i = j$
\end{itemize}

Definition 1.1.12 (End of a graph). Let $G$ be a graph.
We define an equivalence relation on the rays of $G$.
Two rays $q_1 : \mathbb{Z}_{\geq 0} \to V(G)$, $q_2 : \mathbb{Z}_{\geq 0} \to V(G)$ in $G$ are equivalent if there exists $n \in \mathbb{Z}_{\geq 0}$ such that $q_1(n) = q_2(k)$ for some $k \in \mathbb{Z}_{\geq 0}$ and $q_1(n+i) = q_2(k+i)$ for all $i \in \mathbb{Z}_{\geq 0}$.
An end in $G$ is an equivalence class of rays in $G$. 

Figure 1.1: A finite tree
Definition 1.1.13 (Homogeneous). A graph $G$ is homogeneous if the degree of vertices in $G$ only takes upon one value.

A graph is semi-homogeneous if the degree of vertices in $G$ only takes two values.

1.2 Groups

1.2.1 Groups acting on trees

Definition 1.2.1 (Automorphism group of set). Let $S$ be a set.

Then the automorphism group of $S$ denoted $\text{Aut}(S)$ is the set of bijective functions $\phi : S \to S$ with group operation composition.

Definition 1.2.2 (Automorphism group of graph). Let $A$ be a graph.

Then the automorphism group of $A$ denoted $\text{Aut}(A)$ is the set

$$\{ \phi \in \text{Aut}(V(A)) \mid \{v_1, v_2\} \in E(A) \implies \{\phi(v_1), \phi(v_2)\} \in E(A)\}$$

with group operation composition.

So $\text{Aut}(A)$ consists of automorphisms of $V(A)$ which preserve the adjacency of vertices in $A$.

Definition 1.2.3 (Group acting on a graph). Let $A$ be a graph, let $G$ be a group.

Then an action of $G$ on $A$ is a group homomorphism $\varphi : G \to \text{Aut}(A)$.

Note that if $G$ acts on a graph $A$ then it will also have a well defined action on $V(A)$, $E(A)$ and geodesics/paths in $A$.

For $g \in G$ and $v \in V(A)$ we will write $g \cdot v = \varphi(g)(v)$.

Similarly, for $e = \{v_1, v_2\} \in E(A)$ we will write $g \cdot e = \{g \cdot v_1, g \cdot v_2\}$.

And for a geodesic $f : \mathbb{Z} \to V(A)$, $g \cdot f : \mathbb{Z} \to V(A)$, $(g \cdot f)(n) = g \cdot f(n)$ for all $n \in \mathbb{Z}$ and similarly for paths in $G$.

Definition 1.2.4 (Act transitively). Let $S$ be a set and let $G$ act on $S$.

Then $G$ acts transitively on $S$ if given $a, b \in S$ there exists $g \in G$ such that $g \cdot a = b$.

Definition 1.2.5 (Act faithfully). Let $S$ be a set and let $G$ act on $S$.

Then $G$ acts faithfully on $S$ if given $g_1, g_2 \in G$ such that $g_1 \cdot s = g_2 \cdot s$ for all $s \in S$ then $g_1 = g_2$.

Definition 1.2.6 (Act freely). Let $S$ be a set and let $G$ act on $S$.

Then $G$ acts freely on $S$ if given $g_1, g_2 \in G$ such that $g_1 \cdot s = g_2 \cdot s$ for all $s \in S$ then $g_1 = g_2$.

Definition 1.2.7 (Stabilizer). Let $S$ be a set and let $G$ act on $S$.

Let $s \in S$.

Then the stabilizer of $s$ is the set $\{g \in G \mid g \cdot s = s\}$.

Proposition 1.2.8. The stabilizer of any element is a group.

Proof. $id \cdot s = s$ so $id$ is in the stabilizer of $s$. If $g$ stabilizes $s$ then $g \cdot s = s$ so $s = g^{-1} \cdot s \cdot g^{-1}$ stabilizes $s$. If $g_1, g_2$ stabilize $s$ then $g_1 g_2 \cdot s = g_1 \cdot s = s$ so $g_1 g_2$ stabilizes $s$.

Definition 1.2.9 (Fixed point set). Let $S$ be a set and let $G$ act on $S$ by automorphisms.

Let $g \in G$.

Then the fixed point set of $g$ in $S$, denoted $S^g$ is the set $\{s \in S \mid g \cdot s = s\}$.

A group action on a tree is the group action on the underlying graph.

Definition 1.2.10 (elliptic, hyperbolic elements acting on a tree). Let $G$ be a group acting on a tree.

An element $g \in G$ is elliptic if it fixes a vertex in the tree. Otherwise it is hyperbolic.
Definition 1.2.11 (Edge inversion). Let $T$ be a tree, let $G$ act on $T$.

Then $G$ contains an edge inversion if there exists $g \in G$ and an edge $\{v_1, v_2\} \in T$ such that $g \cdot v_1 = v_2$, $g \cdot v_2 = v_1$.

If $g \in G$ is not an edge inversion then we call $g$ type preserving.

Definition 1.2.12 (Type-preserving). Let $G$ act on a tree $T$.

Then $G$ is type-preserving if every element of $G$ is type preserving.

Remark. The type preserving elements of a group form a subgroup.

Definition 1.2.13 (Generated subgroup). Let $G$ act on a tree $T$.

Then $G$ is type-preserving if every element of $G$ is type preserving.

Definition 1.2.14 (Generating set). Let $G$ be a group, let $S \subset G$.

Then $S$ is a generating set for $G$ if $\langle S \rangle = G$.

Definition 1.2.15 (Cayley Graph). Let $G$ be a group and let $S$ be a generating set for $G$.

The Cayley graph of $G$ with respect to $S$, denoted $\Gamma(G,S)$, is the directed graph with

- Vertex set $V(\Gamma) = G$,
- $(g_1,g_2) \in E(\Gamma)$ if and only if $g_2 = g_1 s$ for some $s \in S$.

In addition we label the edges with their corresponding element in $S$. That is label $E(\Gamma)$ by a function $f : E(\Gamma) \rightarrow S$ where for an edge $(g_1,g_1 s) \in E(G)$, $f(g_1,g_1 s) = s$.

1.2.2 HNN-extensions

Definition 1.2.16 (HNN-extension). Let $G$ be a group with presentation $\langle S \mid R \rangle$. Let $\alpha : G \rightarrow G$ be an injective group homomorphism.

Then the HNN-extension of $G$ with respect to $\alpha$ is the group with presentation

$$\langle S, t \mid R, tgt^{-1} = \alpha(g) \forall g \in G \rangle$$

HNN-extensions have many nice properties, we will need the existence of a specific automorphism.

Definition 1.2.17 (induced automorphism). Let $H$ be a group with presentation $\langle S \mid R \rangle$. Let $\alpha : H \rightarrow H$ be an injective group homomorphism.

Let $G$ be the HNN-extension of a group $H$ with presentation

$$\langle S, t \mid R, thth^{-1} = \alpha(h) \forall h \in H \rangle$$

Note that we can consider $H$ as a subgroup of $G$, by considering $H \cong \langle S \rangle \subset G$.

Then the group isomorphism $\hat{\alpha} : G \rightarrow G$, $\hat{\alpha}(g) = tgt^{-1}$ for all $g \in G$ is called the induced isomorphism.

We will also used the existence of special form of elements in HNN-extensions called the reduced word form, this will be defined in section [7.1]

Example 1.2.18. Consider the group with presentation

$$\langle x, y \mid yxy^{-1} = x^2 \rangle$$

we will call this group $G$.

Then $G$ is the HNN-extension of $\langle x \rangle \cong \mathbb{Z}$ under the injective endomorphism $\alpha : \langle x \rangle \rightarrow \langle x \rangle$, $\alpha(x) = x^2$.

To build $G$’s Cayley graph with respect to the above presentation first consider a basic cycle containing the identity generated by the relation:
1.2. GROUPS

Figure 1.2: A basic block

where we have edges with label $x$ have black arrowheads and edges with label $y$ have white arrowheads.

We can attach copies of the block generated by the relation to the sides of this original block to generate a larger portion of the Cayley graph:

Figure 1.3: A ‘sheet’

This is not yet the full Cayley graph; every second vertex along each horizontal line does not have an edge above it labelled $y$.

The full Cayley graph involves attaching a half-sheet on every second vertex of a horizontal line for each horizontal line, so the Cayley graph looks as though it ‘branches’ along each horizontal line as show below:

Figure 1.4: Branching

Then $G$ acts on its Cayley graph by given a vertex $v$ labelled $w$, $g \cdot v = u$ where $u$ is labelled $uw$.

Remark. Every group acts on its Cayley graph in a similar faithful fashion (for example compare [Mei08] Theorem 1.42).
1.2.3 Some final definitions

We will need a few more definitions which are used in the edge stabilizer and HNN-extension sections of chapter 4.

**Definition 1.2.19** (Semi-direct product). Let $N$ and $H$ be groups. Let $\varphi : H \to \text{Aut}(N)$ and for $h \in H$ let $\varphi_h = \varphi(h)$.

Then the semi direct product of $N$ and $H$ with respect to $\varphi$, denoted $N \rtimes_{\varphi} H$, is the set $N \times H$ with group operation

$$(n_1, h_1) \ast (n_2, h_2) = (n_1\varphi_{h_1}(n_2), h_1 h_2)$$

**Remark.** This is indeed a group.

**Definition 1.2.20** (Endomorphism ring). Let $(A, +, \times)$ be an algebra. Then the endomorphism ring of $A$, $\text{End}(A)$ is the set of algebra homomorphisms $\phi : A \to A$ with multiplication defined by composition and addition by pointwise addition in $A$:

$$(\phi_1 + \phi_2)(a) = \phi_1(a) + \phi_2(a) \quad \forall a \in A$$

**Remark.** An endomorphism ring can also be defined similarly for just an abelian group, but the term will only be used in the context of algebras.

**Definition 1.2.21** (Semi-group). A semi-group is a set $G$ with a binary operation $\ast : G \times G \to G$ (we will write $g * g = *(g, g)$) such that $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$ for all $g_1, g_2, g_3 \in G$.

**Definition 1.2.22** (Semi-group homomorphism). Let $G_1, G_2$ be semi-groups. A map $\phi : G_1 \to G_2$ is a semi-group homomorphism if $\phi(g_1 * g_2) = \phi(g_1) * \phi(g_2)$ for all $g_1, g_2 \in G_1$.

**Definition 1.2.23** (Semi-group crossed product). Let $A$ be an algebra and let $G$ be a semi-group. Let $\varphi : G \to \text{End}(A)$ be a semi-group homomorphism and write $\varphi(g) = \varphi_g$.

Then the semi-group crossed product of $A$ and $G$, denoted $A \rtimes_{\varphi} G$, is the set $\bigoplus_{g \in G} A_g$ where $A_g = A$ for all $g \in G$ and the subscript $g$ indexes the copies of $A$.

Addition follows from addition on $A$.

If $a \in A_g = A$ we will write $a = a_g$.

Then given $a_{g_1} \in A_{g_1} \subseteq \bigoplus_{g \in G} A_g$, $b_{g_2} \in A_{g_2} \subseteq \bigoplus_{g \in G} A_g$ define multiplication by

$$a_{g_1} \times a_{g_2} = (a \times \varphi_{g_1}(b))_{g_1 g_2}$$

where $a \times \varphi_{g_1}(b)$ is the multiplication of $a$ and $\varphi_{g_1}(b)$ in $A$.

Multiplication on two general elements in $A \rtimes_{\varphi} G$ then follows by the distributive law.

**Definition 1.2.24** (Twisted tensor product). Let $G$ be a group acting by automorphisms on a ring $R$.

Then the twisted tensor product of the action of $G$ on $R$ is the $\mathbb{Z}$-module $\mathbb{Z}[G] \otimes_{\mathbb{Z}} R$ with multiplication

$$(w \otimes r) \cdot (w' \otimes r') = ww' \otimes (w'^{-1} \cdot r)r'$$

for $w, w' \in G$, $r, r' \in R$.

For a specific action $\alpha$ of $G$ on $R$ we will write $\mathbb{Z}[G] \otimes_{\alpha} R$ for the corresponding twisted tensor product.
1.3 $p$-adics numbers and Laurent series over finite fields

1.3.1 $p$-adics

**Definition 1.3.1** ($p$-adics). Let $p \in \mathbb{Z}_{>0}$ be a prime number.

The $p$-adic numbers, $\mathbb{Q}_p$, are the set of formal series

$$\left\{ \sum_{i=-n}^\infty a_ip^i \mid n \in \mathbb{Z}, a_i \in \{0, 1, \ldots, p - 1\} \forall i, a_{-n} \neq 0 \right\}$$

with addition and multiplication such that the restriction of finite series to $\mathbb{Q}$ (which sends a finite sum to the finite sum in $\mathbb{Q}$) is a field isomorphism.

The $p$-adic integers, $\mathbb{Z}_p$, is the sub ring of elements $\sum_{i=-n}^\infty a_ip^i$ such that $n \in \mathbb{Z}_{\geq 0}$.

**Remark.** The $p$-adic numbers form a field.

**Definition 1.3.2** ($p$-adic valuation). The $p$-adic valuation on $\mathbb{Q}_p$ is the function $\nu : \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$ defined by

$$\nu\left( \sum_{i=-n}^\infty a_ip^i \right) = \begin{cases} 
-n & \text{if } \sum_{i=-n}^\infty a_ip^i \neq 0 \\
\infty & \text{if } \sum_{i=-n}^\infty a_ip^i = 0
\end{cases}$$

**Definition 1.3.3** ($p$-adic norm). The $p$-adic norm on $\mathbb{Q}_p$ is the function $|\cdot|_p : \mathbb{Q}_p \to \mathbb{R}_{\geq 0}$ defined by

$$|\sum_{i=-n}^\infty a_ip^i|_p = \begin{cases} 
p^{-\nu(\sum_{i=-n}^\infty a_ip^i)} & \text{if } \sum_{i=-n}^\infty a_ip^i \neq 0 \\
0 & \text{if } \sum_{i=-n}^\infty a_ip^i = 0
\end{cases}$$

**Remark.** It can be shown that the $p$-adic norm is a norm.

**Proposition 1.3.4** (Invariant factor decomposition).

$$\text{GL}_2(\mathbb{Q}_p) = \text{GL}_2(\mathbb{Z}_p)D \text{GL}_2(\mathbb{Z}_p)$$

where $D = \left\{ \begin{bmatrix} p^\alpha & 0 \\ 0 & p^\beta \end{bmatrix} \mid \alpha, \beta \in \mathbb{Z} \right\}$.

And

$$\text{SL}_2(\mathbb{Q}_p) = \text{SL}_2(\mathbb{Z}_p)E \text{SL}_2(\mathbb{Z}_p)$$

where $E = \left\{ \begin{bmatrix} p^{-\alpha} & 0 \\ 0 & p^\alpha \end{bmatrix} \mid \alpha \in \mathbb{Z} \right\}$.

**Proof.** We give only a sketch of the proof. For more details see for example [Art11] §14.4 or [vdW91].

Note that $\mathbb{Z}_p$ is a principal ideal domain, so we can iteratively do $\mathbb{Z}_p$-invertible row and column operations until the only non zero entry in the first row/column is in the upper left corner. We can assume this element is of the form $p^\alpha$ since for every $q \in \mathbb{Q}_p$, $q = up^\beta$ for unit $u$. Similarly we can assume that the bottom right entry is of the form $p^\beta$. \[ \square \]

This is also called the invariant factor decomposition for $\text{GL}_2(\mathbb{Q}_p)$ or $\text{SL}_2(\mathbb{Q}_p)$. It is also an example of a Cartan decomposition.
1.3.2 Laurent series over finite fields

Definition 1.3.5. $\mathbb{F}_p((t))$ is the set

$$\left\{ \sum_{i=-n}^{\infty} a_i t^i \mid n \in \mathbb{Z}, a_i \in \mathbb{F}_p \forall i, a_{-n} \neq 0 \right\}$$

with addition and multiplication as usual on formal power series.

$\mathbb{F}_p[[t]]$ is the sub ring of elements $\sum_{i=-n}^{\infty} a_i t^i$ such that $n \in \mathbb{Z}_{\geq 0}$.

Remark. $\mathbb{F}_p((t))$ is a field.

Note that addition is different in $\mathbb{F}_p((t))$ and $\mathbb{Q}_p$. For example in $\mathbb{Q}_3$, $1 + 2 = 1 \cdot 3$ but in $\mathbb{F}_3((t))$ (after identifying $\mathbb{F}_3$ with $\mathbb{Z}/3\mathbb{Z}$) $1 + 2 = 0$.

Definition 1.3.6 (Valuation). The valuation on $\mathbb{F}_p((t))$ is the function $\nu : \mathbb{F}_p((t)) \to \mathbb{Z} \cup \{\infty\}$ defined by

$$\nu\left( \sum_{i=-n}^{\infty} a_i t^i \right) = \begin{cases} -n & \text{if } \sum_{i=-n}^{\infty} a_i t^i \neq 0 \\ \infty & \text{if } \sum_{i=-n}^{\infty} a_i t^i = 0 \end{cases}$$

Definition 1.3.7 (Norm). The $p$-adic norm on $\mathbb{F}_p((t))$ is the function $\cdot : \mathbb{F}_p((t)) \to \mathbb{R}_{\geq 0}$ defined by

$$\nu\left( \sum_{i=n}^{\infty} a_i t^i \right) = \begin{cases} 2^{-\nu\left( \sum_{i=n}^{\infty} a_i t^i \right)} & \text{if } \sum_{i=n}^{\infty} a_i t^i \neq 0 \\ 0 & \text{if } \sum_{i=n}^{\infty} a_i t^i = 0 \end{cases}$$
Bass formalised a lot of earlier work on groups acting on trees in \cite{Bas93}.

In it, Bass defines a \textit{graph of groups} and associates a group constructed from free products with amalgamation and HNN-extensions to the graph of groups. The universal cover of the graph of groups is a tree which the group acts upon.

This chapter follows the first section of \cite{Bas93}. 

}\footnotemark
2.1 Graph of Groups

This section follows [Bas93].

2.1.1 Graph of groups

Bass defines a graph slightly differently then we did in chapter 1. One possible correspondence between the definitions is that for each edge \( \{v_1, v_2\} \) in our definition of a graph, Bass’s graph has two edges, one from \( v_1 \) to \( v_2 \) and one from \( v_2 \) to \( v_1 \).

**Definition 2.1.1** (Graph). A graph \( G \) consists of

- A vertex set \( V(G) \)
- An edge set \( E(G) \)
- Endpoint functions \( \partial_0 : E(G) \to V(G) \) and \( \partial_1 : E(G) \to V(G) \)
- A fix point free involution. That is a function \( \sim : E(G) \to E(G) \) such that \( \bar{e} \neq e \) for all \( e \in E(G) \) and \( \bar{e} = e \).

Additionally, we require that \( \partial_0(\bar{e}) = \partial_1(e) \) and \( \partial_1(\bar{e}) = \partial_0(e) \) for all \( e \in E(G) \).

If \( H \) consists of some subset of \( V(G) \) and some subset of \( E(G) \) such that the involution and the endpoint functions of \( G \) are well defined and consistent for these subsets then we will call \( H \) a subgraph of \( G \).

**Definition 2.1.2** (Edge path). Let \( G \) be a graph.

An edge path in \( G \) is a sequence of edges \( p = (e_1, e_2, \ldots, e_n) \), \( e_i \in E(G) \) for all \( e_i \), such that \( \partial_1(e_i) = \partial_0(e_{i+1}) \) for all \( e_i \).

We will also call \( p \) an edge path from \( \partial_0(e_1) \) to \( \partial_1(e_n) \).

The vertex sequence of \( p \) is the sequence of \( n+1 \) elements of \( V(G) \), \( (\partial_0(e_1), \partial_0(e_2), \ldots, \partial_0(e_n), \partial_1(e_n)) \).

**Definition 2.1.3** (Graph of groups). A graph of groups \( G = (G, \mathcal{G}) \) is a graph \( G \) along with some \( \mathcal{G} \) which

- Assigns a group \( A_v \) to each vertex \( v \in V(G) \).
- Assigns a group \( A_e \) to each \( e \in E(G) \) such that \( A_v = A_e \) for all \( e \in E(G) \).
- Assigns an injective homomorphism \( \alpha_e : A_e \to A_{\partial_0 e} \) for each \( e \in E(G) \).

**Definition 2.1.4** (Path group). Let \( G \) be a graph of groups.

Let \( F(E(G)) \) be the free group on the set \( E(G) \).

Then the path group of \( G \), denoted \( \pi(G) \) is defined as

\[
\pi(G) = \langle \ast_{v \in V(G)} A_v \ast F(E(G)) \rangle / R
\]

that is, the free product of every vertex group and \( F(E(G)) \) quotiented by the subgroup \( R \), where \( R \) is the normal subgroup which imposes the relations

- \( \bar{e} = e^{-1} \) for all \( e \in E(G) \)
- \( e \alpha_e(a)e^{-1} = \alpha_e(a) \) for all \( e \in E(G), a \in A_e \).

**Definition 2.1.5** (Paths in the graph of groups). Let \( G = (G, \mathcal{G}) \) be a graph of groups.

A path of length \( n \) in \( G \) is a sequence

\[
\gamma = (g_0, e_1, g_1, e_2, \ldots, g_{n-1}, e_n, g_n)
\]

where \( (e_1, e_2, \ldots, e_n) \) is an edge path in \( G \) with vertex sequence \( (v_0, v_1, \ldots, v_n) \) and \( g_i \in A_{e_i} \) for all \( i \).

We will also call \( \gamma \) a path from \( v_0 \) to \( v_n \).

We will write \( |\gamma| = g_0e_1g_1e_2 \ldots e_ng_n \in \pi(G) \) for the representative of \( \gamma \).
**Definition 2.1.6** (Representative of paths from u to v). Let \( G = (G, \emptyset) \) be a graph of groups.

Let \( u \) and \( v \) be vertices in \( G \).

Then we define

\[
\pi[u, v] := \{ |\gamma| \mid \text{\( \gamma \) is a path from \( u \) to \( v \)}\}
\]

to be the set of representatives of paths from \( u \) to \( v \).

**Example 2.1.7.** Let \( G \) be the graph of groups with a single vertex and two edges \( e \) and \( \bar{e} \) where \( A_v = \langle x \rangle \cong \mathbb{Z} \), \( A_e = A_{\bar{e}} = \langle x \rangle \cong \mathbb{Z} \), \( \alpha_e = \text{id} \), the identity homomorphism on \( \langle x \rangle \), and \( \alpha_{\bar{e}} \) is the map defined by \( \alpha_{\bar{e}}(x) = x^2 \).

![Figure 2.1: The described graph of groups G](image)

Then the path group of \( G \) has presentation:

\[
\langle x, e, \bar{e} \mid e^{-1} = \bar{e}, e\alpha_e(x)e^{-1} = \alpha_{\bar{e}}(x) \rangle = \langle x, e \midexe^{-1} = x^2 \rangle
\]

Consider the paths \( \gamma = (x, e, x, e^{-1}, x) \) and \( \gamma' = (x^4) \).

Then \( |\gamma| = xex^{-1}x = xx^2x = x^4 = |\gamma'| \). So although \( \gamma \) is a path of length 2 there exists a path of length 0 which has the same representative as \( \gamma \).

To define ‘length of a representative’ we would like to look at paths which are minimal length for that representative, this will correspond to *reduced paths*, or paths with no *reversals*.

**Definition 2.1.8** (Reversal). Let \( \gamma = (g_0, e_1, g_1, e_2, \ldots, g_{n-1}, e_n, g_n) \) be a path where \( (e_1, e_2, \ldots, e_n) \) is an edge path with vertex sequence \( (v_0, v_1, \ldots, v_n) \).

If there exists an \( i \) such that \( e_{i+1} = \bar{e}_i \) and \( g_i = \alpha_{e_{i+1}}(h) \) for some \( h \in A_{e_i} \) then \( \gamma \) contains a reversal.

**Definition 2.1.9** (Reduced paths). A path \( \gamma \) of length \( n \) is reduced if either

- \( n > 0 \) and \( \gamma \) does not contain a reversal or
- \( n = 0 \) and \( |\gamma| \neq \text{id} \).

Note that a reduced path can be produced from any path \( \gamma \) by considering \( |\gamma| \), finding the shortest word in \( \pi(G) \) equal to \( |\gamma| \) and then writing that word out letter by letter as a path.

**Theorem 2.1.10.** Let \( G \) be a graph of groups.

If \( \gamma \) is a reduced path in \( G \) then \( |\gamma| \neq \text{id} \).

*Proof.* See [Ser80] Chapter I, Theorem 1.1.

**Corollary 2.1.11** (Invariance of length for reduced paths). Consider two reduced paths from \( a \) to \( b \) of length \( n \) and \( m \) respectively: \( \gamma = (g_0, e_1, \ldots, e_n, g_n) \), \( \gamma' = (g_0', e_1', \ldots, e_m', g_m') \).

If \( |\gamma| = |\gamma'| \) then \( n = m \), \( e_i = e'_i \) for all \( 1 \leq i \leq n \) and there exists \( h_i \in A_{e_i} \) for each \( 1 \leq i \leq n \) such that

\[
\begin{align*}
g_0' &= g_0\alpha_{e_1}(h_1)^{-1} = g_0\alpha_{e_1}(h_1^{-1}) \\
g_i' &= \alpha_{e_i}(h_i)g_i\alpha_{e_{i+1}}(h_{i+1})^{-1} \\
g_n' &= \alpha_{e_n}(h_n)g_n
\end{align*}
\]
Proof. We will proceed by induction on $N := n + m$.

If $N = 0$ then $g_0 = |\gamma| = |\gamma'| = g_0'$, so $\gamma = (|\gamma|) = (|\gamma'|) = \gamma'$.

Now suppose that $N > 0$ and that the corollary holds for all $N < K$.

Consider $\delta = (|\gamma'| \land \gamma^{-1}) := (g_0, e'_1, \ldots, e'_m, g_m, e_n, \ldots, e_1, g_1')$ (i.e. $\delta$ is the concatenation of $\gamma'$ and the ‘inverse’ of $\gamma$), which is a path of length $N$ from $a$ to $a$. Since $|\delta| = |\gamma'| |\gamma|^{-1} = \text{id}_G(\mathcal{G})$, $\delta$ must contain a reversal by Theorem 2.1.10.

But $\gamma$ and $\gamma'$ are reduced and so do not contain reversals. Therefore the only possible reversal is $(e'_m, g_m, g^{-1}_m, e_n)$.

By the corollary, $e_m = e'_m$ and $g_m g^{-1}_m = e_n$ for some $h_n \in A_{e_n} = A_{e'_n}$.

Then $g_{m-1} e'_m g_m = g_{m-1} e_n (h_n) g_n = g_{m-1} e_n (h_n) e_n g_n$, where we have used the fact that $e_n e_n (h_n) = e_n (h_n) e_n$ from the relations in $\pi(G)$.

Now let

$$\gamma_1 = (g_0, e_1, \ldots, e_{n-1}, g_{n-1})$$

$$\gamma'_1 = (g_0, e'_1, \ldots, g_{m-2}, e'_{m-1}, g_{m-1} e_n (h_n))$$

Then $\gamma_1$ and $\gamma'_1$ are reduced paths from $a$ to $\partial_0 (e_n)$.

Additionally, $|\gamma_1| e_n g_n = |\gamma| = |\gamma'_1| e_n g_n$ since $g_{m-1} e'_m g_m = g_{m-1} e_n (h_n) e_n g_n$. So we must have that $|\gamma_1| = |\gamma'_1|$.

So by our induction hypothesis, $n - 1 = m - 1$ (so $n = m$), $e_i = e'_i$ for $1 \leq i < n$ and there exists $h_i \in A_i$ such that:

$$g'_0 = g_0 e_1 (h_1)^{-1} = g_0 e_1 (h_1^{-1})$$

$$g'_i = e_1 (h_i) g_0 e_{i+1} (h_{i+1})^{-1}$$

$$g_{m-1} e_n (h_n) = g_{m-1} e_n (h_n) = e_{n-1} (h_{n-1}) g_{n-1}$$

Then note that,

$$g_{n-1} e_n (h_n) = e_{n-1} (h_{n-1}) g_{n-1}$$

$$\Rightarrow g_{n-1} = e_{n-1} (h_{n-1}) g_{n-1} e_n (h_n)^{-1}$$

\[\square\]

Definition 2.1.12 (Length of a representative). Let $G = (G, \mathcal{G})$ be a graph of groups.

Let $g \in \pi[v_1, v_2] \subseteq \pi(G)$ for some vertices $v_1, v_2$ vertices in $G$.

The length of $g$ is the length of any reduced path $\gamma$ such that $|\gamma| = g$.

Definition 2.1.13 (Representatives of length $n$ from $v_0$ to $v_1$). We will use

$$\pi[v_0, v_1]_n = \{ g \in \pi[v_0, v_1] \mid g \text{ has length } n \}$$

to denote the set of length $n$ representatives of paths from $v_0$ to $v_1$.

Note that in corollary 2.1.11 we have proved more than just the invariance of length for reduced paths with the same representative, we have also proved that these paths are all related in some sense.

We will wish to define uniqueness of paths up to this relation — this will be the concept of $S$-normalized paths. That is, after some choice of normalisation, every representative will correspond to a unique $S$-normalized path.

