Towards Logarithmic D-Type Models

Author: Joel Gill  
Supervisor: Prof. Paul Pearce

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Department of Mathematics and Statistics  
The University of Melbourne
Abstract

Restricted Solid-On-Solid (RSOS) lattice models are constructed by placing heights (spins) on the sites of a square lattice. The height configurations must satisfy an adjacency condition which is encoded by a Dynkin diagram of a classical Lie algebra classified as either A, D or E type (hence they are called A-D-E models). The Yang-Baxter equation determines whether these models are exactly solvable. Usually, the Yang-Baxter integrable A-D-E models are studied for unitary cases for which the face weights are all positive but we are interested in non-unitary cases which necessarily involve some negative face weights. In this thesis, we use the Temperley-Lieb algebra to study the Yang-Baxter equation for non-unitary A- and D-type RSOS (minimal) models. Replacing the height representation with a dense loop representation of the Temperley-Lieb algebra, we also investigate whether the D-type RSOS models are related to D-type logarithmic minimal models in the same way as the A-type RSOS models are known to be related to the A-type logarithmic minimal models of Pearce-Rasmussen-Zuber.
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I dedicate this thesis to my family (Kevin Gill, Marie Gill, Samantha Gill, Christie Gill, Anne Gill and Beverley Fuller), friends, and Ursula Parker.
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Chapter 1

Introduction

1.1 Statistical Mechanics

Statistical mechanics is a branch of physics that studies the macroscopic properties of matter such as: pressure, temperature, density, magnetisation, etc. Statistical mechanics concerns itself with studying systems composed of a number of interacting particles in the order of Avagadro’s number, \(6.02 \times 10^{23}\), thus it is impractical to study each particle, but to study the bulk behaviour. This is accomplished by looking at particular states the system can exist in, and assigning them a probability. A state is assigned a probability (or pdf for systems with continuous states) based on its corresponding energy (Hamiltonian), \(H(s)\), by assigning it a Boltzmann weight \(e^{-\beta H(s)}\), and normalising it by dividing through by a normalisation factor \(Z\) so the probability of the system being in state \(s\), \(P(s)\), is

\[
P(s) = \frac{1}{Z} e^{-\beta H(s)} \tag{1.1.1}
\]

\[
\rho(\sigma) = \frac{1}{Z} e^{-\beta H(\sigma)} \tag{1.1.2}
\]

The normalisation factor, \(Z\), is called the *canonical partition function*, which is a sum (or integral) of the Boltzmann weights over all possible states, i.e.

\[
Z = \sum_{\{s\}} e^{-\beta H(s)} \tag{1.1.3}
\]

\[
Z = \int_{\Gamma} e^{-\beta H(s)} d\Gamma \tag{1.1.4}
\]

The partition function given in (1.1.3) is used in systems with discrete states such as lattices, whereas the partition function given in (1.1.4) has its states exist over phase space, \(\Gamma = \{(q_1, \ldots, q_N, p_1, \ldots, p_N)\}\), where \(q_i\) is the position of particle \(i\), and \(p_i\) is the momentum of particle \(i\). We can use these probabilities to calculate expectation values of observables \(O(s)\) as follows

\[
\langle O(s) \rangle = \sum_{\{s\}} O(s) P(s) = \frac{1}{Z} \sum_{\{s\}} O(s) e^{-\beta H(s)} \tag{1.1.5}
\]

\[
\langle O(\sigma) \rangle = \int_{\Gamma} O(\sigma) \rho(\sigma) d\Gamma = \frac{1}{Z} \int_{\Gamma} O(\sigma) e^{-\beta H(\sigma)} d\Gamma \tag{1.1.6}
\]

The partition function is useful in its own right, as it can be used to calculate useful macroscopic properties of the system such as: pressure, entropy, internal energy, etc. One important area of

\[\beta = (k_B T)^{-1}\] where \(k_B = 1.38 \times 10^{-23} J K^{-1}\) is Boltzmann’s constant. Note: later on \(\beta = 2 \cos \lambda\) is used for the loop fugacity of loop models.
study in statistical mechanics is phase transitions. A common phase transition is boiling water, where it goes from a liquid phase to a gaseous phase. Phase transitions of a system can be found if the partition function is known, as they occur at singularities of the free energy, $\Psi$, which is defined as

$$-\beta \Psi = \log Z_N$$

(1.1.7)

For systems of large $N$, the free energy per site, $\psi$, is

$$-\beta \psi = - \lim_{N \to \infty} \frac{\beta}{N} \Psi = \lim_{N \to \infty} \frac{1}{N} \log Z_N$$

(1.1.8)

### 1.2 Gas Models

Modelling gases is an important to statistical mechanics, as a gas is a system with many particles. Here we will consider three models of gases that increase in sophistication.