Definition 2.1.14 (S-normalized). For each $e \in E(G)$ choose a set $S_e \subseteq A_{\partial_e} \cup A_{\partial_{e^{-1}}}$ of coset representatives for $A_{\partial_e} / A_{\partial_{e^{-1}}}$ such that $\text{id}_{A_{\partial_e}} \in S_e$.

A path $\gamma$ is $S$-normalized relative to these choices of $S_e$ if it has the form $\gamma = (s_0, e_1, \ldots, e_{n-1}, e_n, g)$ where $s_i \in S_{e_i+1}$ for all $1 \leq i < n$, $g \in A_{\partial_e e_n}$ and either $n > 0$ and $\gamma$ is reduced or $n = 0$.

Note that given a $S$-normalized path $\gamma = (s_0, e_1, \ldots, s_{n-1}, e_n, g)$, if $e_i = e_{i-1}$ for some $i$ then $s_i \neq \text{id}_{A_{\partial_{e_i}}}$, since $\gamma$ is reduced.
Corollary 2.1.15 (Uniqueness of S-normalized paths). Let $G = (G, \mathfrak{S})$ be a graph of groups. Given $v_1, v_2 \in V(G)$ every element of $\pi[a, b]$ is represented by a unique S-normalized path from $a$ to $b$ (after some choice of $S_v$ for each $e \in E(G)$).

Proof. Such a path exists since from the comment under definition 2.1.9 given a representative $|\eta| \in \pi[a, b]$ we can always find a reduced path $\gamma$ such that $|\gamma| = |\eta|$.

Suppose $\gamma = (g_0, e_1, g_1, \ldots, e_n, g_n)$. Then for some $s_0 \in S_{e_1}$, $h_0 \in A_{e_1}$, $g_0 = s_0\alpha_{e_1}(h_0)$. Then $g_0e_1g_1 = s_0\alpha_{e_1}(h_0)e_1g_1 = s_0e_1\alpha_{e_1}(h_0)g_1$. Then we can write $\alpha_{e_1}(h_0)g_1 = s_1\alpha_{e_2}(h_1)$ for some $s_1 \in S_{e_2}$, $h_1 \in A_{e_2}$. Continue this process to produce the required path.

Now we prove uniqueness. Consider two S-normalized paths from $a$ to $b$, $\gamma = (g_0, e_1, \ldots, e_n, g_n)$ and $\gamma' = (g_0', e_1', \ldots, e_m', g_n')$ such that $|\gamma| = |\gamma'|$ (we know that they are the same length since they are reduced if $n > 0$).

By corollary 2.1.11

\[
g_0' = g_0\alpha_{e_1}(h_1)^{-1} \\
g_i' = \alpha_{e_i}(h_i)g_i\alpha_{e_{i+1}}(h_{i+1})^{-1}
\]

Since $\gamma$ and $\gamma'$ are S-normalized we know $g_0 = g_0'$ so $\alpha_{e_1}(h_1) = id_{A_{e_1}}$. So $h_1 = id_{A_{e_1}}$ since $\alpha_{e_1}$ is injective.

Proceeding by induction on $i$ we will know that $\alpha_{e_i}(h_i) = \alpha_{e_i}(id_{A_{e_i}})$.

Then $g_i' = g_i\alpha_{e_{i+1}}(h_{i+1})^{-1}$, so $g_i' = g_i$. \qed

2.1.2 The fundamental group and its covering tree

Definition 2.1.16 (Fundamental group). Let $G = (G, \mathfrak{S})$ be a graph of groups. Let $v \in V(G)$.

The fundamental group of $G$ at $v$ is $\pi_1(G, v) = \pi[v, v]$.

Note that the set $\pi[v, v]$ is a subgroup of $\pi(G)$ since the concatenation of two paths from $v$ to $v$ is a path from $v$ to $v$ and the representative of the reversal of a path is the inverse of the representative of the path.

For the following section, given $G = (G, \mathfrak{S})$ a graph of groups and $v_0$ is some distinguished vertex in $G$, if $\gamma$ is a path in $G$ from $v_0$ to $v$ and $g = |\gamma|$ we will write $|\gamma|_v$ or $[g]_v$ for the coset $gA_v \in \pi[v_0, v]/A_v$.

We will also write edges as ordered pairs of vertices, for example if $[g]_{v_1}$ and $[g']_{v_2}$ were vertices for some graph then $e = ([g]_{v_1}, [g']_{v_2})$ would be an edge such that $\delta_0 e = [g]_{v_1}$ and $\delta_1 e = [g]_{v_2}$.

Definition 2.1.17 (Universal cover). Let $G = (G, \mathfrak{S})$ be a graph of groups and let $v_0 \in V(G)$.

The universal cover of $G$ at $v_0$ is the graph denoted $X = (G, v_0)$.

Where

\[ V(X) = \coprod_{v \in V(G)} \pi[v_0, v]/A_v \]

and if $v_1, v_2 \in V(G)$, $([g]_{v_1}, [g']_{v_2})$ is an element of $E(X)$ if and only if $g^{-1}g' \in \pi[v_1, v_2]_1$.

Note that the requirement for two vertices $[g]_{v_1}, [g']_{v_2}$ to be adjacent is equivalent to requiring that $g' = gaeb$ for some $a \in A_{v_1}$, $b \in A_{v_2}$, $e \in E(V)$ such that $\delta_0(e) = v_1$, $\delta_1(e) = v_2$.

Also note that $\pi(G, v)$ acts on $(G, v_0)$ by given $g \in \pi(G, v)$, $g \cdot (g'A_v) = gg'A_v$. This action is without edge inversions.
Definition 2.1.18 (Tree). Let $G$ be a graph.

$G$ is a tree if and only if both of the following hold:

- $V(G) \neq \emptyset$ and for any two vertices $v_1, v_2 \in V(G)$ there exists an edge path from $v_1$ to $v_2$. ($G$ is connected)
- There does not exist an edge path $\gamma = (e_1, \ldots, e_n)$, $n > 0$ such that $\partial_0(e_1) = \partial_1(e_n)$ and $e_{i+1} \neq \overline{e}_i$ for all $1 \leq i < n$. ($G$ has no cycles)

If $H$ is both a tree and a subgraph of a tree $G$ then $H$ is called a subtree of $G$.

A subtree $H$ of a tree $G$ is maximal if for any other subtree $F$ of $G$, if $H$ is a subtree of $F$ then $F = H$.

Theorem 2.1.19. Let $G = (G, \mathcal{F})$ be a graph of groups. Let $v_0 \in V(G)$.

Then the covering graph $X = (\tilde{G}, v_0)$ is a tree.

Proof. Bass proves this in [Bas93] theorem 1.17 by showing that $X$ is an inverse system then referring to [Ser80] Chapter I 5.3.

So far the fundamental group and covering tree depends on the choice of $v_0$, but it turns out that both are independent $v_0$.

Definition 2.1.20 (Fundamental group relative to $T$). Let $G = (G, \mathcal{F})$ be a graph of groups.

Let $T$ be a maximal subtree of $G$.

The fundamental group of $G$ with respect to $T$ is

$$\pi_1(G, T) = \pi(G)/R$$

where $R$ is the normal subgroup which imposes the relations $e = \text{id}$ if $e \in E(T)$.

Remark. Bass calls this the ‘group theoretic contraction of $T$ to a point’.

Proposition 2.1.21. Let $G$ be a graph of groups with some maximal subtree $T$ and some $v \in V(G)$.

The natural surjective homomorphism of graph of groups $q : \pi(G) \to \pi_1(G, T)$ restricts to an isomorphism $q_1 : \pi(G, v) \to \pi_1(G, T)$.

Proof. See [Ser80] Chapter I, prop 20.
Chapter 3

Hecke Algebras

Erwin Hecke introduced certain endomorphisms on the space of entire elliptic modular forms in 1937 [Hec37a], [Hec37b]. These endomorphisms are now known as Hecke operators.

Goro Shimura introduced abstract Hecke Algebras in 1959 [Shi59] which give an algebraic viewpoint to Hecke operators.

Since then work on Hecke algebras has been done in many areas, including work by Iwahori [Iwa64], Matsumoto [Mat64] and Curtis, Iwahori and Kilmoyer [CIK71] on $(B, N)$-pairs, Kazhdan and Lusztig on Coxeter groups [KL79] and works by Iwahori, Matsumoto, Kazhdan and Lusztig on local fields [IM65], [KL87].

The following chapter follows [Kri90] in defining Hecke algebras.
3.1 Hecke Algebras

3.1.1 The free module of double cosets

Recall that given a group \( G \) and a subgroup \( H \) we can define left and right cosets in \( G \) relative to \( H \):

Definition 3.1.1 (Cosets). Let \( G \) be a group. Let \( H \) be a subgroup of \( G \). Let \( g \in G \). The right coset of \( g \) (relative to \( H \)) is the set

\[ Hg = \{hg \mid h \in H\} \]

The left coset of \( g \) is the set

\[ gH = \{gh \mid h \in H\} \]

Double cosets are defined similarly,

Definition 3.1.2. The double coset of \( g \) relative to \( H \) is the set

\[ HgH = \{h_1gh_2 \mid h_1, h_2 \in H\} \]

Also recall that for right (and left) cosets we also have the property that

Proposition 3.1.3. For \( g, g^* \in G \), the following are equivalent:

1. \( Hg = Hg^* \)
2. \( Hg \cap Hg^* \neq \emptyset \)
3. There exist \( h \in H \) satisfying \( hg = g^* \)

And once again similarly for double cosets,

Proposition 3.1.4. For \( g, g^* \in G \), the following are equivalent:

1. \( HgH = Hg^*H \)
2. \( HgH \cap Hg^*H \neq \emptyset \)
3. There exist \( h_1, h_2 \in H \) satisfying \( h_1gh_2 = g^* \)

Proof.

(1 \( \implies \) 2) If \( HgH = Hg^*H \), then \( HgH \cap Hg^*H = HgH \neq \emptyset \).
(2 \( \implies \) 3) If \( HgH \cap Hg^*H \neq \emptyset \) then there exists \( h_1, h_2, h_\alpha, h_\beta \in H \) such that \( h_1gh_2 = h_\alpha g^*h_\beta \implies h_\alpha^{-1}h_1gh_2h_\beta^{-1} = g^* \). Then \( h_\alpha^{-1}h_1, h_3h_\beta^{-1} \in H \) since \( H \) is a group.
(3 \( \implies \) 1) If there exists \( h_1, h_2 \in H \) such that \( h_1gh_2 = g^* \) then

\[
HgH = \{h_\alpha gh_\beta \mid h_\alpha, h_\beta \in H\} \\
= \{h_\alpha h_1^{-1}h_1gh_2h_\beta^{-1}h_\beta \mid h_\alpha, h_\beta \in H\} \\
= \{h_\alpha h_1^{-1}g^*h_2^{-1}h_\beta \mid h_\alpha, h_\beta \in H\} \\
= \{hg^*h_j \mid h_i, h_j \in H\} \\
= Hg^*H
\]

Since every element of \( G \) is in some double coset, the above proposition implies that the double cosets of \( G \) with respect to \( H \) partition \( G \) (similar to left and right cosets). Additionally, since any left coset \( gH \) will wholly be contained in a double coset \( HgH \), left (and similarly right) cosets will partition each double coset as well.
We will wish to sum over left, right and double cosets so we define the following equivalence relations on $G$:

\[
    g \sim_r g' \iff Hg = Hg' \\
    g \sim_l g' \iff gH = g'H \\
    g \sim_d g' \iff HgH = Hg'H
\]

Then the notation $g : H \backslash G$ in a summation will represent summing over all the right cosets of $H$ in $G$. That is, we choose a representative for each equivalence class under $\sim$ and then sum over these representatives, choice of representatives will not matter in any of the places we use this notation.

Similarly $g : G/H$ will denote summing over the left cosets and $g : H \backslash G/H$ will denote summing over the double cosets.

**Definition 3.1.5 (Right coset module).** Let $G$ be a group and $H$ be a subgroup of $G$. Let $R$ be a ring.

Then $R_R(H, G)$ is the free $R$-module with basis rights cosets in $G$ relative to $H$.

That is $R_R(H, G)$ is a free $R$-module with basis

\[
    B_R = \{ Hg \mid g : H \backslash X \}
\]

We will abbreviate $R_R(H, G)$ as $R(H, G)$.

**Remark (Addition and scalar multiplication).**

We can write an element $U \in R(H, G)$ as

\[
    U = \sum_{g : H \backslash G} u(Hg)Hg
\]

where $u(Hg) \in R$ and $u(Hg)$ is non zero for only finitely many right cosets $Hg$ (i.e. $u$ has finite support).

Then

\[
    \alpha U = \sum_{g : H \backslash G} \alpha u(Hg)Hg
\]

Also if we add two elements $U_1, U_2 \in R(H, G)$, $\alpha \in R$, where

\[
    U_1 = \sum_{g : H \backslash G} u_1(Hg)Hg \\
    U_2 = \sum_{g : H \backslash G} u_2(Hg)Hg
\]

where $u_1, u_2 : B \to R$ and $u_1$ and $u_2$ have finite support.

Then

\[
    U_1 + U_2 = \sum_{g : H \backslash G} [u_1(Hg) + u_2(Hg)]Hg
\]

Additionally, we can define an action of $G$ on $R(H, G)$ by multiplying on the right. That is, if $h \in G$ and $U \in R(H, G)$ denote the action of $h$ on $U$ as $Uh$.

Then we define

\[
    Uh := \sum_{g : H \backslash G} u(Hg)Hgh
\]

**Definition 3.1.6 (Invariant subset).** Let $G$ be a group. Let $H$ be a subgroup of $G$.

\[
    R(H, G)^H = \{ U \in R(H, G) \midUh = U \forall h \in H \}
\]

That is, $R(H, G)^H$ is the subset of elements in $R(H, G)$ invariant under the action of every element in $H$. 

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Proposition 3.1.7. $\mathcal{R}(H, G)^H$ with multiplication and addition from $\mathcal{R}(H, G)$ is a submodule of $\mathcal{R}(H, G)$.

Proof. Let $U_1, U_2$ be elements of $\mathcal{R}(H, G)$ and $\alpha$ and element of the ring $R$.
We will prove that $\mathcal{R}(H, G)^H$ is closed under addition and scalar multiplication.
So given some $h \in H$,

$$(U_1 + U_2)h = U_1h + U_2h = U_1 + U_2$$

and

$$(\alpha U_1)h = \alpha(U_1h) = \alpha U_1$$

Definition 3.1.8 (Double coset module). $\mathcal{H}_R(H, G)$ is the free $R$-module of double cosets in $G$ relative to $H$.
That is, $\mathcal{H}_R(H, G)$ has basis the set $B_H = \{HgH \mid g \in H \backslash X / H\}$

We will abbreviate $\mathcal{H}_R(H, G)$ as $\mathcal{H}(H, G)$.

Remark. So $\mathcal{H}(H, G)$ consists of elements:

$$T = \sum_{g \in H \backslash G / H} t(Hg)HgH$$

where $t : B_H \to R$ has finite support.
3.1.2 The product

We will require each double coset to decompose into a finite number of right (or left) cosets. This will be used in identifying \( \mathcal{R}(H,G)^H \) with \( \mathcal{H}(H,G) \) which is a step in defining multiplication in the Hecke Algebra.

**Proposition 3.1.9.** Let \( G \) be a group. Let \( H \) be a subgroup of \( G \). Then for each \( g \in G \)

\[
HgH = \bigsqcup_{a \in \langle H \cap g^{-1}Hg \rangle \setminus H} Hga \quad \text{as disjoint unions.}
\]

**Proof.** Note that \( g^{-1}Hg \) is a group, so \( H \cap g^{-1}Hg \) is a group and the above unions are well defined.

We will show that

\[
HgH = \bigsqcup_{a : (H \cap g^{-1}Hg) \setminus H} Hga
\]

If \( m \in HgH \) then \( m = h_1g_2 \) for some \( h_1, h_2 \in H \). But since \( h_2 \in H \), \( h_2 = g^{-1}h_1 a ga \) for some \( h_1 \in h, a : (H \cap g^{-1}Hg) \setminus H \). Then \( m = h_1g_2 = h_1gg^{-1}h_1a ga = h_1h_2a ga \in Hga \).

On the other hand, if \( m \in \bigsqcup_{a : (H \cap g^{-1}Hg) \setminus H} Hga \), then \( a \in H \) for each \( a : (H \cap g^{-1}Hg) \setminus H \) so \( m \in HgH \).

The union is disjoint since right cosets partition double cosets (see proposition 3.1.3 and the comment after proposition refprop:doublecos).

A similar proof holds for the left coset version. \( \square \)

Then the following equivalence follows from the above proposition:

**Corollary 3.1.10.** Let \( G \) be a group and \( H \) be a subgroup of \( G \). The following conditions are equivalent:

1. For any \( g \in G \), the index of \( H \cap gHg^{-1} \) in \( H \) and \( gHg^{-1} \) is finite.
2. Every \( H \) double coset contains finitely many right \( H \) cosets.
3. Every \( H \) double coset contains finitely many left \( H \) cosets.

**Definition 3.1.11 (Almost normal).** A subgroup \( H \) of a group \( G \) which satisfies any of the above conditions is said to be almost normal in \( G \).

If \( H \) is an almost normal subgroup of \( G \) we say that \( (G,H) \) is a Hecke pair.

Let \( (G,H) \) be a Hecke pair.

We will define a map \( \iota : \mathcal{H}(H,G) \to \mathcal{R}(H,G) \) where \( \iota(T) \) will represent the decomposition of a double coset into its right cosets.

Let \( T \in \mathcal{H}(H,G) \) so

\[
T = \sum_{g : H \setminus G / H} t(HgH)HgH
\]

for some \( t(HgH) \) with finite support.

Then let

\[
\iota(T) := \sum_{g' : H \setminus G / H} \sum_{g : (H \cap g^{-1}Hg) \setminus H} t(Hg'H)Hg'g = \sum_{g : H \setminus G} t(HgH)Hg
\]

where the last equality highlights that after decomposing a sum of double cosets, every right coset has the coefficient of the double coset in which it was contained.

The image of \( \iota \) is a finite sum since \( (G,H) \) is a Hecke pair, and \( t \) is only non zero for a finite number of double cosets.

Additionally, if we choose some different representative \( Hg'H \) for a right coset \( HgH \), then

\[
t(Hg'H) = t(HgH)
\]

since \( Hg'H = HgH \) so the image under \( \iota \) is the same.

So \( \iota \) is well defined.
Proposition 3.1.12. If \((G, H)\) is a Hecke pair, \(\iota\) is an isomorphism between \(R(H, G)^H\) and \(\mathcal{H}(H, G)\).

Proof. First we will show that \(\text{Im}(\iota) \subseteq \mathcal{R}(H, G)^H\), so \(\iota\) is well defined.

Let \(HgH \in \mathcal{H}(H, G)\). Then \(\iota(HgH) = \sum_{i=1}^n Hgg_i\) for \(g : (H \cap g^{-1}Hg) \backslash H\), \(n\) is finite since \((H, G)\) is a Hecke pair and \(HgH = \bigsqcup_{i=1}^n Hgg_i\) where the union is disjoint.

Let \((HgH)h = \{h_1g_ih \mid h_1, h_2 \in H\}\). Note \(HgH = (HgH)h\) (by setting \(h_2 = h_1h^{-1}\)).

Then \(HgH = \iota(HgH)h = \bigcup_{i=1}^n Hgg_ih\) where both unions are disjoint so \(\bigcup_{i=1}^n Hgg_i = \sum_{i=1}^n Hgg_ih\).

So \(\iota(HgH) = \sum_{i=1}^n Hgg_i = \iota(HgH)h\), so \(\iota(HgH) \in \mathcal{R}(H, G)^H\).

This extends by linearity of \(\iota\) to every element of \(\mathcal{H}(H, G)\).

Note that \(\iota\) is a module homomorphism. We now prove that \(\iota\) is an isomorphism.

\(\iota\) is injective since double cosets are disjoint and right cosets partition double cosets.

Then let \(U \in \mathcal{R}(H, G)^H\) where

\[ U = \sum_{g : H \backslash G} u(Hg)Hg \]

Since \(U \in \mathcal{R}(H, G)^H\), given \(h \in H\), \(Uh = U\). So \(u(Hg) = u(Hgh)\) \(\forall h \in H\).

So we can write \(U\) in the form

\[ U = \sum_{g : H \backslash G \setminus H} u(Hg) \sum_{a : (H \cap g^{-1}Hg) \setminus H} Hga \]

which is in the image of \(\iota\) (where \(\iota^{-1}(U) = \sum_{g : H \backslash G \setminus H} u(Hg)HgH\)). \(\square\)

We now define an action of a double coset on a right coset

Given \(g, g^* \in G\), define

\[(HgH) \ast (Hg^*) := \iota(HgH)g^* = \sum_{a : (H \cap g^{-1}Hg) \setminus H} Hgag^*\]

This is well defined since if we choose a different representative of \(Hg^*\), \(g'\), then \(g' = h_1g^*\) for some \(h_1 \in H\), and we proved in proposition \[3.1.12\] that \(\iota(HgH)\) was invariant under the action of any element of \(H\).

We can extend this action to an action of elements in \(\mathcal{H}(H, G)\) on \(\mathcal{R}(H, G)\) by linearity.

That is, if

\[ T = \sum_{g : H \backslash G \setminus H} t(HgH)HgH \in \mathcal{H}(H, G) \]

\[ U = \sum_{g : H \backslash G} u(Hg)Hg \in \mathcal{R}(H, G) \]

Then

\[ T \ast U = \sum_{g : H \backslash G \setminus H} \sum_{g^* : G \setminus H} \sum_{g : H \backslash G} t(HgH)u(Hg^*)[(HgH) \ast (Hg^*)] \]

\[ = \sum_{g : H \backslash G \setminus H} \sum_{g^* : G \setminus H} t(HgH)u(Hg^*)Hgg^* \]

Proposition 3.1.13. If \(U \in \mathcal{R}(H, G)^H\), then \(T \ast U \in \mathcal{R}(H, G)^H\).

Proof. Let

\[ T = \sum_{g : H \backslash G \setminus H} t(HgH)HgH \in \mathcal{H}(H, G) \]

\[ U = \sum_{g : H \backslash G} u(Hg)Hg \in \mathcal{R}(H, G) \]
3.2. EXAMPLE: $S_n$

So

\[(T * U)h = \left( \sum_{g \in H \setminus G} \sum_{g' \in H \setminus G} t(HgH)u(Hg^*)Hgg^* \right) h \]
\[= \sum_{g \in H \setminus G} \sum_{g' \in H \setminus G} t(HgH)u(Hg^*)Hgg^*h \]
\[= \sum_{g \in H \setminus G} \sum_{g' \in H \setminus G} t(HgH)u(Hg^*)[(HgH) * (Hg^*h)] \]
\[= T * (Uh) \]
\[= T * U \quad \text{(Since } U \in \mathcal{R}(H,G)^H) \]

We can now define a product on $\mathcal{H}(H,G)$. Given $T_1, T_2 \in \mathcal{H}(H,G)$, let

\[T_1 \cdot T_2 = \iota^{-1}(T_1 * \iota(T_2))\]

which is well defined by proposition 3.1.13.

**Definition 3.1.14** (Hecke Algebra). The Hecke Algebra of a Hecke pair $(H,G)$ is the module $\mathcal{H}(H,G)$ with the product: if $T_1, T_2 \in \mathcal{H}(H,G)$ then:

\[T_1 \cdot T_2 = \iota^{-1}(T_1 * \iota(T_2))\]

In practice, it is difficult to use the above definition to calculate products.

There is an alternative equivalent characterisation of the product in a Hecke algebra which will be used for calculation.

**Definition 3.1.15.** Let $(G, H)$ be a Hecke pair. Let $HgH, Hg'H \in \mathcal{H}(H,G)$; we can write $HgH = \bigsqcup_{i \in I} g_i H$, $Hg'H = \bigsqcup_{j \in J} g'_j H$ for some set of representatives for left cosets $I, J$, where $g_i, g'_j \in G$.

Then

\[HgH \cdot Hg'H = \sum_{g' \in H \setminus G / H} \alpha(g, g'; x) HxH\]

where $\alpha(g, g'; x) := \text{Card}(\{(i, j) : g_i g'_j \in xH\})$.

Finally, if we choose a ring $R$ with an involution $^\sim : R \to R$ (for example, $\mathbb{C}$) then we can define an involution on $\mathcal{H}_R(H,G)$ by $^\sim rHgH = rHg^{-1}H$.

3.2 Example: $S_n$

(Section 1.5 in [Kri90]) Let $G = S_n$. We will view $S_n$ as the group of bijections of the set $\{1, \ldots, n\}$ to itself with operation composition.

We will let $H = S_{n-1}$ where we will identify $S_{n-1}$ with the subgroup of $S_n$ which map $n$ to $n$.

**Proposition 3.2.1.** There are $n$ right cosets labelled $Hg_i$, $i \in \{1, \ldots, n\}$ where $Hg_i$ is the set of bijections that send $i$ to the integer $n$.

**Proof.** Let $g \in G$ and consider $Hg$. Then $g$ must send some number $i$ to $n$ and every element of $H$ fixes $n$, so we must have that every element of $Hg$ sends $i$ to $n$, so $Hg \subseteq Hg_i$.

On the other hand any bijection which sends $i$ to $n$ can be written in the form $hg$ for some $h \in H$ since $H$ is the full group of permutations on $\{1, \ldots, n-1\}$ so $Hg = Hg_i$. 

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Proposition 3.2.2. There are exactly two double cosets of $H$ in $G$.

Proof. Consider $Hg_i$, $Hg_j$ where $i \neq n$, $j \neq n$.

Let $\pi_{i,j}$ be the bijection that swaps $i$ and $j$ (and keeps all else fixed).

Then $\pi_{i,j} \in H$, since $i \neq n$ and $j \neq n$ so $\pi_{i,j}(n) = n$.

Then given any $f \in Hg_i$, $f \circ \pi_{i,j} \in Hg_j$. So $Hg_i \cap Hg_j \neq \emptyset \implies Hg_i \pi_{i,j} = Hg_j$. So $Hg_i \subseteq Hg_j H$. If we fix some $j \neq n$ we have that $Hg_i \subseteq Hg_j H$ for every $i \neq n$.

Additionally, $Hg_n \nsubseteq Hg_j H$ since $Hg_n$ contains the identity bijection but $Hg_j H$ does not since every element in $Hg_j H$ sends some number which is not $n$ to $j$ which is then sent to $n$ (i.e. no element in $Hg_j H$ fixes $n$).

Therefore, since double cosets partition $G$, if $n > 1$ we must have exactly two double cosets: $Hg_j H$ for some $j \neq n$ which we will label $a$ and $Hg_n H$ which we will label $e$.

Since $\pi_{j,n} \in Hg_j$ we will write $Hg_j = H\pi_{j,n}$. 

\[\square\]
3.2. EXAMPLE: \( S_n \)

Then

\[
a \cdot a = \iota^{-1} \left( \sum_{j=1}^{n-1} \sum_{i=1}^{n-1} H\pi_{j,n} \pi_{i,n} \right)
\]

If \( i = j \), then \( \pi_{j,n} \pi_{i,n} = \text{id}_G \), so \( H\pi_{j,n} \pi_{i,n} = Hg_n \). Otherwise \( \pi_{j,n} \pi_{i,n} \) send \( j \to n \) so \( H\pi_{j,n} \pi_{i,n} = Hg_j \neq Hg_n \) since \( a = Hg_j H \) so \( j \neq n \).

So

\[
\sum_{j=1}^{n-1} \sum_{i=1}^{n-1} H\pi_{j,n} \pi_{i,n} = \sum_{j=1}^{n-1} \sum_{i \neq j} H\pi_{j,n} \pi_{i,n} + (n - 1)Hg_n
\]

\[
= \sum_{j=1}^{n-1} (n - 2)H\pi_{j,n} + (n - 1)Hg_n
\]

\[
= (n - 2)a(a) + (n - 1)e(e)
\]

So

\[
a \cdot a = (n - 2)a + (n - 1)e
\]

Also, \( Hg_n H = Hg_n = H \text{id}_G \), so \( a \cdot e = a \) and \( e \cdot e = e \).

So the Hecke algebra \( \mathcal{H}_2(S_n) \) has two generators \( a \) and \( e \), where \( e \) is the multiplicative identity and \( a \cdot a = (n - 2)a + (n - 1)e \).
Chapter 4

Results from a paper by U. Baumgartner et al.

The following chapter covers some of the results from [BLRW09].

It contains the explicit multiplication tables of the Hecke algebra for the vertex stabilizer and edge stabilizer cases and gives a characterisation for the HNN-extension case.
4.0 Trees and Their Automorphism Groups

**Definition 4.0.1** (Transitively on spheres). Let $\mathcal{T}$ be a tree which has at least 3 edges at each vertex.

Let $\Gamma_0$ be a group which acts on $\mathcal{T}$ and fixes a vertex $o$ in $\mathcal{T}$.

Then $\Gamma_0$ acts transitively on each sphere around $o$ if for any distance $l \in \mathbb{Z}_{\geq 0}$ given $u$, $v \in V(\mathcal{T})$ such that $d(o,u) = d(o,v) = l$ there exists $g \in \Gamma_0$ such that $g \cdot u = v$.

**Definition 4.0.2.** Let $G$ be a group, let $H$ be a subgroup of $G$.

Then we define $L(g)$ to be the cardinality of the set of left cosets contained in $HgH$. In particular, if $(G,H)$ is a Hecke pair then $L(g)$ is the number of left cosets in $HgH$.