#### 1.2.1 Ideal Gas Model

The ideal gas models the particles as $N$ non-interacting point particles (zero radius) of mass $m$. Thus, the Hamiltonian for this system is

$$H(\sigma) = \frac{1}{2m} \sum_{i=1}^{N} p_i^2$$

(1.2.1)

i.e. only kinetic energy contributes to the Hamiltonian, and the potential energy is zero. Now it can be shown that the partition function is given by

$$Z_N = \frac{V^N N!}{(2\pi m \beta)^{3N/2}}$$

(1.2.2)

Using the formula,

$$P = - \frac{\partial \Psi}{\partial V}$$

(1.2.3)

We obtain the well known equation of state for ideal gases

$$PV = Nk_B T$$

(1.2.4)

The ideal gas equation of state is a good approximation, but it can be improved on.

#### 1.2.2 Tonks Gas Model

In 1960 physicists Marvin D. Girardeau and Lewi Tonks introduced the Tonks-Girardeau gas model \[1\]. In the model, the particles are confined to a 1-dimensional line. In this model there’s a pair interaction (hard-core) potential (given in (1.2.5)) that stops any two particles from passing through each other and changing positions

$$\phi(r) = \phi_{hc}(r) = \begin{cases} \infty, & 0 \leq r < a \\ 0, & r \geq a \end{cases}$$

(1.2.5)

If we consider $N$ spheres on the interval $0 \leq x \leq L$ and omit the kinetic energy from each particle, then we obtain the following Hamiltonian,

$$H(x) = \sum_{1 \leq i < j \leq N} \phi_{hc}(|x_i - x_j|)$$

(1.2.6)
It can be shown that the partition function is,
\[ Z_N = \frac{(L - (N - 1)a)^N}{N!} \] (1.2.7)

In the limit \( N, L \to \infty \) such that the volume per particle remains finite (i.e. \( V/N = v \)), it can be shown that the free energy per particle is
\[ \psi = 1 + \log(v - a) \] (1.2.8)

From equations (1.2.8) and (1.2.3), it can be shown that the equation of state is
\[ P = \frac{k_B T}{v - a} \] (1.2.9)

It can be seen that in the limit \( a \to 0 \) we obtain the equation of state for an ideal gas. This model is an improvement, but in reality particles experience an attraction as a result of the Lennard-Jones attractive force.

1.2.3 van der Waals Gas

The van-der Waals gas model is a model introduced by Johannes Diderik van der Waals in his thesis in 1873 [2], in which he one the 1910 Nobel prize for. In van der Waals model, the particles had a non-zero volume and they experience a pairwise attractive force. In 1924 John Lennard-Jones proposed such a potential for the pairwise attractive force [3]
\[ \phi_{LJ}(r) = 4\epsilon \left( \left( \frac{\sigma}{r} \right)^{12} - \left( \frac{\sigma}{r} \right)^6 \right) \] (1.2.10)

Here \( \sigma \) is the distance at which \( \phi_{LJ} = 0 \), and \( \epsilon \) is the depth of the potential well.

If we look at Figure 1.1a, the part of the curve that goes above the \( r \)-axis represents the repulsive forces, and the part of the curve that goes below the \( r \)-axis represents the attractive forces. Now \( r_m \) represents the distance at which the potential is at a minimum. This distance is where the two particle system is at a stable equilibrium, and as a result will oscillate about this distance. A reasonable approximation that we will use for the potential is the following,
\[ \phi(r) = \phi_{hc} - \frac{\alpha}{L}, \quad \alpha > 0 \] (1.2.11)
The first term is from the hard core repulsion term in the Tonks gas, and the second term is the attractive term. A plot of this potential is given in Figure 1.1b. From this new potential, the Hamiltonian now becomes

\[ H = \sum_{1 \leq i < j \leq N} \left[ \phi(|x_i - x_j|) - \frac{\alpha}{L} \right] \]

\[ = H_{\text{Tonks}} - \frac{\alpha N(N - 1)}{2L} \]

From this it can be seen that the partition function is,

\[ Z_N = \exp \left[ \frac{\beta \alpha N(N - 1)}{2L} \right] Z_{N_{\text{Tonks}}} \]

In the limit \( N, V \to \infty \) such that the volume per particle remains finite (i.e. \( V/N = v \)), it can be shown that the free energy per particle is

\[ \psi = 1 + \frac{\beta \alpha}{2v} + \log(v - a) \]

From equations (1.2.14) and (1.2.3), it can be shown that the equation of state is

\[ \left( P + \frac{\alpha}{2v^2} \right)(v - a) = k_B T \]

It can be seen that in the limit \( \alpha = 0 \) we obtain the equation of state for the Tonks gas, and setting \( a = 0 \) gives the equation of state for an ideal gas. Now let’s plot the isotherms (set \( T \) to be constant)

(a) Isotherms according to the van der Waals model

(b) Isotherms of the van der Waals model after Maxwell equal area correction

Figure 1.2
The plot in Figure 1.2a shows the isotherms for three temperatures, it can be seen that as the temperature approaches a value $T_c$, the isotherm gets a point of inflection at $(v_c, P_c)$. Once $T < T_c$, we observe a region where the pressure drops as volume increases, and then it increases as volume increases. This violates thermodynamic stability, i.e. at constant temperature, increasing the pressure on a fluid compresses it, something has gone wrong with this model for low temperatures. James Clerk Maxwell in 1875 proposed a method to fixing this problem [4], he proposed that the oscillatory section of the curve can be replaced with a horizontal line such that the area of the curve above the line is equal to the area of the curve below the line. i.e. In Figure 1.3 we require the area of the region shaded red be equal to that of the area of the region shaded blue.