Similarly, define $R(g)$ to be the be the cardinality of the set of right cosets contained in $HgH$.

Note that $R(g) = L(g^{-1})$ since by taking the inverse of $HgH$.

Also, $R : \mathbb{Z}[\Gamma, \Gamma_0] \rightarrow \mathbb{Z}$ is a ring homomorphism, (it follows (after some work) from the way the Hecke algebra was defined in chapter [3]).

**Lemma 4.0.3** (Lemma 2.1 in [BHRW09]). Let $\Gamma$ be a group which acts by isometries on a graph $X$.

Let $o$ be a vertex in $X$ with stabilizer $\Gamma_0$ in $\Gamma$. Then the following statements are true for the pair $(\Gamma, \Gamma_0)$,

1. $L(g)$ is the cardinality of the orbit of $g \cdot o$ under $\Gamma_0$.
2. If the the sphere with center $o$ and radius $d(o, g \cdot o)$ is finite then $L(g)$ is finite.
3. If the action of $\Gamma_0$ on the sphere with centre $o$ and radius $d(o, g \cdot o)$ is transitive then $L(g) = R(g) = N$ where $N$ is the cardinality of that sphere and $\Gamma_0 g \Gamma_0 = \Gamma_0 g^{-1} \Gamma_0$.
4. If $\Gamma_0$ acts transitively on every sphere around $o$ and every sphere has finite cardinality then the Hecke algebra $\mathbb{C}[\Gamma, \Gamma_0]$ is commutative.

**Proof.**

1. Note that

$$g_1 \in g_2 \Gamma_0 \iff g_2^{-1} g_1 \in \Gamma_0 \iff g_2^{-1} g_1 \cdot o = o \iff g_1 \cdot o = g_2 \cdot o$$

So the elements of a left coset $g \Gamma_0$ are exactly the elements which send $o$ to $g \cdot o$.

Then the left cosets of $\Gamma_0 g \Gamma_0$ are in one to one correspondence with the vertices in the orbit of $g \cdot o$ under $\Gamma_0$. So $L(g)$ is the cardinality of the orbit of $g \cdot o$ under $\Gamma_0$.

2. $\Gamma$ acts by isometries so $\Gamma_0$ preserves distance from $o$. Then $L(g)$ is the cardinality of the orbit of $g \cdot o$ so $L(g)$ is less than the cardinality of the sphere with centre $o$ and radius $d(o, g \cdot o)$.

So if the cardinality of the sphere is finite then $L(g)$ is finite.

3. Since $\Gamma_0$ acts by isometries and fixes $o$, every element of $\Gamma_0 g \Gamma_0$ must send $o$ to some vertex distance $d(o, g \cdot o)$ from $o$, so $\Gamma_0 g \Gamma_0$ is a subset of the elements which map $o$ to some vertex on the sphere with centre $o$ and radius $d(o, g \cdot o)$.

If $\Gamma_0$ is transitive on the sphere with centre $o$ and radius $d(o, g \cdot o)$ then given $h \in \Gamma$ which sends $o$ to some vertex on the sphere with centre $o$ and radius $d(o, g \cdot o)$ there exists $k_1 g k_2 \in \Gamma_0 g \Gamma_0$ where $k_1, k_2 \in \Gamma_0$ such that $h \cdot o = k g \cdot o$.

Then $(k_1 g k_2)^{-1} h \cdot o = o \implies (k_1 g k_2)^{-1} h = k$ for some $k \in \Gamma_0$. Then $h = k k_1 g k_2$ so $h \in \Gamma_0 g \Gamma_0$.

So $\Gamma_0 g \Gamma_0$ is the set of elements of $\Gamma$ mapping $o$ to some vertex in the sphere with centre $o$ an radius $d(o, g \cdot o)$.

Then $d(g^{-1} \cdot o, o) = d(o, g \cdot o)$ so $g^{-1} \cdot o$ is a vertex on the sphere with centre $o$ and radius $d(o, g \cdot o)$. Then $\Gamma_0 g^{-1} \Gamma_0$ is the set of elements in $\Gamma$ which map $o$ to some vertex on the sphere with centre $o$ and radius $d(o, g \cdot o)$ so $\Gamma_0 g^{-1} \Gamma_0 = \Gamma_0 g \Gamma_0$.

Then $R(g) = L(g^{-1}) = L(g)$. 

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4. If every sphere around $o$ is finite and $\Gamma_0$ acts transitively on every sphere then for every basis element $\Gamma_0 g \in \mathbb{C}[\Gamma, \Gamma_0]$ (considering $\mathbb{C}[\Gamma, \Gamma_0]$ as a module), $(\Gamma_0 g)^* = \Gamma_0 g^{-1} \Gamma_0 = \Gamma_0 g \Gamma_0$ from above.

Then given $a, b \in \mathbb{C}[\Gamma, \Gamma_0]$ we know that $ba$ is the $\mathbb{Z}$ sum of basis elements, that is $ba = \sum_{g \in \mathbb{H} \setminus \mathbb{G}/\mathbb{H}} \alpha(g, g'; x) HxH$ where $\text{Im}(\alpha) = \mathbb{Z}$. So $(ba)^* = ba$.

Then $ab = a^* b^* = (ba)^* = ba$

\[ \square \]

### 4.1 Vertex Stabilizer Which Acts Transitively on Spheres

We will require the following lemma

**Lemma 4.1.1** (Lemma 3.2 in [BLRW09]). Let $\mathcal{T}$ be a tree which has at least 3 edges at each vertex. Let $o$ be a distinguished vertex in $\mathcal{T}$.

Suppose that $\Gamma$ is a group acting on $\mathcal{T}$ such that the stabiliser $\Gamma_0$ of $o$ acts transitively on each sphere around $o$.

Then for each even integer $D$ there exists an elliptic element $\delta \in \Gamma$ such that $d(\delta \cdot o, o) = D$.

**Proof.** By [Tit70] proposition 3.4, $\Gamma$ contains a hyperbolic element $\eta$. So $d(\eta^n \cdot o, o)$ tends to infinity as $n$ tends to infinity.

So we can choose some $n$ such that $d(\eta^n \cdot o, o) > D$. Let $\gamma = \eta^n$.

Then let $v$ be the vertex on the path from $o$ to $\eta^n \cdot o$ at distance $D/2$ from $o$ (since $\mathcal{T}$ is a tree, such a path is unique).

Choose a vertex $w$ on the sphere centred at $o$ with radius $D$ such that the intersection of the path from $o$ to $\gamma \cdot o$ and the path from $o$ to $w$ is exactly the path from $o$ to $v$.

Explicitly, at $v$ there exists at least one edge which is not in the path from $o$ to $\gamma \cdot o$ since $\mathcal{T}$ has at least 3 edges at each vertex, call this edge $e$. Take the path from $o$ to $v$ and extend this path along $e$ by adding the vertex adjacent to the edge $e$ on the path $D/2 - 1$ more times. Call the final vertex of this path $w$.

![Figure 4.1: $o$, $v$, $w$ and $\gamma \cdot o$ as described above](image)

We can find some $\pi \in \Gamma_0$ such that $\pi \cdot \gamma \cdot o = w$ since $\Gamma_0$ acts transitively on the sphere with centre $o$ and radius $D$.

Let $\delta = \gamma^{-1} \pi \gamma$. $\delta$ is elliptic since $\delta \cdot (\gamma^{-1} \cdot o) = \gamma^{-1} \pi \gamma^{-1} \cdot o = \gamma^{-1} \pi \cdot o = \gamma^{-1} \cdot o$. Additionally, $d(\delta \cdot o, o) = d(\gamma^{-1} \pi \gamma \cdot o, o) = d(\gamma^{-1} \cdot w, o) = d(w, o) = D$

\[ \square \]
4.1. VERTEX STABILIZER WHICH AIDS TRANSITIVELY ON SPHERES

Proposition 4.1.2 (prop 3.1 in [BLRW09]). Let \( T \) be a tree which has at least 3 edges at each vertex. Let \( o \) be a distinguished vertex in \( T \).

Suppose that \( \Gamma \) is a group acting on \( T \) by automorphisms such that the stabiliser \( \Gamma_0 \) of \( o \) acts transitively on each around \( o \).

Then \( T \) is semi-homogeneous.

Additionally, the action of \( \Gamma \) is either transitive or has two orbits:

\[
\mathcal{T}_{\text{even}} = \{ v \in V(T) \mid d(v, o) \text{ is even} \}, \quad \mathcal{T}_{\text{odd}} = \{ v \in V(T) \mid d(v, o) \text{ is odd} \}
\]

Proof. By lemma 4.1.1 given any even non-negative integer \( D \) there exists an elliptic element in \( \Gamma \) mapping \( o \) to a vertex in the sphere centred at \( o \) with radius \( D \).

Then since \( \Gamma_0 \) acts transitively on each sphere, for any vertex \( v \) in a sphere of even distance from \( o \) there exists an element in \( \Gamma \) which maps \( o \) to \( v \). So all the vertices at even distance from \( o \) must be in the same orbit.

Additionally, given a vertex at an odd distance from \( o \) choose a vertex \( p \) such that \( d(p, o) = 1 \). Then given a vertex \( w \) at odd distance from \( o \), there exists an element \( \gamma \in \Gamma \) which maps \( o \) to a vertex adjacent to \( w \) since the vertices adjacent to \( w \) at even distance from \( o \).

Then since \( \Gamma_0 \) acts transitively on vertices, there exists some \( \beta \in \Gamma_0 \) such that \( \delta \beta \cdot p = w \).

So all the vertices at an odd orbit must also be in the same orbit.

So the action of \( \Gamma_0 \) on \( T \) has either one orbit in which case \( \Gamma \) is transitive or it has two orbits:

\[
\mathcal{T}_{\text{even}} = \{ v \in V(T) \mid d(v, o) \text{ is even} \}, \quad \mathcal{T}_{\text{odd}} = \{ v \in V(T) \mid d(v, o) \text{ is odd} \}
\]

\[\square\]

For both the case where \( \Gamma \) acts transitively on vertices and the case where \( \Gamma \) we can write down the multiplication table for the Hecke Algebra. The details are contained in the following two statements.

Theorem 4.1.3 (Theorem 3.3 in [BLRW09]). Let \( T \) be a tree with at least 3 edges at each vertex.

Let \( \Gamma \) be a group acting by automorphisms of \( T \) and assume that the stabilizer of \( o \), \( \Gamma_0 \) acts transitively on each sphere around \( o \).

Let \( q_0 + 1 \) be the degree of the even vertices and let \( q_1 + 1 \) be the degree of the odd vertices.

For each \( n \in \mathbb{Z}_{\geq 0} \), let \( \Gamma_n = \{ \gamma \in \Gamma \mid d(\gamma \cdot o, o) = n \} \).

Then

1. If \( \Gamma \) acts transitively on vertices let \( q = q_0 = q_1 \). Then:
   - A basis for \( C[\Gamma, \Gamma_0] \) (as a vector space) is the set \( \{ \Gamma_n \mid n \in \mathbb{Z}_{\geq 0} \} \).
   - The map from the complex polynomials in the variable \( T \) to \( C[\Gamma, \Gamma_0] \) which maps \( T \) to \( \Gamma_1 \) is an algebra isomorphism.
   - \( \Gamma_1 \Gamma_n = (q_0 + \delta_{n,1})\Gamma_{n-1} + \Gamma_{n+1} \) where \( \delta \) is the Kronecker delta.

2. If \( \Gamma \) has two orbits of vertices then:
   - A basis for \( C[\Gamma, \Gamma_0] \) is the set \( \{ \Gamma_{2n} \mid n \in \mathbb{Z}_{\geq 0} \} \).
   - The map from the complex polynomials in the variable \( T \) to \( C[\Gamma, \Gamma_0] \) which maps \( T \) to \( \Gamma_2 \) is an algebra isomorphism.
   - \( \Gamma_2 \Gamma_{2n} = (q_0 + \delta_{n,1})q_1\Gamma_{2n-2} + (q_1 - 1)\Gamma_{2n} + \Gamma_{2n+2} \)

Proof. The sets \( \{ \Gamma_n \mid n \in \mathbb{Z}_{\geq 0} \} \) and \( \{ \Gamma_{2n} \mid n \in \mathbb{Z}_{\geq 0} \} \) are bases for their respective cases by lemma 4.0.3 part 3.

- If \( \Gamma \) acts transitively on vertices, given \( \alpha \in \Gamma_1, \beta \in \Gamma_n \),
  \[
d(\alpha \beta \cdot o, o) = d(\beta \cdot o, \alpha^{-1} \cdot o) = \begin{cases} 
n + 1 & \text{if } \alpha^{-1} \cdot o \text{ is on the path from } o \text{ to } \beta \cdot o \\
n - 1 & \text{if } \alpha^{-1} \cdot o \text{ is not on the path from } o \text{ to } \beta \cdot o
\end{cases}
\]

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so either $\alpha \beta \in \Gamma_{n-1}$ or $\alpha \beta \in \Gamma_{n+1}$.

So for each $n \in \mathbb{Z}_{\geq 0}$ we can write $\Gamma_1 \Gamma_n = a_n \Gamma_{n-1} + b_n \Gamma_{n+1}$ for some $a_n, b_n \in \mathbb{Z}_{> 0}$.

Then since $R$ is a ring homomorphism and the $R(\Gamma_n)$ is the number of vertices at distance $n$ from $o$ we have that

$$(q + 1)(q + 1)^{q^{n-1}} = (q + 1)^2 q^{n-1} = \begin{cases} a_1 + b_1(q + 1)q & \text{if } n = 1 \\ a_n(q + 1)q^{n-2} + b_n(q + 1)q^n & \text{if } n \geq 2 \end{cases}$$

For any $n$, if $b_n \geq 2$ then $b_n(q + 1)q^n \geq 2(q + 1)q^n \geq (q + 1)q^n + (q + 1)q^{n-1} = (q + 1)^2 q^{n-1}$ so we must have that $b_n = 1$.

Then substituting $b_n$ into the above equations, $a_1 = q + 1$ if $n = 1$ and $a_n = q$ if $n \geq 2$.

- If $\Gamma$ has two orbits of vertices, given $\alpha \in \Gamma_2$, $\beta \in \Gamma_{2n}$,

$$d(\alpha \beta \cdot o, o) = d(\beta \cdot o, \alpha^{-1} \cdot o) = \begin{cases} 2n - 2 & \text{if the path from } o \text{ to } \alpha^{-1} \cdot o \text{ does not} \\
 & \text{intersect the path from } o \text{ to } \beta \cdot o \\
2n & \text{if the path from } o \text{ to } \alpha^{-1} \cdot o \text{ intersects} \\
 & \text{the path from } o \text{ to } \beta \cdot o \text{ on one edge} \\
2n + 2 & \text{if the path from } o \text{ to } \alpha^{-1} \cdot o \text{ is contained} \\
 & \text{in the path from } o \text{ to } \beta \cdot o \end{cases}$$

so either $\alpha \beta \in \Gamma_{2n-2}$ or $\alpha \beta \in \Gamma_{2n}$ or $\alpha \beta \in \Gamma_{2n+2}$.

So for each $n \in \mathbb{Z}_{\geq 0}$ we can write $\Gamma_1 \Gamma_n = a_n \Gamma_{2n-2} + b_n \Gamma_{2n} + c_n \Gamma_{2n+2}$ for some $a_n, b_n, c_n \in \mathbb{Z}_{> 0}$.

Again since $R$ is a ring homomorphism and the $R(\Gamma_n)$ we have that

$$(q_0 + 1)q_1 \cdot (q_0 + 1) \prod_{j=1}^{2n-1} q_{s(j)} = (q_0 + 1)^2 q_1 \prod_{j=1}^{2n-1} q_{s(j)} = \begin{cases} a_1 + b_1(q_0 + 1)q_1 + c(q_0 + 1)q_0 q_1^2 & \text{if } n = 1 \\
a_n(q_0 + 1) \prod_{j=1}^{2n-3} q_{s(j)} + b_n(q_0 + 1) \prod_{j=1}^{2n-1} q_{s(j)} + c_n(q_0 + 1) \prod_{j=1}^{2n+1} q_{s(j)} & \text{if } n \geq 2 \end{cases}$$

where $s(j) = 1$ if $j$ is odd 0 if $j$ is even.

For any $n$, if $c_n \geq 2$ then $c_n(q_0 + 1) \prod_{j=1}^{2n+1} q_{s(j)} \geq 2(q_0 + 1) \prod_{j=1}^{2n+1} q_{s(j)} = 2(q_0 + 1)q_0 q_1 \prod_{j=1}^{2n-1} q_{s(j)} \geq (q_0 + 1)^2 q_1 \prod_{j=1}^{2n-1} q_{s(j)}$ so we must have that $c_n = 1$.

We also know that given $\alpha \in \Gamma_2$ and $\beta \in \Gamma_{2n}$, $\alpha \beta \in \Gamma_{2n}$ only if the path from $o$ to $\alpha^{-1} \cdot o$ intersects the path from $o$ to $\beta \cdot o$ on exactly one edge. So fixing $\beta \Gamma_2 \subseteq \Gamma_{2n}$ there exists $q_1 - 1$ choices of $\alpha \Gamma_0$ such that $\alpha \beta \in \Gamma_{2n}$.

Then since $R$ is a module homomorphism, $(q_1 - 1)R(\Gamma_{2n}) = b_n R(\Gamma_{2n}) \implies b_n = q_1 - 1$ for all $n$.

Then substituting $b_n$ and $c_n$ into the above equations, $a_1 = q_0 + 1$ if $n = 1$ and $a = q_0$ if $n \geq 2$.

Then given these multiplication tables by setting $n = 1$ we know that $\Gamma_2$ is a polynomial with integer coefficients in $\Gamma_1$ for the vertex transitive case (where we identify $\Gamma_0$ with a constant term) and $\Gamma_4$ is a polynomial in $\Gamma_2$ for the two orbit case. Then by induction on $n$ we have that $\Gamma_n$ and $\Gamma_{2n}$ are polynomials in $\Gamma_1$ and $\Gamma_2$ respectively.

So the maps which map $T \rightarrow \Gamma_1$ and $T \rightarrow \Gamma_2$ are surjective.

The maps are also injective since by the multiplication tables, neither $\Gamma_1$ nor $\Gamma_2$ are nilpotent for their respective cases.

So the maps which send $T$ to $\Gamma_1$ and $T$ to $\Gamma_2$ are algebra isomorphisms.
4.2. STRONGLY TRANSITIVE STABILIZER OF AN EDGE

**Corollary 4.1.4** (Corollary 3.5 in [BLRW09]). Let $\mathcal{T}$ be a tree with at least 3 edges at each vertex.

Let $\Gamma$ be a group acting by automorphisms of $\mathcal{T}$ and assume that the stabilizer of $\alpha$, $\Gamma_0$ acts transitively on each sphere around $\alpha$.

Let $q_0 + 1$ be the degree of the even vertices and let $q_1 + 1$ be the degree of the odd vertices and if $q_0 = q_1$ let $q = q_0 = q_1$.

Then the multiplication table for $\mathbb{C}[\Gamma, \Gamma_0]$ is given by

1. if $\Gamma$ acts transitively on vertices then

$$\Gamma_n \Gamma_m = \Gamma_m \Gamma_n = \Gamma_{m+n} + q^{n-1}(q + \delta_{m,n})\Gamma_{m-n} + (q - 1) \sum_{l=1}^{n-1} q^{l-1}\Gamma_{m+n-2l}$$

2. if $\Gamma$ acts with two orbits on vertices then

$$\Gamma_2 \Gamma_m = \Gamma_m \Gamma_2 = \Gamma_{2(m+n)} + q_0^{n-1}(q_0 + \delta_{m,n})\Gamma_{2(m-n)} + \sum_{l=1}^{n-1} (q_0(i) - 1) \prod_{i=1}^{l-1} q_0(s)\Gamma_{2(m+n-l)}$$

**Proof.** For both cases induct on $m$. The induction for $\Gamma$ acting transitively on vertices is given in the appendix $A.1.1$. □

4.2. STRONGLY TRANSITIVE STABILIZER OF AN EDGE

**Definition 4.2.1** (Strongly transitive). A type preserving group acting on a tree $\mathcal{T}$ is strongly transitive if it acts transitively on the set of doubly-infinite geodesics in $\mathcal{T}$ and the stabilizer of some doubly-infinite geodesic $A$ acts transitively on the set of edges in $A$.

**Lemma 4.2.2** (Lemma 4.2 in [BLRW09]). Suppose that $G$ is a group of type preserving automorphisms of a locally finite tree $\mathcal{T}$.

Then the following are equivalent:

1. $G$ is strongly transitive.
2. $G$ acts transitively on the edge set $E(\mathcal{T})$ and there exists an edge $f \in E(\mathcal{T})$ whose stabilizer acts transitively on the set of all doubly-infinite geodesics containing $f$.
3. $G$ acts transitively on the edge set $E(\mathcal{T})$ and the stabilizer of each edge acts transitively on the set of all doubly-infinite geodesics through that edge.
4. For any two doubly-infinite geodesics $A_1$ and $A_2$ and any two edges $f_1 \in A_1$ and $f_2 \in A_2$ there exists $g \in G$ such that $g \cdot A_1 = A_2$, $g \cdot f_1 = f_2$.

**Proof.** 1 $\implies$ 2.

Every edge in $E(\mathcal{T})$ is an element of some doubly-infinite geodesic in $\mathcal{T}$. So, since $G$ acts transitively on doubly-infinite geodesics, given $e \in E(\mathcal{T})$ there exists $g \in G$ such that $g \cdot e \in A$.

Then given two edges $e_1, e_2 \in E(\mathcal{T})$ we know that there exists $g_1, g_2 \in G$ such that both $g_1 \cdot e_1, g_2 \cdot e_2 \in A$.

Since $G$ acts transitively on edges in $A$, there exists $g_3 \in G$ such that $g_3 \cdot g_1 \cdot e_1 = g_2 \cdot e_2 \implies g_2^{-1}g_3g_1 \cdot e_1 = e_2$. So $G$ acts transitively on $E(\mathcal{T})$.

For the second part of 2, choose any edge $f$ in $A$.

Given a geodesic $B$ through $f$, we can find $g \in G$ such that $g \cdot A = B$ since $G$ is strongly transitive. If $g \cdot f = f$ then $g$ is in the stabilizer of $f$.

Otherwise, consider $g^{-1} \cdot f$. We know that $g^{-1} \cdot f \in A$ since $g \cdot A = B \implies A = g^{-1} \cdot B$. So there exists $h$ in the stabilizer of $A$ such that $h \cdot f = g^{-1} \cdot f$.

Then $gh \cdot A = B$ since $h$ is in the stabilizer of $A$ and $gh \cdot f = gg^{-1} \cdot f = f$.

So the stabilizer of $f$ acts transitively on doubly-infinite geodesics containing $f$. 

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• $2 \implies 3$

We need only prove that the stabilizer of each edge acts transitively on doubly-infinite geodesics through that edge.

Let $f \in E(T)$ be the edge whose stabilizer acts transitively on doubly-infinite geodesics containing $f$.

Let $e \in E(T)$ be an edge. Let $B_1, B_2$ be doubly-infinite geodesics containing $e$.

Since $G$ acts transitively on $E(T)$, there exists $g \in G$ such that $g \cdot e = f$. Then $g \cdot B_1$ and $g \cdot B_2$ are doubly-infinite geodesics containing $f$ so there exists $h$ in the stabilizer of $g$ such that $h \cdot (g \cdot B_1) = g \cdot B_2$.

Then $g^{-1}hg \cdot e = g^{-1}h \cdot f = g^{-1} \cdot f = e$, so $g^{-1}hg$ is in the stabilizer of $e$. And $g^{-1}hg \cdot B_1 = g^{-1} \cdot (g \cdot B_2) = B_2$.

So the stabilizer of $e$ acts transitively on doubly-infinite geodesics through $e$.

• $3 \implies 4$. Given doubly-infinite geodesics $A_1, A_2$ and edges $f_1 \in A_1$, $f_2 \in A_2$ we can find some $g \in G$ such that $g \cdot f_1 = f_2$.

Then $g \cdot A_1$ is a doubly infinite geodesics containing $f_2$ so we can find some $h$ in the stabilizer of $f_2$ such that $h \cdot (g \cdot A_1) = A_2$.

Then $hg \cdot f_1 = f_2$ and $gh \cdot A_1 = A_2$.

• $4 \implies 1$. Since every edge is in some doubly-infinite geodesic $G$ must act transitively on $E(T)$.

Let $e \in E(T)$. Then given two doubly-infinite geodesics $A_1, A_2$ containing $e$ there exists some $g \in G$ such that $g \cdot A_1 = A_2$, $g \cdot e = e$.

So the stabilizer of $e$ acts transitively on doubly-infinite geodesics through $e$.

\[ \square \]

Proposition 4.2.3 (Proposition 4.3 in [BLRW09]). Let $G$ be a group acting on a tree $T$ in such that the subgroup $G^+$ of type-preserving automorphisms is strongly transitive.

For a distinguished edge $e$ of $T$, let $B$ be the subgroup of $G$ fixing $e$ pointwise and let $A$ be a doubly-infinite geodesic containing $e$.

Then $T$ is semi-homogeneous and the action of $G$ on $V(T)$ is either transitive (in the case that $G = G^+$) or else has two orbits (in the case $G \neq G^+$).

If $G \neq G^+$ there exists $i \in G$ which inverts $e$ and stabilizes $A$ which gives a short exact sequence $1 \to G^+ \to G \to \mathbb{Z}/2\mathbb{Z} \to 1$. This short exact sequence splits if and only if $G$ contains an inversion that is an involution.

Additionally, $(G, B)$ and $(G^+, B)$ are Hecke pairs.

Proof. Choose some $e = \{v_1, v_2\} \in T$ and a doubly-infinite geodesic $A$ containing $e$.

If $G$ is type preserving ($G = G^+$), given $v \in V(T)$ choose some edge $f$ such that $v \in f$ (which is possible since $T$ contains more than one vertex and is connected), then there exists $g \in G$ such that $g \cdot f = e$.

So given any $v \in (T)$ either there exists some $g \in G$ such that $g \cdot v = v_1$ or some $g \in G$ such that $g \cdot v = v_2$, so $G$ has at most two orbits in $T$.

If $G$ has only one orbit then there must exist some $h$ such that $h \cdot v_1 = v_2$, but then $h$ inverts $e$ so $G$ is not type preserving. So $G$ has exactly two orbits in $T$.

If $G$ is not type preserving ($G \neq G^+$) then there exists some $i' \in G$ such that $i'$ inverts some $e' \in T$. But $G$ is transitive on edges, so there exists $g \in G$ such that $g \cdot e = e'$, then $g^{-1}ig \in G$ inverts $e$. Additionally, since $G^+$ acts transitively on the set of doubly-infinite geodesics through $e$ by lemma [4.2.2] we can find some type preserving $h$ in the stabilizer of $e$ such that $hg^{-1}ig$ stabilizes $A$.

So, letting $j = hg^{-1}ig$, there exists $j \in G$ which inverts $e$ and stabilizes $A$.

Then given $v_1, v_2 \in V(T)$, there exists some $g \in G$ such that $g \cdot v_1 \in e$ and some $h \in G$ such that $v_2 \in h \cdot e$ since $G$ is transitive on $E(T)$.

Then either $hg \cdot v_1 = v_2$ or $hig \cdot v_1 = v_2$, so $G$ acts transitively on $V(T)$.
4.2. STRONGLY TRANSITIVE STABILIZER OF AN EDGE

In both cases \( \mathcal{T} \) must be semi-homogeneous since every vertex in an orbits must have the same degree.

For the exact sequence where \( G \neq G^+ \), note that since \( G \) acts by automorphisms its action is free. From above, we know that \( G^+ \) has two orbits in \( \mathcal{T} \) and that an element of \( G \) acting on \( \mathcal{T} \) maps vertices between these two orbits if and only if it is an element of \( G \setminus G^+ \) (the set of elements in \( G \) which are not type preserving).

Then the composition of two elements in \( G \setminus G^+ \) acting on \( \mathcal{T} \) does not map vertices between orbits, so the composition of two elements in \( G^+ \) must be an element of \( G^+ \). And the composition of an element in \( G^+ \) with an element in \( G \setminus G^+ \) acting on \( \mathcal{T} \) does map vertices between the two orbits so must be an element of \( G \setminus G^+ \).

So the map \( \varphi : G \to \{ \pm1 \} \cong \mathbb{Z}/2\mathbb{Z} \):

\[
\varphi(g) = \begin{cases} 
1 & \text{if } g \in G^+ \\
-1 & \text{otherwise} 
\end{cases}
\]

is well defined.

This gives an exact sequence \( 1 \to G^+ \to G \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z} \to 1 \) and this exact sequence splits if \( G \setminus G^+ \) contains an involution.

To show that \((G^+, B)\) and \((G, B)\) are Hecke pairs, let \( S \) be the graph with vertices edges of \( \mathcal{T} \) and two vertices in \( S \) defining an edge if they share a vertex in \( \mathcal{T} \).

That is, let \( V(S) = E(\mathcal{T}) \) and if \( e_1, e_2 \in V(S) = E(\mathcal{T}), e_1 \neq e_2 \) then \( \{e_1, e_2\} \in E(S) \) if and only if there exists \( v \in V(\mathcal{T}) \) such that \( v \in e_1 \) and \( v \in E_2 \).

Then if \( G = G^+, B \) fixes \( e \in E(\mathcal{T}) = V(S) \) and every sphere around \( e \) in \( S \) is finite since \( \mathcal{T} \) is locally finite. So by lemma 4.0.3 part 2 \((G^+, B)\) is a Hecke pair.

If \( G \neq G^+ \), then once again by lemma 4.0.3 part 2 \((G, N_G(B))\), where \( N_G(B) \) is the normalizer of \( B \) is a Hecke pair. But from the short exact sequence \( 1 \to G^+ \to G \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z} \to 1 \), the index \( B \) in \( N_G(B) \) is 2. So \((G, B)\) is also a Hecke pair. \( \square \)

**Theorem 4.2.4** (Theorem 4.4 in [BLRW09]).

1. The algebras \( \mathbb{C}[G, B] \) and \( \mathbb{C}[G^+, B] \) are isomorphic to the group algebra of the infinite dihedral group \( D_\infty \).

2. (a) If \( G = G^+ \), let \( N^+ \) be the subgroup of \( G \) stabilizing \( \mathcal{A} \), let \( T \) be the subgroup of \( G \) pointwise stabilizing \( \mathcal{A} \).

   i. \( N^+/T \) is isomorphic to the group of automorphisms of \( \mathcal{A} \) as a bipartite graph. That is, \( N^+/T \) is isomorphic to \( D_\infty \).

   ii. Given some \( n \in N^+ \), for all \( g_1, g_2 \in nT \), \( Bg_1B = Bg_2B \).

      Hence let \( W = N^+/T \). Then we will write \( Bg_1B = Bg_2B = BwB \) for some \( w \in N^+/T \).

   iii. We can write \( G^+ \) as the disjoint union \( G^+ = \bigsqcup_{w \in W} BwB \).

   Denote by \( s \) and \( t \) the two elements in \( W = N^+/T \) which send \( e \) to its two neighbours in \( \mathcal{A} \). Given \( w \in W \) denote by \( \Delta_w \) the double coset \( BwB \in \mathbb{C}[G^+, B] \).

   Then \( \mathbb{C}[G^+, B] \) is generated by \( \Delta_s, \Delta_t \) and \( B = \Delta_{idw} \). In fact, given \( w \in W \) with reduced decomposition \( w = s_1s_2\ldots s_n \) as a word in \( \{s, t\} \) (so \( s_i \in \{s, t\} \) for all \( i \)) then \( \Delta_w = \Delta_{s_1}\Delta_{s_2}\ldots\Delta_{s_n} \).

(b) If \( G \neq G^+ \), let \( N \) be the subgroup of \( G \) stabilizing \( \mathcal{A} \), let \( T \) be the subgroup of \( G \) pointwise stabilizing \( \mathcal{A} \).

   i. \( N/T \) is isomorphic to the group of automorphisms of \( \mathcal{A} \) as a graph and \( N/T \) is isomorphic to \( D_\infty \).

      Additionally, \( N/N^+ \cong \mathbb{Z}/2\mathbb{Z} \) and \( 1 \to N^+/T \to N/T \to N/N^+ \to 1 \) is a short exact sequence.

   ii. Given some \( \tilde{n} \in N \), for all \( \tilde{g}_1, \tilde{g}_2 \in \tilde{n}T \), \( B\tilde{g}_1B = B\tilde{g}_2B \).

      Hence let \( \tilde{W} = N/T \). Then we will write \( B\tilde{g}_1B = B\tilde{g}_2B = B\tilde{w}B \) for some \( \tilde{w} \in N/T \).

   iii. We can write \( G \) as the disjoint union \( G = \bigsqcup_{\tilde{w} \in \tilde{W}} B\tilde{w}B \).
Denote by \( s \) the element in \( W = N^+/T \) which sends \( e \) to one of its neighbours and denote by \( i \) the element in \( W \) which inverts \( e \). Given \( w \in W \) denote by \( \Delta_w \) the double coset \( BwB \in \mathbb{C}[G^+, B] \). Then \( \mathbb{C}[G, B] \) is generated by \( \Delta_s, \Delta_i \) and \( B = \Delta_{id_W} \). Additionally, for every \( \tilde{w} \in \tilde{W}, \Delta_i \tilde{w} = \Delta_{id_{\tilde{w}}} \).

3. (a) For \( G = G^+ \), given \( r \in \{s, t\} \) let \( q_r = R(\Delta_r) \).

Then

\[
\Delta_s \Delta_w = q_r \Delta_{rw} + (q_r - 1) \Delta_w \text{ if the reduced word decomposition of } w \text{ begins with } r
\]

\[
\Delta_i \Delta_w = \Delta_{rw} \text{ if } w \text{ does not begin with } r
\]

\[
\Delta_w \Delta_r = q_r \Delta_{wr} + (q_r - 1) \Delta_w \text{ if } w \text{ ends with } r
\]

Additionally, the relations

\[
\Delta_s^2 - (q_r + (q_r - 1) \Delta_r) = 0 \quad \text{for } r \in \{s, t\}
\]

define a presentation of \( \mathbb{Z}[G^+, B] \) as a unital ring.

(b) If \( G \neq G^+ \) given \( w \in W = N^+/T \) let \( \tilde{w} \) be the image of \( w \) under the automorphism of \( W \) which switches \( s \) and \( t \); viewing \( W \) as a subgroup of \( W^+ \) this automorphism is conjugation by \( i \).

Then \( \mathbb{Z}/2\mathbb{Z} \) acts on \( \mathbb{Z}[G^+, B] \) by sending \( \Delta_w \rightarrow \Delta_{\tilde{w}} \) for any \( w \in W \).

\( \mathbb{Z}[G, B] \) is isomorphic to the twisted tensor product of \( \mathbb{Z}/2\mathbb{Z} \) on \( \mathbb{Z}[G^+, B] \).

Additionally the relations

\[
\Delta_s^2 - (q_r + (q_r - 1) \Delta_r) = 0 \quad \text{for } r \in \{s, t\}
\]

\[
\Delta_i^2 - 1 = 0
\]

\[
\Delta_i \Delta_s \Delta_i - \Delta_i = 0
\]

define a presentation of \( \mathbb{Z}[G, B] \) as a unital ring.

**Proof.** We begin with part 2 of the above theorem.

1. The claim for \( \mathbb{C}[G^+, B] \), will follow from 2a, 3a and the second paragraph after theorem 1.11 in [CIK71].

   The claim for \( \mathbb{C}[G, B] \) will be proved in 3.b.

2. (a) i. Since \( G^+ \) acts strongly transitively on \( T \), \( N^+ \) acts transitively on the edges of \( A \).

   Then, \( T \) is the kernel of the surjective map from \( G^+ \) to the automorphisms group of \( A \) as a bipartite graph, so is a normal subgroup of \( N^+ \).

   So by the first isomorphism theorem \( N^+/T \) is the group of automorphisms of \( A \) as a bipartite graph which is isomorphic to \( D_\infty \).

   ii. Since \( B \) is the group of elements fixing the edge \( e \) and \( e \) is contained in the doubly-infinite geodesic \( A \) we have that \( T \subseteq B \).

   So for all \( g_1, g_2 \in nT, Bg_1B = Bg_2B \).

   iii. The union \( \bigcup_{w \in W} BwB \) covers \( G^+ \) since given \( g \in G \) we can find some \( b \in B \) such that \( bg \cdot e \in \tilde{A} \) and some \( n \in N^+ \) such that \( nbg \cdot e = e \).

   Then \( nbg \in B \), so \( g \in Bn^{-1}B \) for some \( n \in N^+ \). Then from part i, \( g \in BwB \) for some \( w \in W \).

   The union is disjoint since if given \( w = nT, w' = n'T \in W, BwB = Bw'B \) then \( n = b_1n' \cdot b_2 \) for some \( b_1, b_2 \in B \). So \( n \cdot e = b_1n' \cdot b_2 \cdot e = b_1n' \cdot e \).

   But since \( n \cdot e \in A \), \( b_1 \) must map the edge \( n' \cdot e \) to an edge in \( A \) so \( b_1 \in N^+ \). But \( b_1 \) also fixes \( e \) since it is an element of \( B \), so it must fix \( A \) pointwise, so \( b_1 \in T \)
Noting that \( n^{-1} = b_2^{-1} n^{-1} b_2^{-1} \) we can repeat to show that \( b_2^{-1} \in T \) so \( b_2 \in T \).

Then \( n = t_1 t_2 \) for some \( t_1, t_2 \in T \) so \( w = w' \) and \( \bigcup_{w \in W} BwB \) is a disjoint union.

For the final part of 2.a, we know that \( W \) is isomorphic to the group of automorphisms of the bipartite graph \( G \), so \( W \) is generated by \( s \) and \( t \).

So we need only show the decomposition of double cosets \( \Delta_w \) for \( w \in W \) to show that \( \mathcal{C}[G^+ \setminus B] \) is generated by \( \Delta s, \Delta t, \) and \( \Delta_{idw} \).

Let \( w \in W \) with reduced decomposition \( w = s_1 s_2 \cdots s_n \) as a word in \( \{s, t\} \). To show that \( \Delta_w = \Delta s_1 \cdots \Delta s_n \) we proceed by induction on \( n \).

For the base case \( n = 1 \), \( \Delta s = \Delta t \) and \( \Delta t = \Delta t \).

Now assume the induction hypothesis holds for \( w \in W \) with reduced decomposition of length \( n \) or less.

Let \( w \in W \) with reduced decomposition \( w = s_1 \cdots s_{n+1} \). Then let \( w' = s_1 \cdots s_n \).

We will show equality as sets, \( Bw' B_{s_{n+1}} B = Bw' B_{s_{n+1}} B \) and equality of number of right cosets, \( R(Bw' B_{s_{n+1}} B) = R(Bw' B) R(B B_{s_{n+1}} B) \) which will give the equality \( \Delta_w = \Delta_w \Delta_{s_{n+1}} = \prod^n_{i=1} \Delta s_i \).

For the equality of sets, let \( n_{w'}, n_{s_{n+1}} \in N^+ \) be elements in the cosets of \( T \) represented by \( w' \) and \( s_{n+1} \).

We know that \( B w' B_{s_{n+1}} B \subseteq B w' B_{s_{n+1}} B \) since \( id_g \in B \).

We will show that \( n_{w'} B n_{s_{n+1}} \subseteq B w' B_{s_{n+1}} B \) which will imply that \( B w' B_{s_{n+1}} B \subseteq B w' B_{s_{n+1}} B \) since \( T \subseteq B \).

Let \( b \in B \). \( n_{s_{n+1}} \cdot e \) is an edge of distance one away from \( e \) and \( b \) fixes \( e \) and is type-preserving so \( n_{w'} B n_{s_{n+1}} \cdot e \) is an edge \( n + 1 \) vertices away from \( e \) on the same side of \( e \) as \( n_{w'} B n_{s_{n+1}} \cdot e \). So \( n_{w'} B n_{s_{n+1}} = e n_{w'} B n_{s_{n+1}} B \) for some \( e \in B \) (since \( B \) is transitive on geodesics through \( e \)).

For the equality of number of right cosets, \( R(Bw' B_{s_{n+1}} B) \) is the number of paths without backtrack of length \( n + 1 \) starting from either vertex of \( e \) not crossing \( e \) since each right coset \( B w' B_{s_{n+1}} B \) is characterised by the elements of \( G^+ \) which fix such a path.

Then \( R(B B_{s_{n+1}} B) \) is the number of ways to extend a path of length \( n \) to path of length \( n + 1 \) without backtrack which gives the equality of number of right cosets.

(b) i. From proposition \( \ref{prop:2.3} \) we know that \( N \) contains an element \( i \) which inverts \( e \). Since we know from part 2.a.i that \( N^+ / T \) is transitive on edges in \( \mathcal{A} \), \( N \) must be transitive on vertices in \( \mathcal{A} \).

Additionally, \( T \) is the kernel of the map from \( G \) to the automorphism group of \( \mathcal{A} \) (as an ordinary graph).

So \( N^+ / T \) is the group of automorphisms of \( \mathcal{A}, D_{\infty} \).

ii. In part 2.a.ii we proved that \( T \subseteq B \), so once again for all \( g_1, g_2 \in n T, B g_1 B = B g_2 B \).

iii. To show that the union \( \bigcup_{w \in W} B w B \) covers \( G \) note that we showed in 2.a.iii that for all \( g \in G^+ \), \( g \in B n T \) for some \( n \in N^+ \).

Additionally, for any \( g \in G \setminus G^+ \) the map \( \varphi \) in the proof of proposition \( \ref{prop:4.23} \) shows that \( g = h i \) for some \( h \in G^+ \).

We also know that \( i \) is in the normalizer of \( B \) since given \( b \in B \), and a vertex \( v \in e \), \( i^{-1} b \cdot v = i^{-1} b \cdot (i \cdot v) = i^{-1} i \cdot v = v \).

So \( g = h i \in B n B i = B n i B \). We know that \( n i \in N \) as both \( n \) and \( i \) stabilizer \( \mathcal{A} \).

So applying 2.b.ii we know \( g \in B w B \) for some \( w \in W \).

To show that the union is disjoint, assume that \( w = n T, \tilde{w} = n' T \in \tilde{W} \) and that \( B w B = B \tilde{w} B \).

Since \( B \subseteq G^+ \), either both \( w \) and \( \tilde{w} \) are type preserving (i.e. every element of \( \tilde{w} = n T \) is type preserving) in which case the result follows from 2.a.iii or both are not.

If both \( w \) and \( \tilde{w} \) are not type preserving we can write \( \tilde{n} = n i \) and \( \tilde{n}' = n' i \) for some \( n, n' \in N^+ \).

Then \( B n B = B \tilde{n} B \implies B n i B = B \tilde{n} i B \implies B n B i = B \tilde{n} i B i \).

Then from 2.a.iii, \( n T = n' T \). But \( i \) is in the stabilizer of \( T \) since for all vertices \( v \in \mathcal{A}, i \cdot v \in \mathcal{A} \) so \( i^{-1} t i \cdot v = v \).
Corollary 4.2.5 (Corollary 4.6 in [BLRW09]). Given \( w \in W \) denote by \( w^{(i)} \) the last \( i \) letters of \( w \)’s reduced word decomposition and \( w^{(i)} \) the first \( i \) letters of \( w \)’s reduced word decomposition.

Let \( w_{[i]} \) be \( w \)’s reduced word decomposition missing its last \( i \) letters and \( [i]w \) the reduced word decomposition missing its first \( i \) letters.

Given \( r \in \{s, t\} \) let \( u \) be the other element in the set.

Given \( w \) with reduced word decomposition \( s_1 \cdots s_n \) let \( q_w = \prod_{i=1}^{n} q_{s_i} \).

1. For \( C[G^+, B] \), let \( w, w' \in W \) then

   (a) If the last letter of the reduced word decomposition of \( w \) differs from the first letter of the reduced word decomposition of \( w' \) then
   \[ \Delta_w \Delta'_w = \Delta_{ww'} \]

   (b) If the first and last letter coincide let \( m \) be the length of the shorter reduced word decomposition between \( w \) and \( w' \) and let \( s_i \) be the \( i \)th letter of \( w' \).

   Then
   \[ \Delta_w \Delta_{w'} = q_w \Delta_{w^{[m]}w'} + \sum_{i=0}^{m-1} q_{w^{(i)}(q_i - 1)\Delta_{w^{(i)}s_i}w'} \]
2. If $G \neq G^+$ then for $\mathbb{C}[G, B]$ we can use the multiplication table in 1 for $\mathbb{C}[G^+, B]$ along with the following:

$$\Delta_i w \Delta_{w'} = \Delta_i (\Delta_w \Delta_{w'})$$
$$\Delta_w \Delta_{i w'} = \Delta_i (\Delta_w \Delta_{w'})$$
$$\Delta_i w \Delta_{i w'} = \Delta_w \Delta_{w'}$$

Proof. For part 1 use theorem 1.2.4 and induct on $m$.

For part 2 also use the relations given in theorem 1.2.4 parts 2.b. and 3.b.

4.3 HNN Extensions and Stabilizers of an End

**Proposition 4.3.1.** Let $G$ be an HNN-extension of a group $H$ with respect to an endomorphism $\alpha : H \to H$ such that $|H : \alpha(H)|$ is finite.

Then $G$ acts on a tree $T$ and stabilizes end of $T$. Additionally, $H$ is the stabilizer of some distinguished vertex of $T$.

**Proof.** Let $T$ be the universal covering tree associated to the graph of groups of $G$ as an HNN-extension and consider the action of $G$ on $T$. □

The following proposition is stated with proof.

**Proposition 4.3.2.** Let $T$ be a locally finite tree.

Let $\infty$ be an end of $T$ and let $B$ be a group of automorphisms of $T$ which stabilize $\infty$ and which contains at least one hyperbolic element.

Let $s$ be a hyperbolic element in $B$ which has the smallest translation length amongst all hyperbolic elements in $B$ such that $\infty$ is attracting.

Then every hyperbolic element of $B$ can be written in the form $s^ng$ for some $g \in M_0$, $n \in \mathbb{Z}_{\geq 0}$ where $M_0$ is the stabilizer of a distinguished vertex $v \in T$.

**Proposition 4.3.3** (Proposition 5.1 in [BLRW09]). Let $T$ be a locally finite tree.

Let $\infty$ be an end of $T$ and let $B$ be a group of automorphisms of $T$ which stabilize $\infty$ and which contains at least one hyperbolic element.

Let $s$ be a hyperbolic element in $B$ which has the smallest translation length amongst all hyperbolic elements in $B$ such that $\infty$ is attracting.

Let $M$ be the set of elliptic elements of $B$.

Let $o$ be a vertex on the axis of $s$ and let $M_0$ be the stabilizer of $o$ in $B$.

Then,

1. $1 \to M \to B \to \mathbb{Z} \to 1$ is a split exact sequence so $B = M \rtimes \mathbb{Z}$. Additionally either 1 or $-1 \in \mathbb{Z}$ acts on $M$ by conjugation with $s^{-1}$, denote this action $\alpha$.

2. $M_0 \subseteq M$ and $M = \bigcup_{n \in \mathbb{Z}_{\geq 0}} s^n M_0 s^{-n}$.

So $B$ is isomorphic to the HNN-extension of $M_0$ with respect to $\alpha$.

3. The index $|M_0 : \alpha(M_0)|$ is at most equal to $|[x \in V(T) \mod d(x, o) = d(s^{-1} \cdot o, o)]| - 1$ and so finite.

4. $(B, M_0)$ and $(M, M_0)$ are Hecke pairs.

**Proof.**

1. $M$ is a normal subgroup of $B$ since the elliptic elements of a group acting on a tree is normal.

$M$ has infinite cyclic quotient generated by $s$ since every hyperbolic element of $B$ can be written in the form $s^ng$ for some $g \in M_0$, $n \in \mathbb{Z}_{\geq 0}$.

2. Any element of $M$ can be written as the conjugation of some element $M_0$ with an appropriate hyperbolic element. From part 1, we know that every hyperbolic element can be written in the form $s^ng \in M_0$ for some $g \in M_0$, $n \in \mathbb{Z}_{\geq 0}$.

So $M = \bigcup_{n \in \mathbb{Z}_{\geq 0}} s^n M_0 s^{-n}$. 

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3. This follows from $G$ acting freely on $T$.

4. From part 3, $s$ commensurates $M_0$ so since $M = \bigcup s^nM_0s^{-n}$ from part 2, $(M, M_0)$ is a Hecke pair.

Additionally, every element of $B$ can be written in the form $s^n g$ for some $g \in M_0$, $n \in \mathbb{Z}_{\geq 0}$ so $(B, M_0)$ is also a Hecke pair.

\textbf{Theorem 4.3.4} (compare Theorem 5.2 in \cite{BLRW09}). Let $B$ be the HNN-extensions of a group $M_0$ with respect to an endomorphism $\alpha$ of $M_0$ such that $|M_0 : \alpha(M_0)|$ is finite.

Let $\tilde{\alpha}$ be the automorphism of $B$ induced by $\alpha$. Let $M = \bigcup_{n \in \mathbb{Z}_{\geq 0}} \tilde{\alpha}^{-n}(M_0) \subseteq B$.

Then

1. $(B, M_0)$ and $(M, M_0)$ are Hecke pairs.

2. There is an action of $\tilde{\alpha}$ on $\mathbb{C}[M, M_0]$ where given a canonical basis element $M_0 g M_0$,

\[ \tilde{\alpha}(M_0 g M_0) = |M_0 : \alpha(M_0)|^{-1} \sum_{B \in M_0 \tilde{\alpha}^{-1}(g) M_0 \cap \mathbb{C}[M, M_0]} B \]

3. $\mathbb{C}[B, M_0]$ is isomorphic as an algebra to the semigroup crossed product $\mathbb{C}[M, M_0] \rtimes \tilde{\alpha} \mathbb{Z}_{\geq 0}$ where $1 \in \mathbb{Z}_{\geq 0}$ acts on $\mathbb{C}[M, M_0]$ by $\tilde{\alpha}$.

4. $\mathbb{C}[M, M_0]$ is the union of a directed family of finite-dimensional subalgebras.

\textbf{Proof.} 1. This follows from propositions \[4.3.1\] and \[4.3.3\]

2. Theorem 1.9 of \cite{LL05}

3. Theorem 1.9 of \cite{LL05}

4. Since $M = \bigcup_{n \in \mathbb{Z}_{\geq 0}} \tilde{\alpha}^{-n}(M_0)$, $\mathbb{C}[M, M_0]$ is the increasing union of the algebras $\mathbb{C}[\tilde{\alpha}^{-n}(M_0), M_0]$. $\mathbb{C}[\tilde{\alpha}^{-n}(M_0), M_0]$ is a finite dimensional algebra for each $n$ since the index $|\tilde{\alpha}^{-n}(M_0) : M_0| = |\tilde{\alpha}^{-1}(M_0) : M_0|^n$ is finite.

\[\square\]
Chapter 5

The Bruhat-Tits tree of $\text{SL}_2(\mathbb{Q}_p)$

In 1966 Ihara gave a combinatorial proof that every torsion-free subgroup of $\text{SL}_2(\mathbb{Q}_p)$ is a free group \cite{Iha66} by decomposing $\text{SL}_2(\mathbb{Q}_p)$ as an amalgam of two copies of $\text{SL}_2(\mathbb{Z}_p)$.

But another way to prove that a group $G$ is free is to show that it acts freely and without inversions on a tree $X$ and then identify it with the fundamental group of the quotient graph $\pi_1(G\backslash X)$. This lead to Serre’s work on trees and amalgams in \cite{Ser71} and \cite{Ser80} (which was written in collaboration with Bass).

This section describes the Bruhat-Tits tree of $\text{SL}_2(\mathbb{Q}_p)$ as given in \cite{Ser80} and calculates its Hecke algebra through an alternate proof of results given in chapter 4.
5.1 The Bruhat-Tits Tree of $\text{SL}_2(\mathbb{Q}_p)$

5.1.1 The tree

Bill Casselman’s notes on his webpage, [Cas] were of great help in the write up of this section.

**Definition 5.1.1** (Lattice). A lattice in $\mathbb{Q}_p^2$ is the $\mathbb{Z}_p$-span of two $\mathbb{Z}_p$-linearly independent vectors.

**Example 5.1.2.** The $\mathbb{Z}_p$-module generated by $(0, 1)$ and $(1, 0)$ is a lattice.

We can write a lattice $L$ as a matrix $M \in \text{GL}_2(\mathbb{Q}_p)$ by using the basis of $\mathbb{Q}_p^2$ writing two vectors which generate $L$ as the columns of $M$.

For example,

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is a matrix which represents the lattice $\mathbb{Z}_p \cdot (1, 0) + \mathbb{Z}_p \cdot (0, 1)$.

Note that multiplying a matrix $M \in \text{GL}_2(\mathbb{Q}_p)$ on the right by a matrix $N \in \text{GL}_2(\mathbb{Z}_p)$ is the same as acting on $M$ by $\mathbb{Z}_p$-invertible column operations. So $MN$ will represent the same lattice as $M$.

Since every element of a lattice is the $\mathbb{Z}_p$-sum of the two spanning vectors we in fact have a bijection between cosets of $\text{GL}_2(\mathbb{Z}_p)$ in $\text{GL}_2(\mathbb{Q}_p)$ and lattices.

We define an equivalence class on the set of lattices of $\mathbb{Q}_p^2$:

Two lattices $L$ and $L'$ are equivalent if $L = \lambda \cdot L'$ for some $\lambda \in \mathbb{Q}_p$ (where $\lambda \cdot L = \{ \lambda \cdot l \mid l \in L \}$).

We will write $[L]$ for the equivalence class of a lattice $L$, we will sometimes abuse notation; for $M \in \text{GL}_2(\mathbb{Q}_p)$ we will write $M$ for the lattice defined by the coset $M \text{GL}_2(\mathbb{Z}_p)$ and write $[M]$ for the equivalence class of lattices containing the lattice represented by the coset $M \text{GL}_2(\mathbb{Z}_p)$.

**Definition 5.1.3** (The Bruhat-Tits tree of $\text{SL}_2(\mathbb{Q}_p)$). The Bruhat-Tits tree of $\text{SL}_2(\mathbb{Q}_p)$ which we will denote $T$ is the graph with

1. Vertex set the set of equivalence classes of lattices in $\mathbb{Q}_p^2$.

2. A pair of vertices $\{[L], [L']\}$ form an edge if there exists lattices $A \in [L], B \in [L']$ such that $A \supseteq B \supseteq p \cdot A$.

Note that if $A \supseteq B \supseteq p \cdot A$ then (multiplying by $p$) $p \cdot A \supseteq p \cdot B$ so we have that $B \supseteq p \cdot A \supseteq p \cdot B$.

We will call the Bruhat-Tits tree of $\text{SL}_2(\mathbb{Q}_p)$ $T_{p+1}$ (where the tree of $\text{SL}_2(\mathbb{Q}_p)$ will be homogeneous of degree $p+1$). We will shorten this to $T$ when there is no explicit mention of $p$.

**Proposition 5.1.4.** If $\{[L], [L']\} \in E(T)$ then $[L] \neq [L']$.

**Proof.**

To prove $T$ is in fact a tree we will show that $T$ is connected and acyclic.

**Proposition 5.1.5** (Connectedness). The Bruhat-Tits tree of $\mathbb{Q}_p^2$ is connected.

**Proof.** Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ be the identity matrix, so $I \text{GL}_2(\mathbb{Z}_p)$ is the lattice $\mathbb{Z}_p(1, 0) + \mathbb{Z}_p(0, 1)$.

To show that the Bruhat-Tits tree is connected we will show that every vertex has a path to $[I]$.

Let $[L]$ be some vertex in the Bruhat-Tits tree, then there exists some $M \in \text{GL}_2(\mathbb{Q}_p)$.

Then $M$ has an invariant factor decomposition $M = S D T$ where $D = \begin{bmatrix} p^\alpha & 0 \\ 0 & p^\beta \end{bmatrix}$ where $S$, $T \in \text{GL}_2(\mathbb{Z}_p)$, $\alpha, \beta \in \mathbb{Z}$, $\alpha \leq \beta$. Note that such a decomposition exists since $\mathbb{Z}_p$ is a principal ideal domain and every element in $\mathbb{Q}_p$ is of the form $p^\alpha u$ for some invertible $u \in \mathbb{Z}_p$ and $\alpha \in \mathbb{Z}$.

Then $M = S D T \implies MT^{-1} = \frac{1}{p} SD$.

Let $n = \beta - \alpha$. And let $D_i = \begin{bmatrix} p^\alpha & 0 \\ 0 & p^{\beta+i} \end{bmatrix}$ for $0 \leq i \leq n$.

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Then we claim that
\[(|I| = |S| = [SD]_0, [SD]_1, \ldots, [SD]_n, [SD] = [MT^{-1}] = [M] = [L])\]
is a path from \([I]\) to \([L]\).

To show this, first note that \([I] = [S]\) and \([MT^{-1}] = [M]\) since \(S, T \in \text{GL}_2(\mathbb{Z}_p)\) and matrix multiplication on the right does not change the lattice represented. Also note that \([SD]_0 = [p^nS] = [S]\) since equivalence classes are invariant under scaling.

Let \(v_1\) and \(v_2\) be the first and second column of \(S\) respectively. Then \([SD]\) represents the lattice \(p^n v_1 + p^n v_2\). Then \(p^n v_1 + p^{n+i} v_2 \supseteq p^n v_1 + p^{n+i+1} v_2 \supseteq p^{n+1} v_1 + p^{n+i+1} v_2\) so \([SD]_i \supseteq [SD]_{i+1} \supseteq p \cdot [SD]_i\).

So \([SD]_i\) and \([SD]_{i+1}\) are neighbours for \(0 \leq i < n\) and we have a path from \([I]\) to \([L]\). \(\square\)

**Remark.** The above proof shows that \(\beta - \alpha = n\) is an upper bound on the distance from \([M]\) to \([I]\).

In fact in general given \(M_1, M_2 \in \text{GL}_2(\mathbb{Q}_p)\), and invariant factor decomposition \(M_2 M_1^{-1} = SDT, D = \begin{bmatrix} p^n & 0 \\ 0 & p^m \end{bmatrix}\) then \(\beta - \alpha\) is the distance from \([M_1]\) to \([M_2]\).

**Proposition 5.1.6.** Let \(A \in [L]\) be a lattice. Then \(A = p^n \cdot L\) for some \(n \in \mathbb{Z}\).

**Proof.** \(A = \lambda \cdot L\) for some \(\lambda \in \mathbb{Q}_p\). But then we can write \(\lambda = p^n \cdot u\) for some \(n \in \mathbb{Z}, u \in \mathbb{Z}_p\) a unit. For any unit \(u \in \mathbb{Z}_p, u \cdot L = L\) since \(L\) is the \(\mathbb{Z}_p\) span of two basis vectors.