Maxwell justified this rule by saying that the work done on the system to go from $v_g$ to $v_c$ is the same as the work released on going from $v_c$ to $v_l$. Work for a reversible process (a process in which no energy is dissipated from the system) is defined as,

$$W = \int_{V_a}^{V_b} PdV,$$

where: $V = Nv$

$$W < 0$$, then work has been done on the system, if $W > 0$, then work has been done by the system. From this method we get the plot in Figure 1.2b. Now $T_c$ has a physical interpretation, $T_c$ is called the critical temperature, the corresponding critical specific volume and pressure are given by $v_c$ and $P_c$ respectively, it can be shown that they are given by,

$$P_c = \frac{\alpha}{54a^2}, \quad v_c = 3a, \quad T_c = \frac{4\alpha}{27ak_B}$$

From the maxwell construction, the free energy becomes non-analytic at $v = v_l, v_g$, this implies a phase transition occurs for $T < T_c$, where $v_l$ corresponds to the specific volume for the liquid phase, and $v_g$ corresponds to the gaseous phase. The region within the dotted curve is a liquid-gas coexistence region.
1.3 Lattice Models

Lattice models are useful for modelling systems where particles are either fixed or localised in space (such as crystals, magnets, etc), since lattice models discretise space such that vertices become the locations of particles. From this, we observe that there’s no kinetic energy contribution to the Hamiltonian. The interactions between particles can be such that for example only neighbouring particles interact with each other (Ising Model).

1.3.1 Ising Model

In 1925 Ernst Ising proposed the Ising model in his Ph.D thesis. The model is constructed by considering a lattice where each vertex is assigned a value \( \pm 1 \).

These values are called spins, which will be denoted by \( \sigma_i \) (\( \sigma_{i,j} \) in two-dimensions) where \( i \) (or \( i,j \)) refers to which vertex we are looking at. In the Ising model, spins can be interpreted as arrows pointing up or down, where up refers to spins aligned with the external magnetic field, and down refers to spins that are going against the external magnetic field. As a result it’s expected that in the presence of a magnetic field most spins will be aligned with the magnetic field (this is observed when materials are magnetised). The Hamiltonian of the Ising model is given as follows,

\[
H_{1D}(\sigma) = -J \sum_{j=1}^{N} \sigma_j \sigma_{j+1} - h \sum_{j=1}^{N} \sigma_j \tag{1.3.1}
\]

\[
\Rightarrow Z_{1D} = \prod_{\{\sigma\}, j=1}^{N} \exp \left[ \sigma_j \sigma_{j+1} - h \sigma_j \right], \quad \text{where: } K = \beta J \tag{1.3.2}
\]

\[
H_{2D}(\sigma) = -J_1 \sum_{i=1}^{M} \sum_{j=1}^{N} \sigma_{i,j} \sigma_{i+1,j} - J_2 \sum_{i=1}^{M} \sum_{j=1}^{N} \sigma_{i,j} \sigma_{i,j+1} \tag{1.3.3}
\]

\[
\Rightarrow Z_{2D} = \prod_{\{\sigma\}, i=1}^{M} \prod_{j=1}^{N} \exp \left[ K_1 \sigma_{i,j} \sigma_{i+1,j} + K_2 \sigma_{i,j} \sigma_{i,j+1} \right], \quad \text{where: } K_1, K_2 = \beta J_1, \beta J_2 \tag{1.3.4}
\]

where: \( J \) is the interaction strength between sites, (in two-dimensions \( J_1 \) is the interaction strength along horizontal edges, and \( J_2 \) is the interaction strength along vertical edges), \( M \) is the number of rows in the lattice and \( N \) is the number of columns in the lattice.

\[2\]The model was actually invented by his advisor Wilhelm Lenz to which he acknowledged in his thesis.
**Boundary Conditions**

These lattice models require some boundary conditions in order to be solved. Possible boundary conditions are:

- **Free**: No restrictions are put on spins along the boundary
- **Fixed**: The spins along the boundaries are fixed to be either +1 or -1.
- **Periodic**: In one-dimension this corresponds to $\sigma_{N+1} = \sigma_1$. In two-dimensions this can either be:
  - **cylindrical**: where $\sigma_{N+1,j} = \sigma_{1,j}$ or $\sigma_{1,M+1} = \sigma_{i,1}$, and the spins along the other boundary can be free or fixed.
  - **toroidal**: where $\sigma_{N+1,j} = \sigma_{1,j}$ and $\sigma_{i,M+1} = \sigma_{i,1}$.