Then \(A = p^n u \cdot L = p^n \cdot L\). \(\square\)

For the next three propositions, compare [Ser80] II.1 Theorem 1.

**Proposition 5.1.7.** Let \([L]\) and \([L']\) be two vertices connected by an edge in the Bruhat-Tits tree and fix a lattice \(A \in [L]\).

Then there exists a lattice \(B \in [L']\) such that \(A \supseteq B \supseteq p \cdot A\).

**Proof.** Since \([L]\) and \([L']\) are adjacent we can find lattices \(P \in [L]\) and \(Q \in [L']\) such that \(P \supseteq Q \supseteq p \cdot P\).

Since \(A \in [L]\), \(A = \lambda \cdot P\) for some \(\lambda \in \mathbb{Q}_p\).

Let \(B = \lambda \cdot Q \in [L']\).

Then \(\lambda \cdot P \supseteq \lambda \cdot Q \supseteq \lambda p \cdot P\) so \(A \supseteq B \supseteq p \cdot A\). \(\square\)

This proposition implies that given any path \(([L_0], [L_1], \ldots, [L_n])\) find a set of lattices \(\{K_i \in [L_i] \mid 0 \leq i \leq n\}\) for each such that \(K_i \supseteq K_{i+1} \supseteq p \cdot K_i\) for all \(0 \leq i < n\). Explicitly, set \(K_0\) to be any arbitrary lattice in \([L_0]\) then use the above proposition to find an appropriate \(K_1\) and iterate.

For the following propositions recall that a path \((v_0, v_1, \ldots, v_n)\) in a graph is said to contain no backtracks if \(v_i \neq v_{i+1}\) for all \(0 < i < n\).

**Proposition 5.1.8.** Let \(([L_0], [L_1], \ldots, [L_n])\) be a path with no backtracks such that \(L_i \supseteq L_{i+1} \supseteq p \cdot L_i\) for all \(0 \leq i < n\).

Then \(L_n \not\subseteq p \cdot L_0\).

**Proof.** We proceed by induction. For the case \(n = 1\) we know that \(L_0 \supseteq L_1 \supseteq p \cdot L_0\) so \(L_1 \not\subseteq p \cdot L_0\).

For the inductive step we have the following inclusions:

\[
\begin{array}{cccccc}
& & & & & L_{n-2} \\
& & & & \supseteq \\
& & & \supseteq \\
& & L_{n-1} & \supseteq & p \cdot L_{n-2} \\
\supseteq & & & & & \supseteq \\
& & L_n & & & p \cdot L_{n-1} \\
& & & & & \supseteq \end{array}
\]

Figure 5.1: Figure from [Ser80]
We know that $L_{n-1}/L_n$ and $L_{n-1}/p \cdot L_{n-2}$ must be different one dimensional subspaces of $L_{n-1}/p \cdot L_{n-1} \cong \mathbb{F}_p^3$ as otherwise $L_n = p \cdot L_{n-2}$ which would imply that $[L_n] = [L_{n-2}]$ which is a contradiction since we assumed our path contained no backtracks.

But then $L_{n-1}/L_n$ and $L_{n-1}/p \cdot L_{n-2}$ must span $L_{n-1}/p \cdot L_{n-1} \cong \mathbb{F}_p^3$ so we must have that $L_{n-1}/(L_n + pL_{n-2}) = \{0\}$.

So $L_{n-1} = L_n + p \cdot L_{n-2}$. Then $L_{n-1} = L_n \mod pL_0$ since $p \cdot L_{n-2} \subseteq p \cdot L_{n-3} \subseteq \cdots \subseteq p \cdot L_0$.

But from our induction hypothesis $L_{n-1} \not\subseteq L_0$, so $L_n \not\subseteq p \cdot L_0$.

\[ \Box \]

**Proposition 5.1.9 (Acyclicity).** $T$ contains no cycles.

**Proof.** To show that $T$ contains no cycle we will induct on $n$ to show that every path without backtrack of length $n$ in $T$ has different initial and final vertex.

The induction hypothesis hold for the base case $n = 1$ from proposition 5.1.4.

For the inductive step, let $([L_0], [L_1], \ldots, [L_n])$ be a path without backtracking. By making an appropriate choice of label for lattice equivalence classes, we can assume that $L_i \supseteq L_{i+1} \supseteq p \cdot L_i$ for all $0 \leq i < n$ from proposition 5.1.3.

We know that $L_n \not\subseteq L_{n-1} \cdots \not\subseteq L_0$, so if $[L_n] = [L_0]$ then $L_n = p^\alpha L_0$ for some $\alpha \in \mathbb{Z}_{\geq 0}$ ([L_n] = [L_0] $\implies L_n = \lambda \cdot L_0$ for $\lambda \in \mathbb{Z}_2$ but $\lambda = p^\alpha u$ for some $\alpha \in \mathbb{Z}$, $u$ invertible in $\mathbb{Z}_p$).

But by proposition 5.1.8 $L_n \not\subseteq p \cdot L_0$, so $L_n \neq p^\alpha L_0$ for some $\alpha \in \mathbb{Z}_{\geq 0}$ and we cannot have $[L_n] = [L_0]$.

\[ \Box \]

Although the structure of the Bruhat-Tits tree will depend on the value of $p$ for the field $\mathbb{Q}_p$ we will generally call the Bruhat-Tits tree $T$.

We now define an action on the Bruhat-Tits tree by matrix multiplication.

**Definition 5.1.10.** Let $M \in \text{GL}_2(\mathbb{Q}_p)$. Then define the map $\varphi_M : V(T) \to V(T)$ by $\varphi_M([L]) = [ML]$.

We will call this the action of $\text{GL}_2(\mathbb{Q}_p)$ (or some subgroup of $\text{GL}_2(\mathbb{Q}_p)$) on $T$.

We will write $M \cdot [L]$ for $\varphi_M([L])$.

**Proposition 5.1.11.** The above action is well defined.

**Proof.** We need only check that edges are preserved.

Let $[L_1], [L_2]$ be vertices in $T$ such that $([L_1], [L_2])$ is an edge of $T$. Then for some $A \in [L_1], B \in [L_2]$, $A \supseteq B \supseteq p \cdot A$.

Note that since $A \in [L_1], B \in [L_2]$ we know that $MA \in [ML_1], MB \in [ML_2]$.

Then multiplying by $M$ on the left to $A \supseteq B \supseteq p \cdot A$, $MA \supseteq MB \supseteq M(p \cdot A) \implies MA \supseteq MB \supseteq p \cdot MA$.

So $([ML_1], [ML_2])$ is also an edge of $T$.

\[ \Box \]

**Proposition 5.1.12.** Each vertex in $T$ has exactly $p + 1$ neighbours.

**Proof.** Let $[L_1], [L_2] \in V(T)$ such that $([L_1], [L_2]) \in E(T)$ by choosing appropriate lattice representatives of the vertices we can assume that $L_1 \supseteq L_2 \supseteq p \cdot L_1$.

Quotienting $L_1$ by each of the submodules in the above inclusion we have that $L_1/L_1 \subseteq L_1/L_2 \subseteq L_1/p \cdot L_1 \cong \mathbb{F}_p^3$. So $L_1/L_2$ must be a proper non-trivial submodule of $\mathbb{F}_p^3$. The only proper non-trivial submodules of $\mathbb{F}_p^3$ are one dimensional subspaces of $\mathbb{F}_p^3$ and $\mathbb{F}_p^3$ itself, so $L_1/L_2$ is a one dimensional subspace of $\mathbb{F}_p^3$.

There are exactly $p + 1$ one dimensional subspaces of $\mathbb{F}_p^3 \cong \{(a, b) \mid a, b \in \mathbb{F}_p\}$ — each one the $\mathbb{F}_p$-span of one of $(1, 0), \ldots, (1, p - 1)$ and $(0, 1)$. So each vertex can have at most $p + 1$ neighbours.

To show that each vertex has exactly $p + 1$ neighbours we wish to show that each one dimensional subspace is the quotient of $L_1$ by some lattice and that two lattices which quotient to give different subspaces are in different equivalence classes of vertices.

There will exist a lattice $L$ for each one dimensional subspace $W$ such that $L_1/L = W$, to find such an $L$ choose a non zero element $v + pL \in W$ and let $\tilde{L}$ be the $\mathbb{Z}_p$-span of $v$. Then $L_1/(\tilde{L} + pL_1) = W$ so we only need to check that $\tilde{L} + pL_1$ is a lattice.

To check, let $v_1, v_2$ be two vectors whose $\mathbb{Z}_p$-span span $L_1$. Then we can write $v = \lambda_1 v_1 + \lambda_2$ for some $\lambda_1, \lambda_2 \in \mathbb{Q}_p$ such that either $|\lambda_1|_p = p^0 = 1$ or $|\lambda_2|_p = 1$ since $L_1 \supseteq \tilde{L} + pL_1 \supseteq p \cdot L_1$. 41
If $|\lambda_1|_p = 1$ then $p\lambda_1^{-1}v - pv_1 = \gamma v_2$ form some $\gamma \in \mathbb{Z}_p$, $|\gamma|_p \leq p^{-1}$, so $pv_1 = p\lambda_1^{-1}v - (\gamma p^{-1})pv_2$. Then $\{v, pv_1\}$ span $L + pL_1$ and $v$ and $pv_2$ must be linearly independent since $v_1$ and $v_2$ were, so $L + pL_1$ is a lattice. Similarly for $|\lambda_2|_p = 1$.

To prove that each vertex has exactly $p + 1$ vertices it remains to show that for two lattices $L$, $L'$ such that $L_1 \supseteq L \supseteq p \cdot L_1$ and $L_1 \supseteq L' \supseteq p$ and $L_1/L \neq L_1/L'$ then $|L| \neq |L'|$.

If not then $L = \lambda L'$ for some $\lambda \in \mathbb{Q}_p$, since both $L_1 \supseteq L \supseteq p \cdot L_1$ and $L_1 \supseteq L' \supseteq p$ we must have that $|\lambda|_p = 1$, but then $L = \lambda L' = L'$ which contradicts $L_1/L \neq L_1/L'$. \hfill \qed 

So we know that $\mathcal{T}$ is a tree of degree $p + 1$.

![Diagram](image)

Figure 5.2: The ball of radius 3 around $[I]$ in the Bruhat-Tits tree of SL$_2(\mathbb{Q}_2)$

where $M_0 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $M_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, $M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

### 5.2 The Action of PGL$_2(\mathbb{Q}_p)$, PGL$_2(\mathbb{Z}_p)$, PSL$_2(\mathbb{Q}_p)$ and PSL$_2(\mathbb{Z}_p)$

Let $M \in$ GL$_2(\mathbb{Q}_p)$ we will write $\tilde{M} \in$ PGL$_2(\mathbb{Q}_p)$ to denote the equivalence class of matrices in PGL$_2(\mathbb{Q}_p)$ containing $M$.

**Proposition 5.2.1.** PGL$_2(\mathbb{Q}_p)$ acts on $\mathcal{T}$ with action given by: for $[L] \in V(\mathcal{T})$, $M \in$ GL$_2(\mathbb{Q}_p)$, $\tilde{M} \cdot [L] = [ML]$.

**Proof.** We first check the action does not depend on representative for an element in PGL$_2(\mathbb{Q}_p)$.

For any other $N \in$ GL$_2(\mathbb{Q}_p)$ such that $\tilde{N} = M$, $M = \lambda \cdot N$ for some $\lambda \in \mathbb{Q}_p$.

So $[NL] = [\lambda ML] = [ML]$.

Since GL$_2(\mathbb{Q}_p)$ is a group we only still need to check that the action preserves edges in $\mathcal{T}$ for all $M \in$ GL$_2(\mathbb{Q}_p)$.

Let $([L_1],[L_2]) \in E(\mathcal{T})$ such that we have the lattice inclusion $L_1 \supseteq L_2 \supseteq p \cdot L_1$.

Let $\{u_1, u_2\}$ be a $\mathbb{Z}_p$-basis for $L_1$ and $\{v_1, v_2\}$ be a $\mathbb{Z}_p$-basis for $L_2$.

Then the inclusion $L_1 \supseteq L_2 \supseteq p \cdot L_1$ is equivalent to the $\mathbb{Z}_p$-span of $u_1$ and $u_2$ containing the $\mathbb{Z}_p$-span of $v_1$ and $v_2$ which contains the $\mathbb{Z}_p$-span of $p \cdot v_1$ and $p \cdot v_2$.

But then $M$ is a linear transformation so the $\mathbb{Z}_p$-span of $Mu_1$ and $Mu_2$ contains the $\mathbb{Z}_p$-span of $v_1$ and $v_2$ which contains the $\mathbb{Z}_p$-span of $p \cdot v_1$ and $p \cdot v_2$.

So $ML_1 \supseteq ML_2 \supseteq p \cdot ML_1$. \hfill \qed
Proposition 5.2.2. PGL₂(ℚₚ) acts faithfully and transitively on the vertices of \( T \).

Proof. We first check transitivity.

Let \([L_1], [L_2] \in V(T)\). By choosing appropriate \( ℤ_p \)-basis vectors for \( L_1 \) and \( L_2 \) let \( L'_1 \) and \( L'_2 \in \text{GL}_2(ℤ_p) \) be matrix representations for the lattices \( L_1 \) and \( L_2 \). Then let \( M = L'_2L'_1^{-1} \).

So \( M \cdot [L_1] = [M_{L_1}] = [L'_{L_1}L'_{L_1}^{-1}] = [L_2] = [L_2] \).

We now check faithfulness.

Let \( N \in \text{GL}_2(ℚ_p) \) and \( \tilde{N} \in \text{PGL}_2(ℚ_p) \) be the coset containing \( N \).

We wish to show that if \( \tilde{N} \cdot [L] = [L] \) for all \([L] \in V(T)\) then \( \tilde{N} = \text{id}_{\text{PGL}_2(ℚ_p)} = \tilde{I} \). Let \( L' \) be a matrix representation for the lattice \( L \).

If \( \tilde{N} \cdot [L] = [L] \) then \( NL' = \lambda L'S \) for some \( S \in \text{GL}_2(ℤ_p) \), \( \lambda \in ℚ_p \).

So \( L'^{-1}NL' = \lambda S \) for some \( S \in \text{GL}_2(ℤ_p) \), \( \lambda \in ℚ_p \). Then \( T = p^\alpha L'^{-1}NL' \) for some \( \alpha \in ℤ \), \( T \in \text{GL}_2(ℤ_p) \).

We will write \( N = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}_2(ℤ_p) \) and consider the matrix representation of a lattice \( L' \):

\[
L' = \begin{bmatrix} 1 & 0 \\ 0 & p^\alpha \end{bmatrix}.
\]

Then \( L'^{-1}NL' = \begin{bmatrix} a & b \\ p^{-m}c & d \end{bmatrix} \). Assume that \( c \neq 0 \) and let \( m = \nu(c) + 1 \). Then \( p^\alpha (p^{-m}c) \in ℤ_p \) and if only if \( \alpha \geq 1 \) so that we have \( T \in \text{GL}_2(ℤ_p) \) such that \( T = p^\alpha L'^{-1}NL' \) then \( \alpha \geq 1 \).

But then \( \det(p^\alpha L'^{-1}NL') = p^{2\alpha} \det(N) \) which is a not a unit in \( ℤ_p \) unless \( \alpha = 0 \). So we must have \( c = 0 \), similarly letting \( m = -\nu(b) + 1 \) if \( b \neq 0 \) we have that \( b = 0 \).

So \( N = \lambda \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \) for some \( a, d \in ℤ_p \), \( \lambda \in ℚ_p \) and since \( \tilde{N} \in \text{PGL}_2(ℚ_p) \) we know that \( a \) and \( d \) must be units, so we can assume that \( d = 1 \) since \( d^{-1}N \cdot [L] = N \cdot [L] \) for all \([L] \in V(T)\).

If \( a \neq 1 \) then, writing \( a = 1 + \sum_{i=1}^{\infty} a_ip^i \), \( a_i \in ℱ_p \), we can define \( n = \min\{a_i \mid a_i \neq 0 \} \).

Then consider the matrix representing a lattice, \( L = \begin{bmatrix} p^{n+1} & 1 \\ 0 & 1 \end{bmatrix} \), then

\[
NL = \lambda \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} + \sum_{i=1}^{\infty} a_ip^i \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda O
\]

Conduct the following invertible column operations on \( O \): multiply the first column by \( a^{-1} \) then subtract \( \sum_{i=0}^{n} a_ip^i \) times the first column from the second column to obtain the matrix

\[
K = \begin{bmatrix} p^{n+1} & 1 + a_np^n \\ 0 & 1 \end{bmatrix}.
\]

So \([K] = [\lambda K] = [\lambda O] = [NL] \) but

\[
L^{-1}K = \begin{bmatrix} 1 & -1 \\ p^{n+1} & p^{n+1} \end{bmatrix} = \begin{bmatrix} p^{n+1} & 1 + a_np^n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a_np^{-1} \\ 0 & 1 \end{bmatrix}
\]

so \( \det(L^{-1}K) = 1 \) but \( L^{-1}K \notin \text{GL}_2(ℤ_p) \) so \( L^{-1}K \neq \lambda S \) for some \( S \in \text{GL}_2(ℤ_p) \), \( \lambda \in ℚ_p \). So \([K] \neq [L] \).

But we assumed \( N \cdot [L] = [L] \) for all \([L] \in V(T)\) so we must have that \( a = 1 \), so \( N = \lambda I \).

\[\Box\]

Proposition 5.2.3. Let \( M \in \text{GL}_2(ℚ_p) \). Let \( I \) be the identity matrix and \([I]\) be the vertex containing the lattices represented by \( I \).

Then \( M \) is an element of the stabilizer of \([I]\) if and only if \( M = \lambda \cdot N \) for some \( \lambda \in ℚ_p \), \( N \in \text{GL}_2(ℤ_p) \).

Proof. Given any \( N \in \text{GL}_2(ℤ_p) \), \( N \cdot I = NI = I N \), but the lattice \( IN \) is the lattice \( I \) since the lattice represented by \( IN \) has basis vectors represented by some invertible sum of \((1,0)\) and \((0,1)\).

So \( N \cdot [I] = [NI] = [I] \). Then since vertices/equivalence classes are defined as lattices up to scaling by elements of \( ℚ_p \), \( \lambda \cdot N \) is an element of the stabilizer of \([I]\) for every \( \lambda \in ℚ_p \), \( N \in \text{GL}_2(ℤ_p) \).

On the other hand, if \( M \in \text{GL}_2(ℚ_p) \) and \( M \cdot [I] = [I] \) then we must have that \( MI = \lambda \cdot IN \) for some \( \lambda \in ℚ_p \), \( N \in \text{GL}_2(ℤ_p) \), so \( MI = \lambda IN = \lambda NI = \lambda N \cdot I \), so \( M = \lambda N \).

\[\Box\]
Proposition 5.2.4. \( SL_2(\mathbb{Q}_p) \) acts with two orbits on the vertices of \( \mathcal{T} \), one orbit of vertices at even distance from the vertex \([I]\) the other orbit of vertices at odd distance from \([I]\).

Proof. We will first show that all matrices at even distance from the vertex \([I]\) are in the same orbit.

Let \( L \in \text{GL}_2(\mathbb{Q}_p) \) be a lattice. Then we have the invariant factor invariant factor decomposition  
\[
L = S D T \quad \text{where} \quad S, T \in \text{GL}_2(\mathbb{Z}_p) \quad \text{and} \quad D = \begin{bmatrix} p^n & 0 \\ 0 & p^\beta \end{bmatrix} \quad \text{for some} \quad \alpha, \beta \in \mathbb{Z}. \quad \text{Let} \quad n = \alpha + \beta.
\]

Then \( LT^{-1} = SD \) is also a matrix for the lattice \( L \). But \( S \in \text{GL}_2(\mathbb{Z}_p) \), \( S \) is a matrix for the lattice represented by \( L \).

So \( SDS^{-1} \cdot I \cong SDS^{-1} \cdot S = SD \cong LT^{-1} \cong L \).

Then \( \text{det}(SDS^{-1}) = \text{det}(D) \) and \( SD = SDS^{-1} \cdot S \). If \( n \) is even then \( 2m = n \), then \( \text{det}(p^{-m}SDS^{-1}) = \text{det}(p^{-m}D) = p^{\alpha-2m} = 1 \). So \( p^{-m}SDS^{-1} \in \text{SL}_2(\mathbb{Z}_p) \) and \( p^{-m}SDS^{-1} : [I] = p^{-m} \cdot [L] = [L] \).

If \( n \) is odd repeat consider the matrix \( J = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} \) instead of \( I \) and find and consider the invariant factor decomposition of \( LT^{-1} \).

The two orbits must be separate. Otherwise there would exist \( M \in \text{SL}_2(\mathbb{Z}_p) \) such that \( M \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} S = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} T \) for some \( S, T \in \text{GL}_2(\mathbb{Z}_p) \).

But then \( M = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} ST^{-1} \) which contradicts the invariant factor decomposition of matrices in \( \text{SL}_2(\mathbb{Z}_p) \).

\[ \square \]

Proposition 5.2.5. \( SL_2(\mathbb{Z}_p) \) fixes the vertex \([I]\) where \( I \) is the lattice represented by the identity matrix and acts transitively on spheres.

Proof. \( SL_2(\mathbb{Z}_p) \subset \text{GL}_2(\mathbb{Z}_p) \) so \( SL_2(\mathbb{Z}_p) \) fixes the vertex \([I]\).

By the invariant factor decomposition we know that \( SL_2(\mathbb{Q}_p) = SL_2(\mathbb{Z}_p) D SL_2(\mathbb{Z}_p) \) where \( D = \left\{ \begin{bmatrix} p^\alpha & 0 \\ 0 & p^{-\alpha} \end{bmatrix} \big| \alpha \in \mathbb{Z} \right\} \).

But for a given sphere at even distance around \([I]\) each vertex contains a lattice which has a matrix representative of the form \( Q_vM \), for some \( Q_v \in SL_2(\mathbb{Q}_p) \) and some fixed \( M \in D \). By the remark after proposition 5.1.5.

So \( SL_2(\mathbb{Z}_p) \) acts transitively on this sphere so \( SL_2(\mathbb{Z}_p) \) acts transitively on spheres at even distance from \([I] \).

Then \( SL_2(\mathbb{Z}_p) \) must act transitively on all spheres around \([I]\).

\[ \square \]

5.3 Associated Hecke Algebras

We will consider the Hecke algebras \( \mathbb{C}[\text{PGL}_2(\mathbb{Q}_p), \text{PGL}_2(\mathbb{Z}_p)] \) and \( \mathbb{C}[\text{PSL}_2(\mathbb{Q}_p), \text{PSL}_2(\mathbb{Z}_p)] \).

Note that \( \text{PGL}_2(\mathbb{Q}_p) \) acts transitively on vertices in \( \mathcal{T} \) and \( \text{PGL}_2(\mathbb{Z}_p) \) fixes \([I] \in V(\mathcal{T})\) and acts transitively on spheres on \( \mathcal{T} \) so the action of \( \text{PGL}_2(\mathbb{Q}_p) \) on \( \mathcal{T} \) is an example of a vertex stabilizer which acts transitively on spheres (See section 4.1).

As such we expect a basis for \( \mathbb{C}[\text{PGL}_2(\mathbb{Q}_p), \text{PGL}_2(\mathbb{Z}_p)] \) to be the set \( \{T_i \mid i \in \mathbb{Z}_{\geq 0}\} \) where \( T_i \) as a double coset is the set of right cosets/vertices at distance \( i \) from \([I]\) with multiplication table:

\[
T_n T_m = T_{m+n} + p^{n-1}(p + \delta_{m,n}) \Gamma_{m-n} + (p-1) \sum_{l=1}^{n-1} p^{l-1} \Gamma_{m+n-2l}
\]

for \( m \geq n \) (see corollary 4.1.4).

Similarly, \( \text{PSL}_2(\mathbb{Q}_p) \) acts with two orbits on \( \mathcal{T} \) and \( \text{PSL}_2(\mathbb{Z}_p) \) fixes \([I]\) so \( \mathbb{C}[\text{PGL}_2(\mathbb{Q}_p), \text{PGL}_2(\mathbb{Z}_p)] \) has a basis \( \{T_{2i} \mid i \in \mathbb{Z}_{\geq 0}\} \) and has multiplication table

\[
T_{2n} T_{2m} = T_{2(m+n)} + p^{2n-1}(p + \delta_{m,n}) \Gamma_{2(m-n)} + \sum_{l=1}^{2n-1} (p-1)p^{l-1} \Gamma_{2(m+n-2l)}
\]

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5.3.1 The algebra of any group with a similar action

The propositions for the following section all assume the following:

- Let $T$ be an infinite homogeneous tree of degree $q + 1$ where $q \geq 2$.
- Let $v_0 \in V(T)$ be a distinguished vertex.
- Let $G$ be a group acting transitively on $T$ such that the stabilizer of $v_0$ in $G$, which we will label $H$, acts transitively on each sphere of vertices of some fixed distance form $v_0$.

**Proposition 5.3.1.** Let $v \in V(T)$. Let $S_v = \{g \in G \mid G \cdot v_0 = v\}$, the set of elements in $G$ which send $v_0$ to $v$.

Then (as sets) $S_v = gH$ for any $g \in S_v$.

Moreover if $v_1, v_2 \in V(T)$, $S_{v_1} = S_{v_2} \iff v_1 = v_2$.

**Proof.** Let $g \in S_v$.

$(S \subseteq gH,$ let $g' \in S_v,$ then $g' \cdot v_0 = g^{-1} \cdot v = v_0.$ So $g' \cdot v_0 \in H,$ so $g' \in gH.$

$(gH \subseteq S_v,$ let $h \in H.$ Then $gh \cdot v_0 = g \cdot v_0 = v.$ So $gh \in S_v.$

For the final statement, if $v_1 \neq v_2$ then for any $g_1 \in S_{v_1}, g_2 \in S_{v_2}, g_1 \cdot v_1 \neq v_2 = g_2 \cdot v_0$ so $S_{v_1} \neq S_{v_2}.$ On the other hand if $S_{v_1} \neq S_{v_2}$ then there exists $g \in S_{v_1}$ such that $v_1 = g \cdot v_0 \neq v_2.$ So $v_1 \neq v_2.$

**Proposition 5.3.2.** Let $g \in G.$ Let $S_g = \{d(g \cdot v_0, v_0)\}$.

Then we can write $HgH$ as a disjoint union, $HgH = \bigsqcup_{v \in V(T), d(v, v_0) = n} S_v$.

**Proof.** ($HgH \subseteq \bigsqcup_{v \in V(T), d(v, v_0) = n} S_v$) Given $h_1, h_2 \in H$, $h_1 gh_2 \cdot v_0 = h_1 g \cdot v_0$. So $d(h_1 g \cdot v_0, v_0) = n$, since $h_1$ fixes $v_0$. So $h_1 gh_2 \in S_v$ for some $v \in V(T)$, $d(v, v_0) = n$.

$(\bigsqcup_{v \in V(T), d(v, v_0) = n} S_v \subseteq HgH)$ If $g' \in \bigsqcup_{v \in V(T), d(v, v_0) = n} S_v$ then $d(g' \cdot v_0, v) = n$. Since $H$ acts transitively on the sphere of radius $n$ from $v_0$, there exists $h \in H$ such that $h g' \cdot v_0 = g \cdot v_0$.

Then $g^{-1} h g' \cdot v_0 = v_0$ so $g^{-1} h g' \in H$. So $h g' \in HgH$ and $g' \in h^{-1} g \subseteq HgH$.

(Disjoint union) If for $v, w \in V(T)$ we have $g \in S_v \cap S_w$ then $g \cdot v_0 = v = w$. So $S_v = S_w$ since $S_v$ and $S_w$ are left cosets by proposition 5.3.1.

**Corollary 5.3.3.** Every double coset is the disjoint union of a finite number of left cosets.

**Proof.** From propositions 5.3.1 and 5.3.2 we know that $HgH$ is the union of at most

$$\#\{v \in V(T) \mid d(v, v_0) = d(g \cdot v_0, v_0)\}$$

left cosets.

Since every vertex has finite degree in $T$, $\#\{v \in V(T) \mid d(v, v_0) = d(g \cdot v_0, v_0)\}$ is finite.

**Proposition 5.3.4.** Let $g, g' \in G$. Write $HgH$ and $Hg'H$ as the disjoint union of left cosets, $HgH = \bigsqcup_{g_i \in I} g_i H,$ $Hg'H = \bigsqcup_{g_i \in J} g_i H$ for some indexing sets $I, J \subseteq G$.

For $v \in V(T), n \in \mathbb{Z}_{\geq 0}$, let $W_{v, n} = \{u \in V(T) \mid d(v, u) = n\}$.

Fix some $g_i \in I, g'_j \in J$ and let $W = W_{g'_j, v_0, d(g'_j \cdot v_0, v_0)}$.

Then $\{g'_j g_i H \mid g_i \in I, g'_j \in J\} = \bigsqcup_{v \in W} S_v$, where $S_v = \{g \in G \mid g \cdot v_0 = v\}$ (as defined in proposition 5.3.1).