The Ising model in one-dimension is trivial to solve (solved by Ernst Ising) and exhibits no phase transitions, the Ising model in two-dimensions is non-trivial and it has been solved in the zero-field ($h = 0$, the Ising model in two-dimensions is yet to be solved for $h \neq 0$) by several people including:

- **Hendrik Kramers and Gregory Wannier in 1941**. They used the lattice duality to find the critical point as follows: For the original lattice, the spins are labelled by $s_i$ and have interaction strength $J$. A Fourier transform is then performed on the Boltzmann weights (Here the weights will be represented as: $e^{-\beta H(s)} = W(s_i - s_j \mod 2)$ where $W(0) = e^K$, $W(1) = e^{-K}$, and the Fourier transform of the weights will be represented by $\hat{W}(t_{ij})$),

$$Z = \sum_{\{s\}} \prod_{\{i,j\}} W(s_i - s_j \mod 2) = \sum_{\{s\}} \sum_{\{t\}} \prod_{\{i,j\}} e^{\pi i (s_i - s_j) t_{ij}} \hat{W}(t_{ij})$$

(1.3.5)

in which we obtain a variable $t_{ij}$ for each pair of spins $s_i - s_j$ and we obtain the following condition

$$t_{ij} + t_{ik} + t_{il} + t_{im} = 0$$

(1.3.6)

which can be satisfied by the following parameterisation,

$$t_{ij} = \sigma_\kappa - \sigma_\lambda$$
$$t_{ik} = \sigma_\lambda - \sigma_\mu$$
$$t_{il} = \sigma_\mu - \sigma_\nu$$
$$t_{im} = \sigma_\nu - \sigma_\kappa$$

these $\sigma$ variables can be interpreted as the spins of a dual lattice (Figure 1.5) with interaction strength $\tilde{K}$, this gives a similar partition function to the original Ising model,

$$Z = 2^{MN} \sum_{\{\sigma\}} \prod_{\{\mu,\nu\}} \tilde{W}(\sigma_\mu - \sigma_\nu \mod 2)$$

(1.3.7)

where,

$$\tilde{W}(0) = e^\tilde{K}, \quad \tilde{W}(1) = e^{-\tilde{K}}$$

(1.3.8)

and we obtain the following relationships between $W$ and $\tilde{W}$,

$$W(0) = \tilde{W}(0) + \tilde{W}(1)$$

(1.3.9)

$$W(1) = \tilde{W}(0) - \tilde{W}(1)$$

(1.3.10)
We then obtain the following relationship between $\tilde{K}$ and $K$,

$$e^{-2K} = \tanh \tilde{K} \quad (1.3.11)$$

Now recall that a phase transition corresponds to a singularity in the free energy $\Psi$. Because the duality transform is analytic, both the original and dual model have the same critical temperature $T_c$. Thus, it can be obtained by solving the following equation

$$e^{-2K_c} = \tanh \tilde{K}_c \quad (1.3.12)$$

$$\Rightarrow K_c = \frac{1}{2} \log(1 + \sqrt{2}) \quad (1.3.13)$$

$$\Rightarrow T_c = \frac{2J}{k_B \log(1 + \sqrt{2})} \quad (1.3.14)$$

\[m = \begin{cases} 
1 - \left[ \sinh \left( \log(1 + \sqrt{2}) \frac{T_c}{T} \right) \right]^{-4} \frac{1}{2}, & T < T_c \\
0, & T \geq T_c
\end{cases} \quad (1.3.15)\]

In statistical mechanics we define the magnetisation as

$$m = \frac{1}{N} \left\langle \sum_{j=1}^{N} \sigma_j \right\rangle \quad (1.3.16)$$

So looking at $|m|$, it can be seen that: $|m| = 1$ corresponds to a completely ordered system, $0 < m < 1$ corresponds to a system that’s partially ordered, and $m = 0$ corresponds to a system that’s disordered. Now let’s plot the magnetisation against temperature that Onsager derived,
This can be interpreted as follows. For \( T = 0 \), the spins are frozen into place, but as \( T \) increases, they’re free to fluctuate and the magnetisation decreases until \( T = T_c \) in which the system becomes disordered. This is a real phenomena, and for magnets, the temperature at which this occurs is called the \textit{Curie temperature}.

### 1.3.2 Ice-Type Model

Ice-type models or six-vertex models are a family of vertex models for crystal lattices with hydrogen bonds that were solved by Lieb in 1967 [8]. The first such model was introduced by Linus Pauling in 1935 to account for the residual entropy of water ice [9]. Ice in two dimensions can be modelled by constructing a lattice of coordination number four, where an oxygen atom is placed at each vertex and a hydrogen ion is placed in-between each adjacent oxygen atom as shown in Figure 1.6.

![Figure 1.6: Two dimensional lattice model of ice](image)

Now a water molecule consists of an oxygen atom and two hydrogen atoms, so here there will be two hydrogen ions drawn towards an oxygen atom, and two will be drawn away. Slater in 1941 proposed that the ions should satisfy the ice rule [11]:

“Of the four ions surrounding each atom, two are close to it, and two are removed from it, on their respective bonds.”