**Proof.** ($\{g'_j g_i H \mid g_i \in I, g'_j \in J\} \subseteq \bigsqcup_{v \in W} S_v$) Since $G$ acts on the tree we know that for any $h \in H$,

$$d(g'_j g_i h \cdot v_0, g'_j \cdot v_0) = d(g'_j g_i \cdot v_0, g'_j \cdot v_0) = d(g_j \cdot v_0, v_0)$$

So $g'_j g_i h \cdot v_0 \in W$, so $g'_j g_i H \subseteq \bigsqcup_{v \in W} S_v$.

($\bigsqcup_{v \in W} S_v \subseteq \{g'_j g_i H \mid g_i \in I, g'_j \in J\}$) If $f \in \bigsqcup_{v \in W} S_v$ then $d(f \cdot v_0, g_i \cdot v_0) = d(g'_j \cdot v_0, v_0)$. So $d(g'_j^{-1} f \cdot v_0, v_0) = d(g'_j \cdot v_0, v_0)$. 

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Then since $H$ acts transitively on the sphere of vertices at distance $d(g'_1\cdot v_0, v_0)$ from $v_0$ there exists $h \in H$ such that $hg'_1^{-1}f \cdot v_0 = g'_1\cdot v_0$, so $g'_1^{-1}hg'_1^{-1}f \cdot v_0 = v_0$. So $g'_1^{-1}hg'_1^{-1}f \in H$, so $f \in g'_1hg'_1H$. But $hg'_1H = g_1H$ for some $g_1 \in J$ since $Hg_1H = \bigcup_{g_j \in J}g_jH$. So $f \in g'_1g_1H$. Then $f \in \{g'_1g_1H \mid g_1 \in J\}$. 

Then under the same assumptions as the previous proposition we have the following corollary,

**Corollary 5.3.5.** As multisets, $\bigcup_{g_i \in I} \bigcup_{v \in W_{g_i\cdot v_0, d(g_i\cdot v_0, v_0)}} \{S_v\} = \{g, g_jH \mid g_i \in I, g_j \in J\}$

**Proof.** We know that for any $g_i' \in I \{g'_1g_jH \mid g_j \in J\} = \bigcup_{v \in W_{g'_1\cdot v_0, d(g'_1\cdot v_0, v_0)}} \{S_v\}$, so taking the multiset union over $I$ we have the above corollary.

**Corollary 5.3.6.** Let $u \in V(T)$. Then the multiplicity of $S_u$ in the multiset

$$\{g, g_jH \mid g_i \in I, g_j \in J\}$$

does not depend on the group $G$.

**Proof.** Choose some $g'_i \in I$, $g'_j \in J$ and let $m = d(g'_i \cdot v_0, v_0)$, $n = d(g'_j \cdot v_0, v_0)$.

From corollary 5.3.3 we know that $\bigcup_{g_i \in I} \bigcup_{v \in W_{g_i\cdot v_0, d(g_i\cdot v_0, v_0)}} \{S_v\} = \{g, g_jH \mid g_i \in I, g_j \in J\}$.

From proposition 5.3.1 we know that $S_{v_1} = S_{v_2} \iff v_1 = v_2$ so the multiplicity of $S_u$ is exactly the multiplicity of $u$ in the multiset union $\bigcup_{g_i \in I}\{v \in W_{g_i\cdot v_0, d(g_i\cdot v_0, v_0)}\}$.

But since $H$ acts transitively on spheres and each left coset corresponds to a distinct vertex we know that $\bigcup_{g_i \in I}\{v \in W_{g_i\cdot v_0, d(g_i\cdot v_0, v_0)}\} = \bigcup_{u \in V(T), d(u, v_0) = m}\{v \in W_{u, n}\}$ for any $g_i$.

**Corollary 5.3.7.**

- Let $v_0 \in V(T)$ be a distinguished vertex.
- Let $G$, $G'$ be groups acting transitively on $T$ such that the stabilizer of $v_0$ in $G$ and $G'$, which we will label $H$ and $H'$, acts transitively on each sphere of vertices of some fixed distance form $v_0$.

Then the Hecke algebras $C[G, H]$ and $C[G', H']$ are isomorphic.

**Proof.** From proposition 5.3.2 we know that each double coset can be associated to a sphere of vertices at a fixed distance from $v_0$.

Label the double coset associated to vertices at distance $i$ from $v_0$ in $C[G, H]$, $T_i$ and label the same double coset in $C[G', H']$, $U_i$.

Then we claim that map $\phi : C[G, H] \to C[G', H']$, $\phi(T_i) = U_i$ is an algebra isomorphism. We know that the map is a vector space isomorphism so we need only check that it preserves multiplication.

Let $T_m, T_n \in C[G, H], U_m, U_n \in C[G', H']$.

Then choose some $g_m, g_n \in G$, $g'_m, g'_n \in G'$ such that $T_m = Hg_mH$, $T_n = Hg_nH$, $U_m = H'g'_mH'$, $U_n = H'g'_nH'$.

Also choose a sets of representatives, $I$, $J$, $I'$, $J'$ for the right cosets in $T_m, T_n, U_m, U_n$ respectively. So for example, $Hg_mH = \bigcup_{g_i \in I}g_iH$.

Recalling definition 3.1.15 we have that:

$$T_m \cdot T_n = \sum_{k \in \mathbb{Z}_{\geq 0}} \alpha(m, n; k)T_k$$

where $\alpha(m, n; k) := \#(\{(g_i, g_j) \mid g_i \in I, g_j \in J, g_ig_j \in g_kH\})$.

And similarly,

$$U_m \cdot U_n = \sum_{k \in \mathbb{Z}_{\geq 0}} \alpha'(m, n; k)U_k$$

where $\alpha'(m, n; k) := \#(\{(g'_i, g'_j) \mid g'_i \in I', g'_j \in J', g'_ig'_j \in g'_kH'\})$. 

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But $\#\{(g_i, g_j) \mid g_i \in I, g_j \in J, g_i g_j \in g_k H\}$ is exactly the multiplicity of $g_k H$ in the multiset $\{g_i g_j H\}$. Since $g_i g_j \in g_k H \iff g_j g_i H = g_k H$.

From corollary [5.3.6] we know that this does not depend on $G$, and similarly for $\alpha'(m, n; k)$. So we then have that $\alpha(m, n; k) = \alpha'(m, n; k)$ for all $m, n, k \in \mathbb{Z}_{\geq 0}$.

So $\phi$ is an isomorphism.

To write down the full multiplication table of $C[PGL_2(Q_p), PGL_2(Z_p)]$ we now need only find the $\alpha(m, n; k)$.

From corollary [5.3.6] and corollary [5.3.7] we know that to find $\alpha(m, n; k)$ we need to find the multiplicity of $g_k \cdot v_0$ in the multiset $\bigcup_{w \in V(T), d(w, v_0) = m} \{v \in W_{w,n}\}$ for $m, n \in \mathbb{Z}_{\geq 0}$.

This is the aim of the next section.

5.3.2 Calculating multiplication

To calculate the multiplicity of vertices in $\bigcup_{w \in V(T), d(w, v_0) = m} \{v \in W_{w,n}\}$ we associate $T$ with the Cayley graph of the free product of the group $G$ with presentation $\langle a_i \mid a_i^2 = id \rangle$, the free product of $q + 1$ copies of $\mathbb{Z}/2\mathbb{Z}$.

That is, we label $v_0$, id. Then we label the vertices of $V(T)$ inducting on the sphere of vertices at distance $n$ from $v_0$ as follows. For a vertex $v$ with label $a_{i_1} a_{i_2} \ldots a_{i_n}$ we label the $q$ vertices without labels adjacent to $v$ with $a_{i_1} a_{i_2} \ldots a_{i_n} b$ where $b$ varies over the $q$ generators of $G$ which are not $a_{i_n}$.

Let $\gamma_v$ be the label of a vertex $v \in V(T)$.

Then to find the labels for each element of the multiset $\bigcup_{w \in V(T), d(w, v_0) = m} \{v \in W_{w,n}\}$, we can consider the multiset $\{\gamma_w \gamma_v \in G \mid w \in V(T), d(w, v_0) = m, d(v, v_0) = n\}$ where $\gamma_w \gamma_v$ corresponds to the multiplication of $\gamma_w$ and $\gamma_v$ in $G$.

This is because the labels of the vertices $v$ at distance $n$ from $w$ (i.e. labels of $v \in W_{w,n}$) can be found by multiplying the label of $w$ with the labels of vertices at distance $n$ from $v_0$ since $v_0$ is labelled with id and the labels of vertices corresponds to a Cayley graph.

Figure 5.3: The ball of radius 3 around the vertex labelled $a_0$

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Proposition 5.3.8.  

- \( \alpha(m, n, m + n) = 1 \)
- \( \alpha(m, n, m + n - 2l) = (q - 1)^{l-1} \) if \( 0 < l < \min(m, n) \)
- \( \alpha(m, n, m - 2l) = (q + \delta_{m,n})^{l-1} \) for \( l = \min(m, n) \)
- \( \alpha(m, n, k) = 0 \) otherwise

Proof. To find \( \alpha(m, n, k) \) consider the vertex \( g_k \cdot v_0 \) at distance \( k \) from \( v_0 \) and consider its label. Its label will be a reduced word of length \( k \) in \( G \), call this element \( \omega_k \).

We need to find its multiplicity in \( \{ \gamma_w \gamma_v \mid w \in V(T), d(w, v_0) = m, d(v, v_0) = n \} \).

- If \( k = m + n \) then \( \gamma_u \) must be the first \( m \) letters of \( \omega_k \) and \( \gamma_v \) must consist of the last \( n \) letters of \( \omega_k \).
  
  So the multiplicity of \( \omega_k \) in \( \{ \gamma_w \gamma_v \mid w \in V(T), d(w, v_0) = m, d(v, v_0) = n \} \) is 1.

- If \( 0 < l < \min(m, n) \), for \( \gamma_m \gamma_n \) to be a reduced word of length \( m + n - 2l \) the last \( l \) letters of \( \gamma_m \) must be the inverse of the first \( l \) letters of \( \gamma_n \).

  Then the first \( m - l \) letters of \( \gamma_m \) must be the first \( m - l \) letters of \( \omega_{m+n-2l} \) and the last \( n - l \) letters of \( \gamma_n \) must be the last \( n - l \) letters of \( \omega_{m+n-2l} \).

  It will be shown that there are exactly \( (q - 1)^{l-1} \) choices for the last \( l \) letters of \( \gamma_m \).

  First note that the \( m - l + 1 \)th letter of \( \gamma_m \) has \( p - 1 \) choices.

  This is because it cannot be the \( m - l \)th letter of \( \gamma_m \) as \( \gamma_m \) is reduced and it also cannot be the \( l + 1 \)th letter of \( \gamma_n \) since \( \gamma_n \) is reduced and the \( m - l + 1 \)th letter of \( \gamma_m \) is the \( l \)th letter of \( \gamma_n \).

  Note that the \( m - l \)th letter of \( \gamma_m \) and the \( l + 1 \)th letter of \( \gamma_n \) are different since \( \omega_{m+n-2l} \) is a reduced word of length \( m + n - 2l \).

  For every letter of \( \gamma_m \) after the \( m - l + 1 \)th letter there are exactly \( q \) choices; each letter cannot be the letter before it since \( \gamma_m \) is reduced.

  So there are \( (q - 1)^{l-1} \) choices for the last \( l \) letters of \( \gamma_m \). Since the first \( l \) letters of \( \gamma_n \) are the inverse of the last \( l \) letters of \( \gamma_m \) we have that

  \[ \alpha_{m+n-2l} = (q - 1)^{l-1} \] for \( 0 < l < \min(m, n) \).

- For \( l = \min(m, n) \), assume first that \( m = \min(m, n) \). Then if \( m \neq n \) we have exactly \( p^l \) choices for the word \( \gamma_m \), the first letter of \( \gamma_m \) must not be the \( l + 1 \)th letter of \( \gamma_n \) and every letter afterwards cannot be the letter previous.

  If \( m = n \) then there are instead \( (q + 1)^{l-1} \) choices of letters for the word \( \gamma_m \), the first letter of \( \gamma_m \) can be any letter, and every letter afterward cannot be previous letter.

  So \( \alpha_{m+n-\min(m,n)} = (q + \delta_{m,n})^{l-1} \) where \( \delta \) is the Kronecker delta.

\[ \Box \]

Note that the above calculation did not depend on the reduced word of length \( k \) that \( \omega_k \) was. Also the calculations are the same after swapping \( m \) and \( n \) so the algebra is commutative.

Corollary 5.3.9. So, assuming that \( m \geq n \) so that \( \min(m,n) = n \), the full multiplication table is

\[ T_m \cdot T_n = T_n \cdot T_m = T_{m+n} + (p + \delta_{m,n})p^{n-1}T_{m-n} + (p - 1) \sum_{l=0}^{n} p^{l-1}T_{m+n-2l} \]

Below is a visualisation of \( T_1 \cdot T_2 \).
CHAPTER 5. THE BRUHAT-TITS TREE OF $\text{SL}_2(\mathbb{Q}_p)$

Figure 5.4: $T_1, T_2$ (striped)

Figure 5.5: $T_1 \cdot T_2$

Figure 5.6: $T_1 \cdot T_2 = 2T_1 + T_3$
5.3.3 Multiplication tables

Proposition 5.3.10 (The full multiplication table of \(\mathbb{C}[\text{PGL}(Q_p), \text{PGL}(\mathbb{Z}_p)]\)). For \(i \in \mathbb{Z}_{\geq 0}\) let 
\[ M_i = \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \in \text{GL}_2(Q_p). \] Let \( \tilde{M}_i \in \text{PGL}_2(Q_p) \) be the equivalence class containing \( M_i \).

Let \( T_i = \text{PGL}_2(Z_p) \tilde{M}_i \text{PGL}_2(Z_p) \).

Then the Hecke algebra \( \mathbb{C}[\text{PGL}_2(Q_p), \text{PGL}_2(Z_p)] \) is the \( \mathbb{C} \) vector space with basis \( \{ T_i \mid i \in \mathbb{Z}_{\geq 0} \} \) and, assuming \( m \geq n \), the full multiplication table is given by

\[
T_m \cdot T_n = T_{m+n} = T_{m+n} + (p + \delta_{m,n})p^{n-1}T_{m-n} + (p-1) \sum_{l=0}^{n} p^{l-1}T_{m+n-2l}
\]

Proof. Let \([I] \in \text{PGL}_2(Q_p)\) be the identity element and let \( v_0 = [I] \). Then from propositions 5.2.2 and 5.2.3 we know that \( \text{PGL}_2(Q_p) \) acts transitively on the tree and the stabilizer of \( v_0 \) is \( \text{PGL}_2(Z_p) \) which acts transitively on spheres around \( v_0 \).

From the comment after proposition 5.1.5 we know that \( M_i \) sends the vertex \([I]\) to a vertex at distance \( i \) from \([I]\), so from 5.3.2 and the fact that double cosets partition a group we know that the \( \{ T_i \} \) form a basis for \( \mathbb{C}[\text{PGL}_2(Q_p), \text{PGL}_2(Z_p)] \).

Then the multiplication table follows follows from subsection 5.3.1 and corollary 5.3.9.

Proposition 5.3.11 (The full multiplication table of \(\mathbb{C}[\text{PSL}(Q_p), \text{PSL}(\mathbb{Z}_p)]\)). For \( i \in \mathbb{Z}_{\geq 0} \) let 
\[ N_{2i} = \begin{pmatrix} p^{-i} & 0 \\ 0 & p^{i} \end{pmatrix} \in \text{SL}_2(Q_p). \] Let \( \tilde{N}_i \in \text{PGL}_2(Q_p) \) be the equivalence class containing \( M_i \).

Let \( S_{2i} = \text{PGL}_2(Z_p) \tilde{N}_{2i} \text{PGL}_2(Z_p) \).

Then the Hecke algebra \( \mathbb{C}[\text{PSL}_2(Q_p), \text{PSL}_2(Z_p)] \) is the \( \mathbb{C} \) vector space with basis \( \{ S_{2i} \mid i \in \mathbb{Z}_{\geq 0} \} \) and, assuming \( m \geq n \), the full multiplication table is given by

\[
S_{2m} \cdot S_{2n} = S_{2n} \cdot S_{2m} = S_{2m+2n} + (p + \delta_{2m,2n})p^{2n-1}S_{2m-2n} + (p-1) \sum_{l=0}^{2n} p^{l-1}S_{2m+2n-2l}
\]

Proof. From the comment after proposition 5.1.5 we know that \( N_{2i} \) sends the vertex \([I]\) to a vertex at distance \( 2i \) from \([I]\).

From propositions 5.2.4 and 5.2.5 we know that \( \text{SL}_2(Z_p) \) acts transitively on spheres around \( v_0 \) and that the action of \( \text{SL}_2(Q_p) \) can only send \( v_0 \) to a vertex at even distance from \( v_0 \).

So then there exists a map from \( \mathbb{C}[\text{PSL}_2(Q_p), \text{PSL}_2(Z_p)] \) to the subalgebra of \( \mathbb{C}[\text{PGL}_2(Q_p), \text{PGL}_2(Z_p)] \) spanned by \( \{ T_{2i} \mid i \in \mathbb{Z}_{\geq 0} \} \) from which the results follow.
Chapter 6

\[ \text{SL}_2(\mathbb{F}_p((t))) \]

\( \text{SL}_2(\mathbb{F}_p((t))) \) is a Tits system or \((B, N)\)-pair.


This chapter gives the decomposition of \( \text{SL}_2(\mathbb{F}_p((t))) \) as an amalgamated free product mentioned in [Ser80] and then gives its associated Bass-Serre tree as defined in chapter 2.

It then proves the results on edge stabilizers from chapter 4 with a slightly different method.
6.1 The tree

**Definition 6.1.1.** The group $\text{SL}_2(\mathbb{F}_p((t)))$ is the matrix group with elements

\[
\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{F}_p((t)), \ |ad - bc| = 1 \right\}
\]

We will label the group $\text{SL}_2(\mathbb{F}_p((t)))$, $G$.

The group $\text{SL}_2(\mathbb{F}_p((t)))$ has a decomposition as follows (compare [Ser80]).

Let (compare [Ser80] §1.7 example 1)

\[
P_1 = \text{SL}_2(\mathbb{F}_p[[t]]) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{F}_p[[t]], \ ad - bc = 1 \right\}
\]

\[
P_2 = \left\{ \begin{bmatrix} a & t^{-1}b \\ tc & d \end{bmatrix} \mid a, b, c, d \in \mathbb{F}_p[[t]], \ ad - bc = 1 \right\}
\]

\[
B = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{F}_p[[t]], \ ad - bc = 1, \ c \equiv 0 \pmod{t} \right\}
\]

Where if we were to write $c = \sum_{i=0}^{\infty} a_i t^i$ then $c \equiv 0 \pmod{t} \iff a_0 = 0$.

Then

**Proposition 6.1.2.** $G = P_1 \ast_B P_2$

*Proof.* See [Ser80] §1.7 Theorem 7

---

![Figure 6.1: The graph of groups associated to $G = P_1 \ast_B P_2$](image)

**Proposition 6.1.3.** Let $s_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Let $s_2 = \begin{bmatrix} 0 & -t^{-1} \\ t & 0 \end{bmatrix}$.

Let for $i \in \{0, 1, \ldots, p-1\}$ let $a_i = \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}$ and let $b_i = \begin{bmatrix} 1 & 0 \\ it & 1 \end{bmatrix}$.

Then

\[
P_1 = B \bigsqcup_{i=0}^{p-1} a_i s_1 B
\]

\[
P_2 = B \bigsqcup_{i=1}^{p-1} b_i s_2 B
\]

In particular, $B$ has index $p + 1$ in $P_1$ and $P_2$

*Proof.* (P1 union is disjoint) Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in B$. So $c \equiv 0 \pmod{t}$. Then

\[
a_i s_1 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} i & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ia - c & -ib - d \\ a & b \end{bmatrix}
\]

Note that $ia - c \equiv ia \pmod{t}$ so if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in a_i s_1 B$ then $\alpha \gamma^{-1} \equiv i \pmod{t}$, so the $a_i s_1 B$ are disjoint from each other since $\alpha \gamma^{-1} (\pmod{t})$ is an invariant.

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B is disjoint from each $a_is_1B$ since $ad - bc = 1 \implies a \not\equiv 0 \pmod{t}$ so for $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in a_is_1B$, $\gamma \neq 0 \pmod{t}$.

- (Union covers $P_1$) Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in P_1$. If $c \equiv 0 \pmod{t}$ then $M \in B$.

Otherwise, let $i = ac^{-1} \pmod{t}$. Let $\alpha = c$, $\beta = d$, $\gamma = ic - a$ and let $\delta = id - b$.

Then $\alpha \delta - \beta \gamma = cid - bc + ad - dic = ad - bc = 1 \implies \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in B$.

Then $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} i & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = a_is_1 \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$.

So $M \in B \bigcup_{i=1}^p a_is_1B$

- (Union covers $P_2$) Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in P_2$. If $b \equiv 0 \pmod{1}$ then $M \in B$.

Otherwise, let $i = -(tb)^{-1}d \pmod{t}$. Let $\alpha = t^{-1}(c - ita)$, $\beta = t^{-1}(d - itb)$, $\gamma = -ta$ and let $\delta = -tb$.

Then $\alpha \delta - \beta \gamma = bc + itab + ad - aitb = ad - bc = 1 \implies \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in B$.

Then $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & -t^{-1} \\ t & -i \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = b_is_2 \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$.

So $M \in B \bigcup_{i=1}^p b_is_2B$}

\end{proof}

**Corollary 6.1.4.** The Bass-Serre tree associated to $P_1 *_B P_2$ is infinite and homogeneous with degree $p + 1$.

\begin{proof}
 We know that the Bass-Serre tree associated to $P_1 *_B P_2$ is an infinite semi-homogeneous tree with degrees the index of $B$ in $P_1$ and $P_2$.

From proposition 6.1.3 both these indexes are $p + 1$ so the Bass-Serre tree is infinite homogeneous with degree $p + 1$.
\end{proof}

So we have a homogeneous degree $p + 1$ Bass-Serre tree associated to $P_1 *_B P_2$.

Call the Bass-Serre tree associated to $P_1 *_B P_2$, $T$. 

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6.2 Action on the tree

Proposition 6.2.1. \(G\) acts transitively on edges of \(T\) without inversions.

Proof. We know that the a group \(G\) acts without inversions on its Bass-Serre tree.

We prove that \(G\) acts transitively on edges by showing that for every \(e \in E(T)\) there exists \(g \in G\) such that \(g \cdot e_0 = e\).

Let \(e = \{v_1, v_2\} \in E(T)\), then we know that one vertex is labelled \(w_1P_1\) and the other is labelled with a coset of \(w_2P_1\) for some \(w_1, w_2\), words in coset representatives of \(B\) in \(P_1\) and \(P_2\). Assume that \(v_1\) is labelled \(w_1P_1\) and \(v_2\) is labelled \(w_2P_2\).

Additionally, we know that of \(w_1\) and \(w_2\) one word is the the other word appended at the end with one extra letter.

If \(w_1\) is longer than \(w_2\) then the extra letter is some coset representative, for example \(b_1\), of \(B\) in \(P_2\). So we can write \(w_1 = w_2q\) for some \(q \in P_2\).

So if \(w_1\) is longer than \(w_2\) consider \(w_1 \cdot e_0\).

Then for the vertex \(u_1 \in e_0\) labelled \(P_1\), \(w_1 \cdot u_1 = v_1\). For the vertex \(u_2 \in e_0\) labelled \(P_2\), \(w_1 \cdot u_2 = w_2q \cdot u_2 = w_2 \cdot u_2\) since \(q \in P_2\) and \(u_2\) is labelled \(P_2\). Then \(w_2 \cdot u_2 = v_2\).

And similarly if \(w_2\) is longer than \(w_1\).

Proposition 6.2.2. For any vertex \(v \in V(T)\) the stabilizer of \(v\) in \(G\) acts transitively on the vertices at distance one from \(v\).

Proof. Let \(e_0 = \{u_1, u_2\}\) and let \(u_1\) be labelled \(P_1\), \(u_2\) be labelled \(P_2\).

We prove that the vertex stabilizers of the \(v_1\) and \(v_2\) act transitively on the vertices at distance from \(v_1\) and \(v_2\) respectively.

Then since for every \(v \in V(T)\) there exists \(g \in G\) such that \(g \cdot v \in e_0\), every vertex stabilizer is a conjugate of one of the above two vertex stabilizers (and \(T\) is homogeneous) and so also acts transitively on the respective vertices at distance 1.

From proposition 6.1.3 we know that vertices around \(P_1\) are labelled by \(P_2\) and \(a_i s_1 P_2\) for \(i \in \{0, 1, \ldots, p-1\}\). Similarly the vertices around \(P_2\) are labelled \(P_1\) and \(b_i s_2 P_1\) for \(i \in \{0, 1, \ldots, p-1\}\).
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But $a_i = \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix} \in P_1$ for all $i$ and $s_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in P_1$. So the stabilizer of $v_1$, $P_1$ acts transitively on the sphere of radius one around $v_1$.

Similarly $b_i \in P_2$ for all $i$ and $s_2 \in P_2$. So $P_2$ acts transitively on the sphere of radius one around $v_2$.

\begin{definition}[edge distance] (edge distance) We define a function $d : E(T) \times E(T) \to \mathbb{Z}_{\geq 0}$ by, given $e$, $e' \in E(T)$,

$$d(e, e') = \begin{cases} 0 & \text{if } e = e' \\ \max\{d(v, v') \mid v \in e, v' \in e'\} - 1 & \text{otherwise} \end{cases}$$

where $d(v, v')$ is the distance function on vertices.

We call this function the edge distance function. It will also be called the distance function and when meaning is clear from context.

\end{definition}

\begin{definition}[sgn] Let $e = \{v_1, v_2\} \in E(T)$.

Let $e' \in E(T)$, then

$$\text{sgn}_e(e') = \begin{cases} v_1 & \text{if for every } v' \in e', d(v', v_1) < d(v', v_2) \\ v_2 & \text{if for every } v' \in e', d(v', v_2) < d(v', v_1) \end{cases}$$

We will call sets of the form $\{e \in E(T) \mid d(e, e_0) = n, \text{sgn}_{e_0}(e) = v\}$ for some $n \in \mathbb{Z}$, $v \in e_0$ half spheres around $e_0$.

\end{definition}

\textbf{Remark}. sgn is not a standard definition on trees, but it will be used in the proofs in the next subsection.

\begin{proposition} $B$ is the stabilizer of some edge $e_0$ in $T$ and acts transitively on half spheres around $e_0$.

\end{proposition}

\begin{proof} From the Bass-Serre Tree construction we know that $B$ is the stabilizer of the edge $\{u_1, u_2\}$ where $u_1$ is labelled $P_1$ and $u_2$ is labelled $P_2$.

We prove that $B$ acts transitively on half spheres by induction on the distance $n$ of half spheres around $e_0$.

For $n = 1$ the half sphere with sgn of $u_1$ consists of edges of the form $\{u_1, u\}$ such that the label of $u$ is of the form $\{a, s_1\}$ for $a \in \{0, \ldots, p - 1\}$.

We know that for each $i$ $a_i = \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix} \in B$ so $a_i a_j^{-1} \in B$ for all $i, j$ so $B$ acts transitively on the $a_i s_1 P_2$, so $B$ acts transitively on the half sphere with sgn of $u_2$ at distance 1 from $e_0$.

Similarly for the half sphere with sgn of $u_1$, the edges are of the form $\{u_1, u\}$ where the label of $u$ is of the form $b_i s_2 P_1$.

Then we know that $b_i = \begin{bmatrix} 1 & 0 \\ it & 1 \end{bmatrix} \in B$, so $b_i b_j^{-1} \in B$ for all $i, j$ so $B$ also acts transitively on this half sphere.

Now assume that $B$ acts transitively on all half spheres less than equal to $n$.

Then we only need to prove that $B$ acts transitively on each set of $p$ edges at distance $n + 1$ from $e_0$ which share a vertex.

This is because given that $B$ acts transitively on the half spheres at distance $n$. So given two edges $e, e'$ in a half sphere at distance $n + 1$ we can find some $b \in B$ such that $b \cdot e$ shares a vertex with $e'$.

So consider a set of $p$ edges at distance $n + 1$ from $e_0$ sharing a vertex. First assume that the shared vertex has a label which is a coset of $P_1$.

Then we know each edge is of the form $\{v, v_i\}$ where $v$ is the shared vertex and each $v_i$ has label $q a_i s_1 P_2$ where $q P_1$ is the label of $v$.

Then we know that $q a_i s_1 a_j^{-1} q^{-1} v_i = v_j$. Then $q a_j s_1 a_i^{-1} q^{-1} = q a_i q^{-1}$, so to show that $B$ acts transitively on the $p$ edges we need to show that $q a_j a_i^{-1} q^{-1} \in B$.

We know that $q = q' b_i s_2$ for some $i$, so $q a_j a_i^{-1} q^{-1} = q' b_i s_2 a_i^{-1} q'$. Thus $B$ acts transitively on the $p$ edges at distance $n + 1$ from $e_0$.

\end{proof}
We also know that if \( M \in B \) is of the form \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\] since \( s_2 \begin{bmatrix} a & b \\ c & d \end{bmatrix} s_2^{-1} = \begin{bmatrix} d & -b t^2 \\ -c & a \end{bmatrix} \), then \( s_2 Ms_2^{-1} \) is a matrix in \( B \) of the form \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

Similarly, if \( M \in B \) is of the form \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\] then \( s_1 Ms_1^{-1} \) is a matrix in \( B \) of the form \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

since \( s_1 \begin{bmatrix} a & b \\ c & d \end{bmatrix} s_1^{-1} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \).

Also, matrices of the form \[
\begin{bmatrix}
1 & * \\
0 & 1
\end{bmatrix}
\] commute with each other as do matrices of the form \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\].

Note that \( a_i a_i^{-1} = \begin{bmatrix} 1 & j - i \\ 0 & 1 \end{bmatrix} \) is an element of \( B \) of the form \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\].

So \( s_2 a_i a_i^{-1} s_2^{-1} \in B \) and is of the form \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]. Then since the matrices commute, \( q^i b_i s_2 a_i a_i^{-1} s_2^{-1} b_i^{-1} q^{-1} = q^i s_2 a_i a_i^{-1} s_2^{-1} q^{-1} \).

But we know \( q^i = q^i a_i s_1 \), and \( s_1 (s_2 a_i a_i^{-1} s_2^{-1}) s_1^{-1} \in B \) and is a matrix of the form \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\], so \( a_i \) commutes with \( s_1 (s_2 a_i a_i^{-1} s_2^{-1}) s_1^{-1} \).

So after iterating the above process we have that \( q a_j s_1 s_1^{-1} a_i^{-1} q^{-1} = s a_j a_i^{-1} s^{-1} \in B \) where \( S \) is an alternating word in \( \{s_1, s_2\} \).

So \( B \) acts transitively on the \( p \) edges so \( B \) acts transitively on half spheres.

\[\Box\]

\section{The algebra for any group acting on a tree}

The action of \( G \) on half spheres will correspond to the action of \( D_\infty \), the infinite dihedral group, on an infinite geodesic so we will wish to index our double cosets by words from \( D_\infty \).

The propositions for the following section all assume the following:

\begin{itemize}
  \item Let \( T \) be an infinite homogeneous tree of degree \( q + 1 \) where \( q \geq 2 \).
  \item Let \( e_0 \in E(T) \) be a distinguished edge.
  \item Let \( G \) be a group acting transitively on the edge set of \( T \) without inversions such that the stabilizer of \( e_0 \) in \( G \), which we will label \( B \), acts transitively on doubly infinitely geodesics through \( e_0 \) and the stabilizer of each vertex \( v \) in \( G \) acts transitively on the sphere of vertices at distance 1 from \( v \).
\end{itemize}

The requirements for \( G \) seem to differ from the condition of strongly transitive in chapter \ref{ch:strongly_transitive} but I believe they imply each other.

**Proposition 6.2.6.** Let \( c \in E(T) \). Let \( S_c = \{ g \in G \mid G \cdot e_0 = e \} \), the set of elements in \( G \) which send \( e_0 \) to \( v \).

Then (as sets) \( S_c = gB \) for any \( g \in S_c \).

Moreover if \( e_1, e_2 \in E(T) \), \( S_{e_1} = S_{e_2} \iff e_1 = e_2 \).

**Proof.** Let \( g \in S_c \).

\((S_c \subseteq gB), \) let \( g' \in S_c \), then \( g^{-1} g' \cdot e_0 = g^{-1} \cdot e = e_0 \). So \( g^{-1} g' \in B \), so \( g' \in gB \).

\((gB \subseteq S_c), \) let \( b \in B \). Then \( g b \cdot e_0 = g \cdot e_0 = e \). So \( g b \in S_c \).

For the final statement, if \( e_1 \neq e_2 \) then for any \( g_1 \in S_{e_1}, g_2 \in S_{e_2} \), \( g_1 \cdot e_0 = e_1 \neq e_2 = g_2 \cdot e_0 \) so \( S_{e_1} \neq S_{e_2} \). On the other hand if \( S_{e_1} \neq S_{e_2} \) then there exists \( g \in S_{e_1} \) such that \( e_1 = g \cdot e_0 \neq e_2 \). So \( e_1 \neq e_2 \).

\[\Box\]
**Proposition 6.2.7.** Let \( g \in G \). Let \( n = d(g \cdot e_0, e_0) \).

Then we can write \( BG = \bigcup_{e \in E(T)} d(v, e_0) = n, \text{sgn}_{e_0}(e) = \text{sgn}_{e_0}(g \cdot e) \), \( S_e \).

**Proof.** \( BG \subseteq \bigcup_{e \in E(T)} d(v, e_0) = n, \text{sgn}_{e_0}(e) = \text{sgn}_{e_0}(g \cdot e) \). Given \( b_1, b_2 \in B \), \( b_1gb_2 \cdot e_0 = b_1g \cdot e_0 \). So \( d(b_1g \cdot e_0, c_0) = n \), since \( b_1 \) fixes \( c_0 \). So \( b_1gb_2 \in S_e \) for some \( e \in E(T) \), \( d(e, c_0) = n, \text{sgn}_{e_0}(e) = \text{sgn}_{e_0}(g \cdot e) \) (since \( B \) does not invert edges).

\[
\left( \bigcup_{e \in E(T)} d(v, e_0) = n, \text{sgn}_{e_0}(e) = \text{sgn}_{e_0}(g \cdot e) \right) \subseteq BG
\]

then \( d(g' \cdot e_0, c_0) = n \). Since \( B \) acts transitively on half spheres around \( e_0 \), there exists \( b \in B \) such that \( bg' \cdot e_0 = g \cdot e_0 \).

Then \( g^{-1}bg' = e_0 = 0 \) so \( g^{-1}bg' \in B \). So \( bg' \in gB \) and \( g' \in b^{-1}gB \). (Disjoint union) If for \( e_1, e_2 \in E(T) \) and \( g \in S_{e_1} \cap S_{e_2} \) then \( g \cdot e_0 = e_1 = e_2 \). So \( S_{e_1} = S_{e_2} \) since \( S_{e_1} \) and \( S_{e_2} \) are left cosets by proposition 6.2.6.

**Corollary 6.2.8.** Every double coset is the disjoint union of a finite number of left cosets.

**Proof.** From propositions 6.2.6 and 6.2.7 we know that \( BgB \) is the union of at most \(|\{ e \in E(T) \mid d(e, c_0) = d(g \cdot e_0, e_0), \text{sgn}_{e_0}(e) = \text{sgn}_{e_0}(g \cdot e) \}|\) left cosets.

Since every vertex has finite degree in \( T \),

\(|\{ e \in E(T) \mid d(e, c_0) = d(g \cdot e_0, e_0), \text{sgn}_{e_0}(e) = \text{sgn}_{e_0}(g \cdot e) \}|\)

is finite.

**Proposition 6.2.9.** Let \( g, g' \in G \). Write \( BgB \) and \( Bg'B \) as the disjoint union of \( BgB = \bigcup_{g \in T} gB \) for some indexing sets \( I, J \subseteq G \).

For \( e, e' \in E(T) \), \( n \in \mathbb{Z}_{>0} \), let \( W_{e,n,\text{sgn}_{e}(e')} = \{ f \in E(T) \mid d(f, e) = n, \text{sgn}_{e}(f) = \text{sgn}_{e}(e') \} \).

Fix some \( g_i \in I, g'_i \in J \) and let \( W = W_{g_i,0,d(g_i \cdot e_0, e_0), \text{sgn}_{g_i \cdot e_0}(g'_i \cdot e_0)} \).

Then \( \{ g'g_iB \mid g_i \in J \} = \bigcup_{e \in W} S_e \), where \( S_e = \{ g \in G \mid g \cdot e_0 = e \} \) (as defined in proposition 6.2.6).

**Proof.** \( \{ g'g_iB \mid g_i \in J \} \subseteq \bigcup_{e \in W} S_e \). Since \( G \) acts on the tree we know that for any \( b \in B \),

\[
d(g'g_iB \cdot e_0, g'_i \cdot e_0) = d(g'g_i \cdot e_0, e_0) = d(g'_i \cdot e_0, e_0)
\]

And \( \text{sgn}_{e_0}(g'g_i \cdot e_0) = \text{sgn}_{e_0}(g'_i \cdot e_0) \) since \( \text{sgn}_{e_0}(g \cdot e_0) = \text{sgn}_{e_0}(g'_i \cdot e_0) \).

So \( g'g_iB \in W \), so \( g'g_iB \in \bigcup_{e \in W} S_e \). (since by proposition 6.2.6 sets of the form \( S_e \) are left cosets and left cosets partition \( G \).)

\[
\left( \bigcup_{e \in W} S_e \right) \subseteq \{ g'g_iB \mid g_i \in J \}
\]

If \( a \cdot e_0 = g \cdot e_0 \) then \( d(a \cdot e_0, g \cdot e_0) = d(g' \cdot e_0, e_0) \). So \( d(g' \cdot e_0, e_0) \).

Additionally, \( \text{sgn}_{e_0}(a \cdot e_0) = \text{sgn}_{e_0}(g'_i \cdot e_0) \) so \( \text{sgn}_{e_0}(g'_{i-1} \cdot e_0) \).

Then since \( N \) acts transitively on the half sphere of edges at distance \( d(g' \cdot e_0, e_0) \) from \( e_0 \) there exists \( b \in B \) such that \( bg'_{i-1} \cdot e_0 = g'_i \cdot e_0 \), so \( g'_{i-1}b = e_0. \)

But \( h'g_iB = g_iB \) for some \( g_i \in J \) since \( Hg_iB = \bigcup_{g_i \in J} g_iB \). So \( g'g_iB \in \{ g'g_iB \mid g_i \in J \} \).

Then under the same assumptions as the previous proposition we have the following corollary,

**Corollary 6.2.10.** As multisets,

\[
\bigcup_{g_i \in I \in W_{g_i,0,d(g_i \cdot e_0, e_0), \text{sgn}_{g_i \cdot e_0}(g'_i \cdot e_0)}} S_e = \{ g, g_iH \mid g_i \in I, g_j \in J \}
\]

**Proof.** We know that for any \( g' \in I \) \( \{ g'g_iH \mid g_i \in J \} = \bigcup_{e \in W_{g' \cdot e_0,d(g' \cdot e_0, e_0), \text{sgn}_{g' \cdot e_0}(g'_i \cdot e_0)}} S_e \), so taking the multiset union over \( I \) we have the above corollary.

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Proposition 6.2.11. Let $G$, $G'$ be groups acting as defined above and let $g_i, g_j \in G$, $g'_i, g'_j \in G'$ such that $g_i \cdot e_0 = g'_i \cdot e_0$ and $g_j \cdot e_0 = g'_j \cdot e_0$.

Then $\text{sgn}_{g_i, e_0}(g_ig_j \cdot e_0) = \text{sgn}_{g'_i, e_0}(g'_ig'_j \cdot e_0)$.

Proof. We will show that since $G$ acts without inversions, there exists only one value which $\text{sgn}_{g_i, e_0}(g_ig_j \cdot e_0)$ can take, and so since $G'$ also acts without inversions $\text{sgn}_{g_i, e_0}(g_ig_j \cdot e_0) = \text{sgn}_{g'_i, e_0}(g'_ig'_j \cdot e_0)$.

There exists an action of $G$ without inversion which sends $e_0$ to $g_i \cdot e_0$ by rotating around vertices. That is, let $(v_0, v_1, \ldots, v_n)$ be the longest path such that $v_i \in e_0, v_n \in g_i \cdot e_0$. Then let $h_i$ be the element in the stabilizer of $v_i$, which sends $v_i$ to $v_{i+1}$ for $1 \leq i \leq n - 1$. Then $h_{n-1}h_{n-2} \cdots h_1 \cdot e_0 = g_i \cdot e_0$.

Note that these rotations act nicely with $\text{sgn}$ that is, if $h$ stabilizes a vertex $v$ in an edge $e$ then $h \cdot \text{sgn}_e(f) = \text{sgn}_{h \cdot e}(h \cdot f)$.

We must have that $\text{sgn}_{g_i, e_0}(g_ig_j \cdot e_0) = \text{sgn}_{g_i, e_0}(h_{n-1}h_{n-2} \cdots h_1 g_i \cdot e_0)$ as otherwise we would have an inversion of the edge $g_ig_j \cdot e_0$.

But we know that $\text{sgn}_{g_i, e_0}(h_{n-1}h_{n-2} \cdots h_1 g_i \cdot e_0) = h_{n-1}h_{n-2} \cdots h_1 \cdot \text{sgn}_{e_0}(g_j \cdot e_0)$.

So $\text{sgn}_{g_i, e_0}(g_ig_j \cdot e_0) = h_{n-1}h_{n-2} \cdots h_1 \cdot \text{sgn}_{e_0}(g_j \cdot e_0)$. But $h_{n-1}h_{n-2} \cdots h_1 \cdot \text{sgn}_{e_0}(g_j \cdot e_0)$ is uniquely determined by the path $(v_0, v_1, \ldots, v_n)$ since $\text{sgn}_{e_0}(g_j \cdot e_0)$ is a vertex in $e_0$. So taking the same process for $G'$ we have that $h'_{n-1}h'_{n-2} \cdots h'_1 \cdot \text{sgn}_{e_0}(g'_j \cdot e_0) = h_{n-1}h_{n-2} \cdots h_1 \cdot \text{sgn}_{e_0}(g_j \cdot e_0)$.

So $\text{sgn}_{g_i, e_0}(g_ig_j \cdot e_0) = \text{sgn}_{g'_i, e_0}(g'_ig'_j \cdot e_0)$.

Note that along a doubly infinite geodesic, rotating around the vertices in the geodesic corresponds to the action of $D_\infty$ on the geodesic.

Corollary 6.2.12. Let $f \in E(T)$. Then the multiplicity of $S_f$ in the multiset

$$\{g_ig_jB \mid g_i \in I, g_j \in J\}$$

does not depend on $G$.

Proof. Choose some $g'_i \in I$, $g'_j \in J$ and let $m = d(g'_i \cdot e_0, e_0)$, $n = d(g'_j \cdot e_0, e_0)$.

From corollary 6.2.10 we know that $\bigcup_{g_i \in I} \bigcup_{e \in W_{g_i, e_0, d(g_i \cdot e_0, e_0)} \cdot g_{\text{sgn}_{g_i, e_0}(g_i \cdot e_0)}} \{S_e \mid g_i \in I, g_j \in J\}$.

From proposition 6.2.6 we know that $S_{e_1} = S_{e_2} \iff e_1 = e_2$ so the multiplicity of $S_f$ is exactly the multiplicity of $f$ in the multiset union $\bigcup_{g_i \in I} \{e \in W_{g_i, e_0, d(g_i \cdot e_0, e_0)} \cdot g_{\text{sgn}_{g_i, e_0}(g_i \cdot e_0)} \mid e \in E(T), d(w, e_0) = m, g_{\text{sgn}_{g_i, e_0}(g_i \cdot e_0)}(f) \in W_{e, n, g_{\text{sgn}_{g_i, e_0}(g_i \cdot e_0)}} \}$ for some fixed $g_i \in I, g_j \in J$.

And since $\text{sgn}_{e}(g_ig_j \cdot e_0)$ does not depend on $G$ by 6.2.11 the multiplicity of $S_f$ also does not depend on $G$.

Corollary 6.2.13. Let $T$ be an infinite homogeneous tree of degree $q + 1$ where $q \geq 2$.

- Let $e_0 \in E(T)$ be a distinguished vertex.

- Let $G$, $G'$ be groups acting transitively on $T$ such that the stabilizer of $e_0$ in $G$ and $G'$, which we will label $B$ and $B'$, acts transitively on each half sphere around $e_0$. And let every stabilizer of a vertex $v$ in $G$ and $G'$ act transitively on the sphere of vertices at distance one from $v$.

Then the Hecke algebras $\mathbb{C}[G, B]$ and $\mathbb{C}[G', B']$ are isomorphic.

Proof. From proposition 6.2.7 we know that each double coset can be associated to half spheres of edges around $e_0$.

We will label the double cosets of $\mathbb{C}[G, B]$ by elements of $\{\Delta_w \mid w \text{ is a word in } \{r, t\}\}$ as follows. Label the double coset corresponding to $e_0$, $\Delta_{d_0}$. For the two half spheres at distance 1 from $e_0$ label one $\Delta_{t}$ and the other $\Delta_r$. 

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Then label iteratively; to label a half sphere at distance $n + 1$ from $e$ the adjacent half sphere at distance $n$ from $e$ will have a label $\Delta_{w'}$. If $s$ is the last letter of $w$ then label the half sphere at distance $n + 1$, $\Delta_{w'}$ otherwise label it $\Delta_{w''}$.

Similarly label the double cosets of $\mathbb{C}[G', B']$ by elements of $\{ \Delta_{w'} \mid w \text{ is a word in } \{r, t\}\}$.

Then we claim that map $\phi : \mathbb{C}[G, B] \to \mathbb{C}[G', B']$, $\phi(\Delta_w) = \Delta'_w$ is an algebra isomorphism.

We know that the map is a vector space isomorphism so we need only check that it preserves multiplication.

Let $\Delta_{w_1}, \Delta_{w_2} \in \mathbb{C}[G, B]$, $\Delta'_{w_1}, \Delta'_{w_2} \in \mathbb{C}[G', B']$.

Then choose some $g_m, g_n \in G$, $g'_m, g'_n \in G'$ such that $\Delta_{w_1} = Bg_mB$, $\Delta_{w_2} = Bg_nB$, $\Delta'_{w_1} = B'g'_mB'$, $\Delta'_{w_2} = B'g'_nB'$.

Also choose a sets of representatives, $I, J, I', J'$ for the left cosets in $\Delta_{w_1}, \Delta_{w_2}, \Delta'_{w_1}, \Delta'_{w_2}$ respectively. So for example, $Bg_mB = \bigsqcup_{i \in I} g_iH$.

Recalling definition 3.1.15 we have that:

$$\Delta_{w_1} \cdot \Delta_{w_2} = \sum_{w \text{ is a word in } \{r, t\}} \alpha(w_1, w_2; w) \Delta_w$$

where $\alpha(w_1, w_2; w) := \#(\{(g_i, g_j) \mid g_i \in I, g_j \in J, g_ig_j \in g_kB\} )$ where $\Delta_w = Bg_kB$.

And similarly,

$$\Delta'_{w_1} \cdot \Delta'_{w_2} = \sum_{w \text{ is a word in } \{r, t\}} \alpha'(w_1, w_2; w) \Delta'_w$$

where $\alpha'(w_1, w_2; w) := \#(\{(g'_i, g'_j) \mid g'_i \in I', g'_j \in J', g'_ig'_j \in g'_kB'\} )$ where $\Delta_w = Bg_kB$.

But $\#(\{(g_i, g_j) \mid g_i \in I, g_j \in J, g_ig_j \in g_kB\} )$ is exactly the multiplicity of $g_kB$ in the multiset $\{g_i,g_j\}$. Since $g_i,g_j \in \{g_kB \iff g_i g_j \in \{g_kB\}$. From corollary 6.2.12 we know that this does not depend on $G$, so we then have that $\alpha(w_1, w_2; w) = \alpha'(w_1, w_2; w)$ for all $w_1, w_2, w$.

So $\phi$ is an isomorphism.

$\square$

To write down the full multiplication table of $\mathbb{C}[G, B]$ we now need only find the $\alpha(w_1, w_2; w)$ for all $w_1, w_2, w$ words in $\{s, t\}$.

From corollary 6.2.12 and corollary 6.2.13 we know that to find $\alpha(w_1, w_2; w)$ we need to find the multiplicity of $g_k^{-1}$ in the multiset $\bigcup_{e \in E(T)} d_{\{w, e\}} = m, sgn_{\{w\}}(e) = sgn_{\{g_k\}}(g, e) \{f \in W_{e, n, sgn_{\{g, g_k^{-1}\}}\{f \in W_{e, n, sgn_{\{g, g_k^{-1}\}}\}}\}$. This is the aim of the next section.
To calculate the multiplicity of vertices in \( \bigcup_{e \in E(T), d(w,e_0) = m, \text{sgn}_e(e) = \text{sgn}_n(g,e)} \{ f \in W_{e,n,\text{sgn}_n(g,e)} \} \) we will label the edges of \( T \) as follows.

First label the half spheres with elements of the set \( \{ \Delta_w \mid w \text{ is a word in } \{ r,t \} \} \).

We label \( e_0, \varnothing \), the empty word. Then we label the edges in the half sphere \( \Delta_r, r_1 \) and \( r_2 \) and label the edges in the half sphere \( \Delta_t, t_1 \) and \( t_2 \).

Then we label iteratively, given an edge \( e \) at length \( n \) from \( e_0 \) with label \( \omega \) where the last letter in the word \( \omega \) is \( r_i \) for some \( i \) we label the \( p \) edges at distance \( n + 1 \) adjacent to \( e \), \( \omega t_i \), \( i \in \{1,2,\ldots,p\} \).

Similarly if the last letter in the word \( \omega \) is \( t_i \) we label the \( p \) edges by \( \omega r_i \), \( i \in \{1,2,\ldots,p\} \).

Then to find the labels for each element of the multiset

\[
\bigcup_{e \in E(T), d(w,e_0) = m, \text{sgn}_e(e) = \text{sgn}_n(g,e)} \{ f \in W_{e,n,\text{sgn}_n(g,e)} \}
\]

we concatenate of the edges in \( Bg_iB \) and \( Bg_jB \) and then act on each of the resulting words using the following rules:

- If in the word we have \( r_i r_j \) with \( i \neq j \) we replace \( r_i r_j \) with \( r_j \).
- If in the word we have \( t_i t_j \) with \( i \neq j \) we replace \( t_i t_j \) with \( t_j \).
- If we have \( r_i r_j \) or \( t_i t_j \) in any of the words we delete both letters.

until none of the above conditions occur in the word.

This gives the appropriate multiset union of half spheres of edges around edges in \( Bg_iB \).

**Proposition 6.2.14.**  
- If the last letter of \( w_1 \) is different to the first letter of \( w_2 \) then we have that \( \alpha(w_1, w_2, w) = 1 \) if \( k = w_1w_2 \) and \( \alpha(w_1, w_2, w) = 0 \) otherwise.

- If the last letter of \( w_1 \) is the same as the first letter of \( w_2 \) let \( n \) be the minimum of the length of \( w_1 \) and \( w_2 \).

Let \( w[i] \) be the word \( w \) without its last \( i \) letters and \( [i]w \) be the word without its first \( i \) letters.

Let \( l_i \) be the \( i \)th letter of \( w_2 \) (which is equal to the \( i \)th-last letter of \( w_1 \)). Then \( \alpha(w_1, w_2, w) = q^{n-1}(q - 1) \) if \( w = w_1[i]l_i[i]w_2 \) if \( 1 \leq i \leq n - 1 \) and \( \alpha(w_1, w_2, w) = q^n \) if \( w = w_1[n][n]w_2 \). Otherwise, \( \alpha(w_1, w_2, w) = 0 \).
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\[ \text{Proof.} \quad \bullet \text{ If the last letter of } w_1 \text{ is different to the first letter of } w_2 \text{ then none of the rules will be applied to the concatenation of labels. So we get exactly one label of each edge in } \Delta_{w_1w_2}. \]

\[ \bullet \text{ Assume that the last letter of } w_1 \text{ is the same as the first letter of } w_2. \]

We wish to concatenate a label \( \gamma_{w_1} \) of an edge in the half sphere corresponding to \( \Delta_{w_1} \) and \( \gamma_{w_2} \) an edge in the half sphere corresponding to \( \Delta_{w_2} \) to get a label \( \gamma \) of an edge in the half sphere corresponding to \( \Delta_{w_1[i],i}w_2 \) for \( 1 \leq i \leq n - 1 \).

Then there are \( q^{i-1} \) choices for the last \( i - 1 \) letters of \( \gamma_{w_1} \) and the first \( i - 1 \) letters of \( \gamma_{w_2} \) must be the last \( i - 1 \) letters of \( \gamma_{w_1} \) in reverse. The \( i \)th letter of \( \gamma_{w_2} \) must be the \( l(w_1) - i + 1 \)th letter of \( \gamma \) where \( l(w_1) \) is the length of \( w_1 \), then the \( i \)th last letter of \( \gamma_{w_1} \) can be any of the remaining \( q - 1 \) choices.

So we have \( (q - 1)q^{i-1} \) choices of \( (\gamma_{w_1}, \gamma_{w_2}) \) such that \( \gamma_{w_1} \gamma_{w_2} = \gamma \) after applying the concatenation rules. So \( \alpha(w_1, w_2, w) = (q - 1)q^{i-1}. \)

If \( \gamma \) is a label of an edge in the half sphere corresponding to \( \Delta_{w_1[i],i}w_2 \) then we can take any choice of word for the shorter word, giving \( n \) choices and the first/last \( n \) letters of the other must be the reverse of this word. So then \( \alpha(w_1, w_2, w) = q^n. \)

These cover all cases where we apply the rules and since the last letter of \( w_1 \) is the same as the last letter of \( w_2 \) there must be cancellation from applying the rules, so we must have that \( \alpha(w_1, w_2, w) = 0 \) for all other cases.

So we have that

\[ \Delta_{w_1} \cdot \Delta_{w_2} = \begin{cases} \Delta_{w_1w_2} & \text{if the last letter of } w_1 \text{ is different to the first letter of } w_2 \\ q^n\Delta_{w_1[n],i}w_2 + (q - 1) \sum_{i=1}^{n-1} q^{i-1}\Delta_{w_1[i],i}w_2 & \text{otherwise} \end{cases} \]

6.2.3 Multiplication tables

Proposition 6.2.16 (The full multiplication table of \( \mathbb{C}[G,B] \)). Recall that \( G = \text{SL}_2(\mathbb{F}_p((t))) \) and

\[ B = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{F}_p[[t]], \ ad - bc = 1, \ c \equiv 0 \pmod{t} \right\} \]

Let \( w \) be a word in \( \{s_1, s_2\} \) where \( s_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \) and \( s_2 = \begin{bmatrix} 0 & -t^{-1} \\ t & 0 \end{bmatrix} \).

Let \( \Delta_w = BwB. \)

Then the Hecke algebra \( \mathbb{C}[G,B] \) is the \( \mathbb{C} \) vector space with basis \( \{\Delta_w \mid w \text{ is a word in } \{s_1, s_2\}\} \) and the full multiplication table is given by

\[ \Delta_{w_1} \cdot \Delta_{w_2} = \begin{cases} \Delta_{w_1w_2} & \text{if the last letter of } w_1 \text{ is different to the first letter of } w_2 \\ p^n\Delta_{w_1[n],i}w_2 + (p - 1) \sum_{i=1}^{n-1} p^{i-1}\Delta_{w_1[i],i}w_2 & \text{otherwise} \end{cases} \]

where \( n \) is the minimum of the length of \( w_1 \) and \( w_2 \), \( w_1[i] \) is the word \( w_1 \) without its last \( i \) letters, \( i|w_2 \) is \( w_2 \) without its first \( i \) letters and \( l_i \) is the \( i \)th letter of \( w_2 \).

\[ \text{Proof.} \]

Let \( e_0 \in E(T) \) be the edge with vertices labelled \( P_1 \) and \( P_2 \). Then from propositions 6.2.1 and 6.2.2 we know that \( G \) acts transitively on the edges of \( T \) without inversion and each vertex stabilizer of a vertex \( v \) acts transitively on the sphere of vertices at distance one from \( v \).

From proposition 6.2.3 we know that stabilizer of \( e_0 \) is \( B \) which acts transitively on half spheres.

By construction of the Bass-Serre tree we know that the action of words in \( \{s_1, s_2\} \) map \( e_0 \) to an edge in any half-sphere. Then from 6.2.7 and the fact that double cosets partition a group we know that the \( \{\Delta_w\} \) form a basis for \( \mathbb{C}[\text{PGL}_2(\mathbb{Q}_p), \text{PGL}_2(\mathbb{Q}_p)] \).

Then the multiplication table follows from subsection 6.2.1 and corollary 6.2.15 \( \square \)

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Chapter 7

Baumslag-Solitar Groups

The Baumslag-Solitar groups inherit their name from a paper written by Gilbert Baumslag and Donald Solitar in 1962 [BS62] which used the groups as examples of groups which had the property of being non-Hopfian.

Baumslag protested against this naming having himself first seen a similar group in a paper by Graham Higman [Hig51] but to little effect as evidenced by the title of this chapter.

Baumslag-Solitar groups of a specific form are HNN-extensions. This chapter constructs the corresponding Bass-Serre trees for these specific Baumslag-Solitar groups and then computes their Hecke algebra.
7.1 Baumslag-Solitar groups

Definition 7.1.1. The Baumslag-Solitar group $BS(m, n)$ is the group with presentation

$$\langle x, y \mid yx^m y^{-1} = x^n \rangle$$

Note that if $m = 1$ then $BS(1, n)$ is the HNN-extension of $(x) \cong \mathbb{Z}$ with respect to the injective endomorphism $\alpha : \mathbb{Z} \to \mathbb{Z}$, $\alpha(x) = x^n$.

For this section we will consider only the groups $BS(1, n)$ although appendix [3] contains some information about the general case.

Proposition 7.1.2 (Normal form). Let $g \in G$, and $g$ not of the form $y^i$ for $i \in \mathbb{Z}$. Then we can write $g$ uniquely as $g = y^{-\alpha} x^c y^{\beta}$ where $\alpha, \beta \in \mathbb{Z}_{\geq 0}$, $c \in \mathbb{Z}$ and $n$ does not divide $c$.

Proof. See chapter IV of [4,5] for an actual proof.