Now every configuration that satisfies the ice rule has a particular energy which will be given by \( E \), thus we can construct a partition function for this model.

\[
Z = \sum_{\text{configurations}} \exp \left( -\frac{E}{k_bT} \right) \quad (1.3.17)
\]

For ice, \( E \) is the same for every arrangement, thus

\[
\exp \left( -\frac{E}{k_bT} \right) = C = \text{constant} \quad (1.3.18)
\]

With a suitable choice of the zero in the energy scale, we can choose \( E = 0 \) and thus find that the partition function, \( Z \), is the total number of possible configurations. From this, we find that the residual entropy, \( S \), is

\[
S = k_b \log Z > 0, \quad \text{since: } Z > 1 \quad (1.3.19)
\]

The oxygen molecule will now be omitted, and arrows will now be used to describe whether a hydrogen ion is drawn towards an oxygen atom or drawn away from an oxygen atom. Using this new arrow notation, it can be seen that it is required that two arrows point towards the vertex, and
two arrows point away from the vertex. There’s exactly six different allowed vertex configurations (hence why it’s called the six-vertex model), they are:

![Figure 1.7: The six allowed vertex configurations](image)

Now each of these configurations have their own energy given by $\varepsilon_i$ where $i = 1, \ldots, 6$. From this it can be seen that any two-dimensional lattice that satisfies the ice rule, has total energy

$$E = \sum_{i=1}^{6} n_i \varepsilon_i$$  \hspace{1cm} (1.3.20)

$$\Rightarrow Z = \sum_{\{n_i\}} \prod_{i=1}^{6} e^{-\beta n_i \varepsilon_i} = \sum_{\{n_i\}} \prod_{i=1}^{6} w_i^{n_i}, \quad w_i = e^{-\beta \varepsilon_i}$$  \hspace{1cm} (1.3.21)

Where $n_i$ is the number of vertices of the $i^{th}$ configuration, and $\{n_i\}$ are the number of vertices of type $l$ where $l = 1, \ldots, 6$. If we impose periodic boundary conditions on the lattice (here instead of spins, we have arrows on the boundary point in the same direction), then $n_5 = n_6$ and we can choose $\varepsilon_5 = \varepsilon_6 = \varepsilon_c$. If we also assume invariance under arrow reversal, this implies configuration 1 is indistinguishable from configuration 2, and configuration 3 is indistinguishable from configuration 4. Thus, $n_1 = n_2$ and $n_3 = n_4$, and we can choose $\varepsilon_1 = \varepsilon_2 = \varepsilon_a$, and $\varepsilon_3 = \varepsilon_4 = \varepsilon_b$. The Boltzmann weights can now be written as follows,

$$a = e^{-\beta \varepsilon_a} = w_1 = w_2$$  \hspace{1cm} (1.3.22)

$$b = e^{-\beta \varepsilon_b} = w_3 = w_4$$  \hspace{1cm} (1.3.23)

$$c = e^{-\beta \varepsilon_c} = w_5 = w_6$$  \hspace{1cm} (1.3.24)

If the lattice has $M$ rows and $N$ columns, we can now write the partition function as,

$$Z = \sum_{r=1}^{M} \sum_{\{m_l\}} q^{m_1+m_2} q^{m_3+m_4} e^{m_5+m_6}$$  \hspace{1cm} (1.3.25)

Where we are now summing over the rows, and hence $m_l^r$ is the number of vertices of type $l$. The ice model is obtained by setting $\varepsilon_a = \varepsilon_b = \varepsilon_c = 0$. More interesting models have arisen from the six-vertex model, such as the KDP model which models two dimensional potassium dihydrogen phosphate, KH$_2$PO$_4$ crystal lattices at low temperatures with $\varepsilon_a = 0$, $\varepsilon_b = \varepsilon_c > 0$ and the F model which is a model of anti-ferroelectrics by choosing $\varepsilon_a = \varepsilon_b > 0$, $\varepsilon_c = 0$. 
1.4 Exactly Solvable Models

Exactly solvable models are of great interest in statistical mechanics. An exactly solvable model is a model where the Hamiltonian is simple enough that the partition function can be solved to give a closed form solution\[10\]. In the introduction, we have discussed two exactly solvable lattice models: The Ising Model in one- and two-dimensions, and the ice-type (or six-vertex) model in two-dimensions.

1.4.1 Transfer Matrices

The calculation of the partition functions of these models is attributed to using the method of Transfer Matrices. If we can find a transfer matrix, then the partition function becomes,

\[ Z = \text{Tr} T^N \]  

(1.4.1)

also, if the transfer matrix is diagonalisable, then (1.4.1) becomes,

\[ Z = \sum_{i=1}^{k} \lambda_i^N \]

(1.4.2)

where \( k \) is the dimension of the transfer matrix. Instead of trying to diagonalise a single transfer matrix, we try to diagonalise a whole family of commuting transfer matrices. If we can find such transfer matrices that commute, the system is called exactly solvable. To understand the relationship between a transfer matrix, and the partition function, we will look at a solution to the six-vertex model using transfer matrices. Here the following notation is used for the a vertex.

\[ L_{\alpha' \gamma'} = \alpha \frac{\gamma'}{\gamma} \alpha' \]  

(1.4.3)

Where \( \alpha \in \{+,-\} \) (+ corresponds to an arrow pointing to the right, and - corresponds to an arrow pointing to the left), \( \gamma \in \{+,-\} \) (+ corresponds to an arrow pointing up, and - corresponds to an arrow pointing down). So: \( L_{++} \) corresponds to configuration 1 and \( L_{--} \) corresponds to configuration 2, thus \( L_{++} = L_{--} = a \); \( L_{+-} \) corresponds to configuration 3 and \( L_{-+} \) corresponds to configuration 4, thus \( L_{+-} = L_{-+} = b \); \( L_{+} \) corresponds to configuration 5 and \( L_{+} \) corresponds to configuration 6, thus \( L_{+} = L_{+} = c \) (Recall that \( a, b, c \) are the Boltzmann weights given in (1.3.22) - (1.3.24)), and by the ice rule, all other values of \( L \) are 0. we can now construct the following matrix representation for \( L \)

\[ L = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \]  

(1.4.4)

We can now construct the transfer matrix, \( T_{\gamma'}^\gamma \), for one row by gluing \( N \) of these \( L \) objects together, i.e.

\[ T_{\gamma'}^\gamma = \alpha_1 \frac{\gamma'}{\gamma} \alpha_2 \frac{\gamma'}{\gamma} \alpha_3 \frac{\gamma'}{\gamma} \alpha_4 \ldots \frac{\gamma'}{\gamma} \alpha_N \]

(Recall that \( a, b, c \) are the Boltzmann weights given in (1.3.22) - (1.3.24)).
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Where $\gamma, \gamma'$ is shorthand for $\gamma_1\gamma_2, \ldots, \gamma_N, \gamma'_1\gamma'_2, \ldots, \gamma'_N$. The transfer matrix can also be expressed as,

$$T_\gamma^{\gamma'} = \sum_{\alpha_1} \cdots \sum_{\alpha_N} L_{\alpha_1 \gamma_1}^{\alpha_2 \gamma_2'} \cdots L_{\alpha_N \gamma_N}^{\alpha_1 \gamma_1'}$$  \hspace{1cm} (1.4.5)

We now wish to construct a commuting family of transfer matrices. Consider introducing a transfer matrix, $(T')_\gamma^{\gamma'}$, with Boltzmann weights $a', b', c'$. Our goal is to find under what conditions (if any) are required for the transfer matrices to commute, i.e

$$[T_\gamma^{\gamma'}, (T')_\gamma^{\gamma'}] = 0$$  \hspace{1cm} (1.4.6)

The product of two transfer matrices is vertical concatenation, i.e

$$(TT')_\gamma^{\gamma'} = \begin{array}{cccccc}
\gamma'_1 & \gamma'_2 & \gamma'_3 & \cdots & \gamma'_N \\
\beta_1 & \beta_2 & \beta_3 & \cdots & \beta_N \\
\gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_N \\
\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_N \\
\end{array}$$

The product of transfer matrices is actually a sum over all $\gamma''$,

$$(TT')_\gamma^{\gamma'} = \sum_{\gamma''} (T')_\gamma^{\gamma'} T_\gamma^{\gamma''}$$  \hspace{1cm} (1.4.7)

Out of this transfer matrix product we get a new object which we will call $S_{\alpha, \beta}^{\alpha', \beta'}(\gamma', \gamma)$, which can be represented diagrammatically as,

$$S_{\alpha, \beta}^{\alpha', \beta'}(\gamma', \gamma) = \begin{array}{ccc}
\gamma' & \beta & \beta' \\
\gamma & \alpha & \alpha' \\
\end{array}$$

Algebraically, $S_{\alpha, \beta}^{\alpha', \beta'}(\gamma', \gamma)$ is expressed as,

$$S_{\alpha, \beta}^{\alpha', \beta'}(\gamma', \gamma) = \sum_{\gamma''} (L_{\alpha \gamma})^{\alpha'}_{\gamma''} L_{\beta \gamma}^{\gamma'' \beta'}$$  \hspace{1cm} (1.4.8)

and the product of transfer matrices in terms of $S$ is,

$$(TT')_\gamma^{\gamma'} = \sum_{\alpha_1, \ldots, \alpha_N} \sum_{\beta_1, \ldots, \beta_N} \prod_{j=1}^N S_{\alpha_j \beta_j}^{\alpha_{j+1} \beta_{j+1}}(\gamma_j, \gamma'_j)$$

$$= \text{Tr} [S(\gamma_1, \gamma'_1) \cdots S(\gamma_N, \gamma'_N)]$$  \hspace{1cm} (1.4.9)

we can also reverse the transfer matrix product to produce a similar object $S'$, such that

$$(T'T)_\gamma^{\gamma'} = \text{Tr} [S'(\gamma_1, \gamma'_1) \cdots S'(\gamma_N, \gamma'_N)]$$  \hspace{1cm} (1.4.10)
1.4.2 Yang-Baxter Equation