Corollary 7.1.3. Let $S = \{y^{-\alpha} x^c y^{\beta} \mid \alpha, \beta \in \mathbb{Z}_{\geq 0}, n \nmid c, c \in \{1, 2, \ldots, n^\beta - 1\}\} \cup \{y^i \mid i \in \mathbb{Z}\}$ then we can write $G = \bigsqcup_{g \in S} gH$.

Proof. (Covers) Let $g = y^{-\alpha} x^c y^{\beta}, n \nmid c$.

Then $c = q + kn^\beta$ for some $q \in \{1, 2, \ldots, n^\beta - 1\}$. So using our relations we can write $y^{-\alpha} x^c y^{\beta} = y^{-\alpha} x^q y^{kn^\beta} x^{k} = y^{-\alpha} x^q y^{kn^\beta} y^{\beta} H$.

(Disjoint Union) Consider two left cosets, $y^{-\alpha_1} x^{c_1} y^{\beta_1} H, y^{-\alpha_2} x^{c_2} y^{\beta_2} H$. If they were not disjoint then $y^{-\alpha_1} x^{c_1} y^{\beta_1} = y^{-\alpha_2} x^{c_2} y^{\beta_2} = y^{-\alpha_2} x^{q_2 + kn^\beta} y^{\beta_2}$ for some $k \in \mathbb{Z}$.

But since $c_2$ does not divide $n$, neither does $c_2 + kn^\beta$ so by our normal form theorem we must have that $\alpha_1 = \alpha_2, \beta_1 = \beta_2$ and $c_1 = c_2 + kn^\beta = c_2 + kn^\beta$ but we must have that $c_1, c_2 \in \{1, 2, \ldots, n^\beta - 1\}$ so we have $k = 0$ and $c_1 = c_2$.

Corollary 7.1.4. Let $T = \{y^{-\alpha} x^c y^{\beta} \mid \alpha, \beta \in \mathbb{Z}_{\geq 0}, n \nmid c, c \in \{1, 2, \ldots, n^\beta - \alpha - 1\}\} \cup \{y^i \mid i \in \mathbb{Z}\}$ then we can write $G = \bigsqcup_{g \in T} gH$.

We saw in example [1,2,13] that $BS(1, n)$ is an HNN-extension. Chapter [2] gives an associated Bass-Serre tree which is infinite homogeneous of degree $n + 1$.

Note that if we compare the Cayley graph of $BS(1, 2)$ in example [1,2,18] to the above tree, the above tree can be seen as the Cayley graph under the equivalence relation which identifies vertices along the same horizontal line, or as viewing the Cayley graph ‘side on’.

If we choose the vertex labelled $\langle x \rangle$ as our distinguished vertex then the vertex stabilizer of $\langle x \rangle$ is $(x)$.

Then we wish to find the Hecke algebra $\mathbb{C}[G, H]$.

Proposition 7.1.5 (Double coset structures). Let $g = y^{-\alpha} x^c y^{\beta}$ where $\alpha, \beta \in \mathbb{Z}_{\geq 0}$, then

Let $L = \beta - \alpha$. Then the double coset $H_gH$ is the disjoint union of 1 left coset if $L \leq 0$ and $n^L$ left cosets if $L > 0$.

Similarly if $g = y^i$ then $H_gH$ is the disjoint union of 1 left coset if $L \leq 0$ and $n^L$ left cosets if $L > 0$.

Proof. First assume that $g = y^{-\alpha} x^c y^{\beta}$ where $\alpha, \beta \in \mathbb{Z}_{\geq 0}$.

The case $L \leq 0$ follows from the uniqueness of normal form of words (proposition [7,1,2]). Assume that $L > 0$.

We claim that $H_gH = \bigsqcup_{i \in \{0, 1, \ldots, n^L - 1\}} x^i gH$ where $k \in \{0, 1, \ldots, n^L - 1\}$.

Disjoint Disjointness again follows from the uniqueness of normal forms.

(Covers) We know that $H_gH = \bigsqcup_{i \in \mathbb{Z}} x^i gH$.

For any $i$, we can write $j = q + kn^\beta - \alpha$ for some $q \in \{0, 1, \ldots, n^L - 1\}$.

From our relations, we can write $x^i gH$ as $x^i y^{-\alpha} x^{c+k n^\beta} y^{\beta} H = x^i y^{-\alpha} x^c y^{\beta} x^k H = x^i y^{-\alpha} x^c y^{\beta} H$.

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We will call $L$ the level of a vertex. Note that the map from the Hecke algebra with multiplication to the group $\text{Z}$ with addition which sends a double coset to its level is almost a group homomorphism. That is, if $H g_1 H$ is in level $L_1$ and $H g_2 H$ is in level $L_2$ then elements in $H g_1 H \cdot H g_2 H$ are of level $L_1 + L_2$.

Also note that the double coset in level $L$ contains exactly $n^L$ left cosets.

Figure 7.1: Double cosets near id $H$ for $L = 0, 1, 2$

The following three prosopisitons calculate the results of $H g_1 H \cdot H g_2 H$ where $g_1 = y^{-\alpha_1} x^c y^{\beta_1}$, $g_2 = y^{-\alpha_2} x^c y^{\beta_2}$. Let $L_1 = \beta_1 - \alpha_1$, $L_2 = \beta_2 - \alpha_2$.

Also recall definition 3.1.15

**Definition 7.1.6.** Let $(G, H)$ be a Hecke pair. Let $H g H, H g' H \in H(\text{H}, G)$; we can write $H g H = \prod_{i \in I} g_i H, H g' H = \prod_{j \in J} q_j H$ for some set of representatives for left cosets $I, J$, where $g_i, q_j \in G$.

Then

$$H g H \cdot H g' H = \sum_{g \in H \setminus G / H} \alpha(g, g'; x) H x H$$

where $\alpha(g, g'; x) := \text{Card}(\{(i, j) : g, g_j \in x H\})$.

**Proposition 7.1.7** (Case $L_1 < 0$, $L_2 > 0$). If $L_1 < 0$ and $L_2 > 0$ then $H g_1 H \cdot H g_2 H = \sum_{i = 1}^{n} H g_1 x^i g_2 H$

Proof. Since $L_1 < 0$ and $L_2 > 0$ the action of the $g + i$ decreases the level, so we know that $H g_1 H \cdot H g_2 H$ must split into a number of different double cosets.

Now consider left cosets of the form $x^{a_1} g_1 x^{a_2} g_2 H$ for $a_1 \in \{0, 1, \ldots, n^{L_1 - 1}\}, a_2 \in \{0, 1, \ldots, n^{L_2 - 1}\}$.

Since $x^{a_1} g_1 x^{a_2} g_2 H = g_1 x^{a_1 - a_2} g_2 H$, so $H g_1 x^{a_1} g_2 H = H g_1 x^{a_2 \text{mod}(n^{L_1 - 1})} g_2 H$. Then the product $H g_1 H \cdot H g_2 H$ contains at most $\min(-L_1, L_2)$ given by double cosets of the form $H g_1 x^i g_2 H$, $i \in \{0, 1, \ldots, n^{\min(-L_1, L_2)}\}$.

Then each of these double cosets are distinct by the normal form of words. 

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Proposition 7.1.8 (Case $L_1 > 0$, $L_2 < 0$). \( Hg_1 H \cdot Hg_2 H = n^{\min(L_1, -L_2)} Hg_1 g_2 H \)

Proof. Since $L_1 > 0$ and $L_2 < 0$, all left cosets of the form $g_1 g_j H$ are elements of $Hg_1 g_2 H$ since the action of $g_1$ on vertices increases the level of vertices. And for an action which increases the level, all the vertices in one double coset are sent to a single double coset.

So $Hg_1 H \cdot Hg_2 H = \alpha Hg_1 g_2 H$.

We also know that $Hg_1 g_2 H$ contains exactly $n^{L_1 + L_2}$ left cosets since it is in level $L_1 + L_2$.

Then since the map which sends double cosets to the number of left cosets is a ring homomorphism, $n^{L_1} = \alpha n^{\max(L_1, -L_2)}$, so $\alpha = n^{L_2}$ if $L_1 + L_2 > 0$ and $\alpha = n^{L_1}$ if $L_1 + L_2 \leq 0$.

Proposition 7.1.9 (Any other case). For any other values of $L_1$, $L_2$, $Hg_1 H \cdot Hg_2 H = Hg_1 g_2 H$.

Proof. This follows from the normal form of words.

So from propositions 7.1.8, 7.1.7 and 7.1.9 we have the full multiplication table.

Proposition 7.1.10 (The multiplication table). Let $g_1 = y^{-i_1} x^{a_1} y^{j_1}$, $g_2 = y^{-i_2} x^{a_2} y^{j_2}$.

Then

$$Hg_1 H \cdot Hg_2 H = \begin{cases} \sum_{i=1}^{n^{\min(-L_1, -L_2)}} Hg_1 x^i g_2 H & \text{if } L_1 < 0, L_2 > 0 \\ n^{\min(L_1, -L_2)} Hg_1 g_2 H & \text{if } L_1 > 0, L_2 < 0 \\ Hg_1 g_2 H & \text{otherwise} \end{cases}$$

Multiplication of $Hg_1 H \cdot Hg_2 H$ can actually be seen as concatenating the representatives of left cosets in $Hg_1 H$ with representatives of left cosets in $Hg_2 H$ and looking at the resulting left cosets. We give two diagrams this picture for multiplication.

![Diagram 1](image1)

Figure 7.3: $H y^{-1} H \cdot H y^2 H = H y H + H y^{-1} x y^2 H$

![Diagram 2](image2)

Figure 7.4: $H y^2 H \cdot H y^{-1} H = 2 H y H$
Appendix A

A.1 Induction in lemma 4.1.4

Corollary A.1.1 (Corollary 3.5 in [BLRW09]). Let $T$ be a tree with at least 3 edges at each vertex.

Let $\Gamma$ be a group acting by automorphisms of $T$ and assume that the stabilizer of $a$, $\Gamma_0$ acts transitively on each sphere around $a$.

Let $q_0 + 1$ be the degree of the even vertices and let $q_1 + 1$ be the degree of the odd vertices and if $q_0 = q_1$ let $q = q_0 = q_1$.

Then the multiplication table for $\mathbb{C}[\Gamma, \Gamma_0]$ is given by

1. if $\Gamma$ acts transitively on vertices then

$$\Gamma_n \Gamma_m = \Gamma_m \Gamma_n = \Gamma_{m+n} + q^{n-1}(q + \delta_{m,n}) \Gamma_{m-n} + (q - 1) \sum_{l=1}^{n-1} q^{l-1} \Gamma_{m+n-2l}$$

2. if $\Gamma$ acts with two orbits on vertices then

$$\Gamma_{2n} \Gamma_{2m} = \Gamma_{2n} \Gamma_{2m} = \Gamma_{2(m+n)} + q_1^{n-1}(q_0 + \delta_{m,n}) \Gamma_{2(m-n)} + \sum_{l=1}^{2n-1} (q_{s(l)} - 1) \prod_{i=1}^{l-1} q_{s(i)} \Gamma_{2(m+n-l)}$$

Proof. • If $\Gamma$ acts transitively on vertices: We proceed by induction on $m$.

The case where $m = 1$ is proven in theorem 4.1.3.

Assume that the required formula holds for all $m < M$.

We can assume without loss of generality that $n \geq M$ since we know that multiplication is commutative.

If $n > m$ then

$$T_M \cdot T_n = (T_{M-1} \cdot T_1 - qT_{M-2})T_n$$
$$= T_{M-1}T_1T_n - T_{M-2}T_n$$
$$= (T_{M-1}(qT_{n-1} + T_{n+1})) - qT_{M-1}T_n$$
$$= qT_{M-1}T_{n-1} + T_{M-1}T_{n+1} - qT_{M-2}T_n \quad (n > 2)$$

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If $n > M$ and $M > 3$ then,

$$T_{M-1}T_{n+1} + qT_{M-1}T_{n-1} - qT_{M-2}T_n = T_{M+n} + q^{M-2}(q + 0)T_{n-M+2} + \sum_{l=1}^{M-2} q^{l-1}T_{M+n-2l}$$

$$+ q \left( T_{M+n-2} + q^{M-2}(q + 0)T_{n-M} + \sum_{l=1}^{M-2} q^{l-1}T_{M+n-2l-2} \right)$$

$$- q \left( T_{M+n-2} + q^{M-3}(2 + 0)T_{n-M+2} + \sum_{l=1}^{M-3} q^{l-1}T_{M+n-2l-2} \right)$$

$$= T_{M+n} + q^{M-2+1}T_{n-M+2} + \sum_{l=1}^{M-2} q^{l-1}T_{M+n-2l} + q \cdot q^{M-2}(q + 0)T_{n-M}$$

$$+ q \cdot q^{M-2+1}T_{M+n-2l(M-2)} - q \cdot q^{M-3+1}T_{n-M+2}$$

$$= T_{M+n} + q^{M-1}(q + 0)T_{n-M} + \sum_{l=1}^{M-2} q^{l-1}T_{M+n-2l} + (q^{M-1} + q^{M-2} - q^{M-1})T_{n-M+2}$$

$$= T_{M+n} + q^{M-1}(q + 0)T_{n-M} + \sum_{l=1}^{M-1} q^{l-1}T_{M+n-2l} + q^{M-2}T_{n-M+2}$$

$$= T_{M+n} + q^{M-1}(q + 0)T_{n-M} + \sum_{l=1}^{M-1} q^{l-1}T_{M+n-2l}$$

$$= T_{M+n} + q^{M-1}(q + \delta M)T_{n-M} + \sum_{l=1}^{M-1} q^{l-1}T_{M+n-2l}$$

If $M = 3$ the above applies if $\sum_{l=1}^{M-3} q^{l-1}T_{M+n-2l-2}$ is replaced with 0.

If $n > M$ and $M = q$

$$T_{M-1}T_{n+1} + qT_{M-1}T_{n-1} - qT_{M-2}T_n = T_1T_{n+1} + qT_1T_{n-1} - qT_0T_n$$

$$= qT_n + T_{n+q} + q^2T_{n-q} + qT_n - qT_n$$

$$= T_{n+2} + q^2T_{n-2} + qT_n$$

$$= T_{M+n} + q^{M-1}(q + \delta M)T_{n-M} + \sum_{l=1}^{M-1} q^{l-1}T_{M+n-2l}$$
If $n = M, M > 3$

$$T_{M-1}T_{n+1} + qT_{M-1}T_{n-1} - qT_{M-2}T_n$$

$$= T_{M+n} + q^{M-2}(q + 0)T_{n-M+2} + \sum_{l=1}^{M-2} q^{l-1}T_{M+n-2l}$$

$$+ q \left( T_{M+n-2} + q^{M-2}(q + 1)T_{n-M} + \sum_{l=1}^{M-2} q^{l-1}T_{M+n-2l-2} \right)$$

$$- q \left( T_{M+n-2} + q^{M-2}(q + 0)T_{n-M+2} + \sum_{l=1}^{M-3} q^{l-1}T_{M+n-2l-2} \right)$$

$$= T_{M+n} + q^{M-2+1}T_{n-M+2} + \sum_{l=1}^{M-2} q^{l-1}T_{M+n-2l} + q \cdot q^{M-2}(q + 0)T_{n-M}$$

$$+ q \cdot q^{M-2-1}T_{M+n-2(M-2)} - q \cdot q^{M-3+1}T_{n-M+2}$$

$$= T_{M+n} + q^{M-2}(q + 1)T_{n-M} + \sum_{l=1}^{M-2} q^{l-1}T_{M+n-2l} + (q^{M-1} + q^{M-2} - q^{M-1})T_{n-M+2}$$

$$= T_{M+n} + q^{M-2}(q + 1)T_{n-M} + \sum_{l=1}^{M-2} q^{l-1}T_{M+n-2l} + q^{M-2}T_{n-M+2}$$

$$= T_{M+n} + q^{M-2}(q + 1)T_{n-M} + \sum_{l=1}^{M-1} q^{l-1}T_{M+n-2l}$$

$$= T_{M+n} + q^{M-2}(q + \delta_M)T_{n-M} + \sum_{l=1}^{M-1} q^{l-1}T_{M+n-2l}$$

If $M = 3$ the above applies if $\sum_{l=1}^{M-3} q^{l-1}T_{M+n-2l-2}$ is replaced with 0.

If $n = M = 2$

$$T_{M-1}T_{n+1} + qT_{M-1}T_{n-1} - qT_{M-2}T_n$$

$$= T_1T_3 + qT_1T_1 - qT_0T_2$$

$$= qT_2 + T_4 + q(q + 1)T_0 + qT_n - qT_n$$

$$= T_4 + q(q + 1)T_0 + 2T_2$$

$$= T_{M+n} + q^{M-1}(q + \delta_M)T_{n-M} + \sum_{l=1}^{M-1} q^{l-1}T_{M+n-2l}$$

- If $\Gamma$ acts with two orbits on vertices:
Appendix B

BS(m, n) case

B.1 Example: BS(2, 3)

\[ G = \langle x, y \mid yx^2y^{-1} = x^3 \rangle \]
\[ H = \langle x \rangle \]

B.1.1 Right Cosets

BS(2, 3) with the subgroup \( \langle x \rangle \) has an infinite tree structure whose vertices correspond to disjoint subsets of the Cayley graph.

Each vertex in the tree is of degree 5, where we consider three of the incident edges rising up from the vertex and three of the edges travelling down from the vertex.

We choose the vertex in the tree corresponding to the subset of the Cayley graph containing the identity vertex to be the root vertex. We label this vertex id \( H \).

![Tree Diagram]

Figure B.1:

B.1.2 Double Cosets

Double cosets of the form \( Hy^nH \)

We first consider double cosets of the form \( Hy^nH \), \( n \in \mathbb{Z} \).

These double cosets will look very similar to the double cosets in BS(1, 2).

Example B.1.1 (\( HyH \)). Consider \( HyH \).

As a set, we have \( HyH = \bigcup_i x^i yH \).

But from the relation of the group we have that \( x^3y = yx^2 \).

So,

\[ HyH = \bigcup_i x^i yH \]
and we can calculate that

$$x^i y H = \begin{cases} 
y H & \text{if } i \equiv 0 \pmod{3} \\
x y H & \text{if } i \equiv 1 \pmod{3} \\
x^2 y H & \text{if } i \equiv 2 \pmod{3} \
\end{cases}$$

So $H y H = y H \sqcup x y H \sqcup x^2 y H$.

**Example B.1.2** ($H y^2 H$). Similar to $H y H$, for $H y^2 H$ we have

$$H y^2 H = \bigcup_i x^i y^2 H$$

and we can calculate that

$$x^i y^2 H = \begin{cases} 
y^2 H & \text{if } i \equiv 0 \pmod{9} \\
x y^2 H & \text{if } i \equiv 1 \pmod{9} \\
x^2 y^2 H & \text{if } i \equiv 2 \pmod{9} \\
y x^2 y H & \text{if } i \equiv 3 \pmod{9} \\
x y x^2 y H & \text{if } i \equiv 4 \pmod{9} \\
x^2 y x^2 y H & \text{if } i \equiv 5 \pmod{9} \\
y x^4 y H = y x y H & \text{if } i \equiv 6 \pmod{9} \\
x y x y H & \text{if } i \equiv 7 \pmod{9} \\
x^2 y x y H & \text{if } i \equiv 8 \pmod{9} \
\end{cases}$$

So $H y^2 H = y^2 H \sqcup x y^2 H \sqcup x^2 y^2 H \sqcup y x^2 y H \sqcup x y x^2 y H \sqcup x^2 y x^2 y H \sqcup y x y x H \sqcup x y x y H \sqcup x^2 y x y H$.

**Example B.1.3** ($H y^{-1} H$). Similarly again,

$$H y^{-1} H = \bigcup_i x^i y^{-1} H$$

and we can calculate that

$$x^i y^{-1} H = \begin{cases} 
y^{-1} H & \text{if } i \equiv 0 \pmod{2} \\
x y^{-1} H & \text{if } i \equiv 1 \pmod{2} \
\end{cases}$$

So $H y^{-1} H = y^{-1} H \sqcup x y^{-1} H$.

So, in general, if $n > 0$ we have $H y^n H$ made up of $3^n$ right cosets, and if $n < 0$ then $H y^n H$ is made up of $2^n$ right cosets.

**Remark.** Note that all the vertices connected to the identity residing in the same level, (e.g. $y^{-1} H$ and $x y^{-1} H$) happen to be in the same double coset because the greatest common divisor of 2 and 3 is 1, i.e. 2 and 3 are coprime.

We will see for the group $BS(2, 4)$ that not all vertices in the same level adjacent to the identity (or any set of vertices corresponding to a double coset) will be in the same double coset.

![Figure B.2: Double cosets adjacent to the root vertex of the tree](image-url)
Alternating between levels

Example B.1.4 \((H_{yxy}^{-1}H)\). We have

\[ H_{yxy}^{-1}H = \bigcup_i x^i yxy^{-1}H \]

and we can calculate that

\[ x^i yxy^{-1}H = \begin{cases} 
  yx^{2n}xy^{-1}H & \text{for some } n \in \mathbb{N} \quad \text{if } i \equiv 0 \pmod{3} \\
  xy^{2n}xy^{-1}H & \text{for some } n \in \mathbb{N} \quad \text{if } i \equiv 1 \pmod{3} \\
  x^2xy^{2n}xy^{-1}H & \text{for some } n \in \mathbb{N} \quad \text{if } i \equiv 2 \pmod{3} 
\end{cases} \]

\[ = \begin{cases} 
  xy^{-1}H & \text{for some } n \in \mathbb{N} \quad \text{if } i \equiv 0 \pmod{3} \\
  xxy^{-1}H & \text{for some } n \in \mathbb{N} \quad \text{if } i \equiv 1 \pmod{3} \\
  x^2xy^{-1}H & \text{for some } n \in \mathbb{N} \quad \text{if } i \equiv 2 \pmod{3} 
\end{cases} \]

\[(\text{Since } x^2y^{-1} = y^{-1}x^3)\]

So \(Hy = yxy^{-1}H \cup xyxy^{-1}H \cup x^2yxy^{-1}H\).

So if we alternate between levels, the number of right cosets in the double coset stays constant.

Rising or falling by two or more levels

Example B.1.5 \((H_{xyy}^{-2}H)\). We have

\[ H_{xyy}^{-2}H = \bigcup_i x^i yxy^{-2}H \]
B.1. Example: $BS(2,3)$

$x^i y x y^{-2} H = \begin{cases} 
  y x^{2n} y x y^{-2} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 0 \pmod{6} \\
  x y x^{2n} y x y^{-2} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 1 \pmod{6} \\
  x^2 y x^{2n} y x y^{-2} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 2 \pmod{6} \\
  y^{x^{2(2n+1)}} y x y^{-2} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 3 \pmod{6} \\
  x y x^{2(2n+1)} y x y^{-2} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 4 \pmod{6} \\
  x^2 y x^{2(2n+1)} y x y^{-2} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 5 \pmod{6} 
\end{cases}

\Rightarrow \begin{cases} 
  y^{x^{2n}} y x^{2n} y^{-1} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 0 \pmod{6} \\
  x y x^{2n} y x^{2n} y^{-1} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 1 \pmod{6} \\
  x^2 y x^{2n} y x^{2n} y^{-1} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 2 \pmod{6} \\
  y x y^{-1} x^{3(2n+1)} y^{-1} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 3 \pmod{6} \\
  x y x^{2(2n+1)} y x^{2(2n+1)} y^{-1} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 4 \pmod{6} \\
  x^2 y x^{2(2n+1)} y x^{2(2n+1)} y^{-1} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 5 \pmod{6} 
\end{cases}

\Rightarrow \begin{cases} 
  y x y^{-1} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 0 \pmod{6} \\
  x y x^{-1} y x^{-1} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 1 \pmod{6} \\
  x^2 y x^{-1} y x^{-1} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 2 \pmod{6} \\
  y x y^{-1} y x^{-1} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 3 \pmod{6} \\
  x y x^{-1} y x^{-1} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 4 \pmod{6} \\
  x^2 y x^{-1} y x^{-1} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 5 \pmod{6} 
\end{cases}

So $H x y y^{-2} H = y x y^{-2} H \sqcup x y x y^{-2} H \sqcup x^2 y x y^{-2} H \sqcup y x y^{-1} x y^{-1} H \sqcup x y y^{-1} y x^{-1} H \sqcup x^2 y y^{-1} y x^{-1} H$.

So once we stop alternating between levels, the number of right cosets multiply similar to the central double cosets of form $H y^k H$. i.e. they triple when rising in levels and double when falling in levels.

**Example B.1.6** ($H x y y^{-1} H$). We have

$$H x y y^{-1} H = \bigcup_i x^i y x y^{-1} H$$

and we can calculate that

$$x^i y x y^{-1} H = \begin{cases} 
  y x^{2n} y x y^{-1} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 0 \pmod{6} \\
  x y x^{2n} y x y^{-1} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 1 \pmod{6} \\
  x^2 y x^{2n} y x y^{-1} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 2 \pmod{6} \\
  y x y^{-1} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 3 \pmod{6} \\
  x y x^{-1} y x^{-1} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 4 \pmod{6} \\
  x^2 y x^{-1} y x^{-1} H \text{ for some } n \in \mathbb{Z} & \text{if } i \equiv 5 \pmod{6} 
\end{cases}$$

So $H x y y^{-1} H = y x y^{-1} H \sqcup x y x y^{-1} H \sqcup x^2 y x y^{-1} H$.

So if we alternate between levels, the number of right cosets in the double coset stays constant.
B.1.3 Multiplication

Multiplication works similar to $BS(1, 2)$.

That is, to multiply two basis elements in our algebra, e.g. $H_y H \cdot H_{yx y^{-1}} H$, we take vertices of the first element $(yH, xyH, x^2yH)$, and lift the tree of the second element, $H_{yx y^{-1}} H$ to each of the vertices in the first element. In the case of $H_y H \cdot H_{yx y^{-1}} H$ this results in 9 vertices, corresponding to the vertices contained in $H_y^2 x y^{-1} H$. 
B.2 Example: \( BS(2, 4) \)

B.2.1 Right Cosets

The right cosets of \( BS(2, 4) = \langle x, y \mid yx^2y^{-1} = x^4 \rangle \) with the subgroup \( \langle x \rangle \) form a tree with each vertex having degree 6: 4 edges to 'level above' and 2 edges to the 'level below'.

\[
\begin{align*}
&yH \\
&xyH \\
&x^2yH \\
&x^3yH \\
&xH \\
&y^{-1}H \\
&xy^{-1}H
\end{align*}
\]

Figure B.5:

B.2.2 Double Cosets

Double cosets of the form \( Hy^nH \)

Example B.2.1. Consider the double coset \( Hy^2H \).

Then

\[
Hy^2H = \bigcup_{i \in \mathbb{Z}} x^iy^2H
\]

and we can calculate that

\[
x^i y^2 H = \begin{cases} 
  y^2 H & \text{if } i \equiv 0 \pmod{8} \\
  xy^2 H & \text{if } i \equiv 1 \pmod{8} \\
  x^2y^2 H & \text{if } i \equiv 2 \pmod{8} \\
  x^3y^2 H & \text{if } i \equiv 3 \pmod{8} \\
  x^4y^2 H & \text{if } i \equiv 4 \pmod{8} \\
  x^5y^2 H & \text{if } i \equiv 5 \pmod{8} \\
  x^6y^2 H & \text{if } i \equiv 6 \pmod{8} \\
  x^7y^2 H & \text{if } i \equiv 7 \pmod{8}
\end{cases}
\]

So \( Hy^2H = y^2H \cup xy^2H \cup x^2y^2H \cup x^3y^2H \cup yx^2yH \cup x^2yx^2yH \cup x^3yx^2yH \).

Note that right cosets of the form \( x^iyxyH \) and \( x^i yx^2yH \) are not contained in \( Hy^2H \), since \( x^4y = yx^2 \). And since the subgroup generated by \( 2 \) has an index of 2 in \( \mathbb{Z}/4\mathbb{Z} \),
Since the subgroup generated by 2 has index 2 in \( \mathbb{Z}/\mathbb{Z}_4 \), and the subgroup generated by 4 has index 2 in \( \mathbb{Z}/\mathbb{Z}_2 \), when rising or falling by two or more levels, there will be two double cosets adjacent to a double coset instead of 1.

\[
\begin{align*}
\text{id} & \quad H \\
yH & \quad xyH \\
x^2yH & \quad x^3yH \\
\end{align*}
\]

Figure B.6: Grey-filled vertices form one double cosets while white-filled vertices form another.

\[
\begin{align*}
\text{id} & \quad H \\
y^{-1}H & \quad xy^{-1}H \\
\end{align*}
\]

Figure B.7: Grey-filled vertices form one double cosets while white-filled vertices form another.

**Alternating between levels**

Alternating between levels again preserves the number of right cosets within a double coset.

\[
\begin{align*}
\text{id} & \quad H \\
y^{-1}H & \quad xy^{-1}H \\
\text{Hy}^{-1}xyH \\
\end{align*}
\]

Figure B.8:

**Rising or falling by two or more levels**

Rising or falling by two or more levels results in the same splitting that occurs in trees. That is, the set of vertices adjacent to a single double coset will consist of two double cosets with an equal number of vertices.

**B.2.3 Multiplication**

Multiplication can gain be visualised by considering growing trees from the vertices of the first double coset in a multiplication.

**Example B.2.2.**

\[
Hy^{-1}H \cdot Hy^2H = HyH + Hy^{-1}xy^2H + Hxy^{-1}x^2y + Hy^{-1}x^2y^2H
\]
Bibliography