In order to show that the system is integrable, we want to show that \( [T^\gamma, (T')^\gamma] = 0 \). Suppose there exists a non-singular matrix \( R \), such that \( S'(\gamma, \gamma') = R^{-1} S(\gamma, \gamma') R \), then we observe that

\[
(TT')^\gamma = \text{Tr} \left[ S(\gamma, \gamma_1) \cdots S(\gamma_N, \gamma_N') \right]
= \text{Tr} \left[ R^{-1} S'(\gamma, \gamma_1) R \cdots R^{-1} S'(\gamma_N, \gamma_N') R \right]
= \text{Tr} \left[ S'(\gamma, \gamma_1) \cdots S'(\gamma_N, \gamma_N') RR^{-1} \right]
= (T'T)^\gamma \tag{1.4.11}
\]

i.e. the existence of this \( R \) matrix allows for the transfer matrices to commute. We can re-express this transformation as follows,

\[
RS(\gamma, \gamma') = S'(\gamma, \gamma') R \\
\Rightarrow \sum_{\alpha''\beta'',\gamma''} R_{\alpha\beta\gamma}^{\alpha''\beta''\gamma''} (L)^{\beta''\gamma''}_{\beta\gamma} L_{\alpha\gamma''}^{\alpha''\gamma'} = \sum_{\alpha''\beta'',\gamma''} L_{\alpha\gamma''}^{\alpha''\gamma''} (L)^{\beta''\gamma''}_{\beta\gamma} R_{\alpha''\beta''\gamma''} \tag{1.4.12}
\]

Equation (1.4.12) is called the **Yang-Baxter Equation** (or star-triangle relation). The equation is coined after works by C.N. Yang in 1967 [12] and R.J. Baxter in 1972 [13]. Diagrammatically this equation is

![Yang-Baxter Equation](image)

This diagram shows us that the \( R \)-matrix takes the \( S(\lambda, \lambda') \) matrix, and it glues the arrows on the horizontal together \( \alpha'', \beta'' \), but it doesn’t touch the vertical arrows. An ansatz is now made, we assume that the structure of \( R \) is similar to the \( L \)-matrix, and thus we assign it it’s own Boltzmann weights \( a'', b'', c'' \). It can be shown that in order for the Yang-Baxter equation to be solved, we require the following relation to hold between the weights,

\[
\frac{a^2 + b^2 - c^2}{ab} = \frac{a'^2 + b'^2 - c'^2}{a'b'} \tag{1.4.13}
\]

From this we define a quantity, \( \Delta \), which must be invariant under the choice of \( L \)

\[
\Delta := \frac{a^2 + b^2 - c^2}{2ab} \tag{1.4.14}
\]

This relation can be satisfied under the following parameterisation,

\[
a = \rho \sin (u + \gamma), \quad b = \rho \sin u, \quad c = \rho \sin \gamma \tag{1.4.15}
\]

it can be shown that,

\[
\Delta = \cos \gamma \tag{1.4.16}
\]
Here $u$ is a parameter that corresponds to choice of weights $a, b, c$ and it’s defined as the spectral parameter. $\gamma$ is some parameter that’s independent of weights $a, b, c$, and $\rho$ is here purely for normalisation. For different spins $a', b', c'$, we can assign the matrix a different spectral parameter $v$. Now that we have made a a parameterisation that satisfies (1.4.2), we obtain a family of commuting transfer matrices defined as $T(u)$, i.e.

$$[T(u), T(v)] = 0$$

(1.4.17)

The weights $a'', b'', c''$ for the $R$-matrix also must satisfy, as we made an ansatz that it has a similar structure to the $L$-matrix, and it can be shown that it’s corresponding spectral parameter is $v - u$. From this parameterisation, we can re-express the Yang-Baxter equation in (1.4.12) in terms of the spectral parameters as follows,

$$R(v - u)L_n(v)L_n(u) = L_n(u)L_n(v)R(v - u)$$

(1.4.18)

where $n$ refers to the corresponding row for the $L$-operator (the parameterisation turns the matrices into operators).

### 1.5 A-D-E Lattice Models

The A-D-E lattice models [17] are exactly solvable Restricted Solid-on-Solid (RSOS) models on a square lattice. Each site of the square lattice is associated with a local degree of freedom in the form of an integer height $a$ which lives on a graph of A, D or E-type as shown in Figure 1.9. These are the Dynkin diagram of the the classical A, D and E Lie algebras. The integer heights $(a, b)$ on any two neighboring sites must be adjacent on the given Dynkin diagram [18]. This restriction allows us to define an adjacency matrix $A$

$$A_{ab} = \begin{cases} 
1, & a \sim b \\
0, & \text{otherwise} 
\end{cases}$$

(1.5.1)

where $a \sim b$ denotes $a$ is adjacent to $b$ in the corresponding A, D or E Dynkin diagram. The eigenvalues of their adjacency matrices are of the form

$$\Lambda_j = 2 \cos \lambda_j, \quad \lambda_j = \frac{m_j \pi}{h}$$

(1.5.2)

where the Coxeter number $h$ and the Coxeter exponents $m_j$ take the values shown in Table 1.1. For each Lie algebra, the corresponding Perron-Frobenius eigenvectors, for the unitary model with $m_j = 1$, is shown in Table 1.2.

The face weights of the A-D-E models are given by

$$W \left( \begin{array}{c|c} a & b \\ d & c \\ \end{array} \right) = \frac{s_1(-u)\delta(a, c) + s_0(u)\epsilon_a h_{ba} h'_{dc} \delta(b, d)}{\epsilon_a}$$

(1.5.3)

where

$$s_r(u) = \frac{\sin(u + r\lambda)}{\sin(\lambda)}$$

(1.5.4)

and $\epsilon_a = \frac{1}{\epsilon_a} = \pm 1$ are appropriately chosen sign factors. For the A-type models, we can take $\epsilon_a = 1$. For unitary models, the crossing parameter $\lambda$ is defined as

$$\lambda = \frac{\pi}{h}, \quad m_j = 1$$

(1.5.5)
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A
1 2 3

L

D
1 2 3

L

E
6

L − 1

E
1 2 3

4 5

E
7

1 2 3 4

5 6

E
8

1 2 3 4

5 6 7

Figure 1.9: A-D-E Dynkin diagrams

Coxeter Exponents

<table>
<thead>
<tr>
<th>Lie Algebra</th>
<th>h</th>
<th>m_j</th>
</tr>
</thead>
<tbody>
<tr>
<td>AL</td>
<td>L + 1</td>
<td>1, 2, 3, ..., L</td>
</tr>
<tr>
<td>DL</td>
<td>2L − 2</td>
<td>L − 1, 1, 3, 5, ..., 2L − 3</td>
</tr>
<tr>
<td>E6</td>
<td>12</td>
<td>1, 4, 5, 7, 8, 11</td>
</tr>
<tr>
<td>E7</td>
<td>18</td>
<td>1, 5, 7, 9, 11, 13, 17</td>
</tr>
<tr>
<td>E8</td>
<td>30</td>
<td>1, 7, 11, 13, 17, 19, 23, 29</td>
</tr>
</tbody>
</table>

Table 1.1: Coxeter numbers and exponents of the classical A-D-E algebras.

Eigenvectors

<table>
<thead>
<tr>
<th>Lie Algebra</th>
<th>( S_a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AL</td>
<td>( \left( \sin \frac{m_j \pi}{L + 1}, \sin \frac{2m_j \pi}{L + 1}, ..., \sin \frac{Lm_j \pi}{L + 1} \right) ), ( m_j = 1, 2, ..., L )</td>
</tr>
<tr>
<td>DL</td>
<td>( \left( 2 \cos \frac{m_j (L − 2) \pi}{2L − 2}, 2 \cos \frac{m_j \pi}{2L − 2}, 2 \cos \frac{m_j \pi}{2L − 2}, 1, 1 \right) ), ( m_j = 1, 3, 5, ..., 2L − 3 )</td>
</tr>
<tr>
<td>E6</td>
<td>( \left( \sin \frac{\pi}{12}, \sin \frac{\pi}{6}, \sin \frac{\pi}{4}, \sin \frac{\pi}{3} - \sin \frac{\pi}{2 \cos \frac{\pi}{12}}, \sin \frac{5\pi}{12} - \sin \frac{\pi}{4}, \sin \frac{\pi}{2 \cos \frac{\pi}{12}} \right) )</td>
</tr>
<tr>
<td>E7</td>
<td>( \left( \sin \frac{\pi}{18}, \sin \frac{\pi}{9}, \sin \frac{\pi}{6}, \sin \frac{\pi}{2 \cos \frac{\pi}{18}}, 2 \cos \frac{\pi}{18}, \sin \frac{\pi}{3} - \sin \frac{\pi}{2 \cos \frac{\pi}{18}} \right) )</td>
</tr>
<tr>
<td>E8</td>
<td>( \left( \sin \frac{\pi}{30}, \sin \frac{\pi}{15}, \sin \frac{\pi}{10}, \sin \frac{\pi}{6}, \sin \frac{\pi}{2 \cos \frac{\pi}{30}}, \sin \frac{7\pi}{30} - \sin \frac{\pi}{2 \cos \frac{\pi}{30}} \right) )</td>
</tr>
</tbody>
</table>

Table 1.2: Entries \( S_a \) of the eigenvectors of the A-D-E adjacency matrices. The \( DL \) eigenvector for \( m_j = L − 1 \) is \((0, ..., 0, −1, 1)\). Only the Perron-Frobenius eigenvector with \( m_j = 1 \) is shown for the exceptional algebras \( E_{6,7,8} \). For non-unitary cases with \( m_j \neq 1 \), some entries of \( S_a \) are negative.
More generally, for non-unitary models,

\[ \lambda = \lambda_j = \frac{m_j \pi}{h}, \quad m_j > 1 \]  

(1.5.6)

The Forrester-Baxter [19, 20] RSOS models are A-type models with heights \( a = 1, 2, \ldots, m' - 1 \), that is, \( a \in A_L \) with \( L = m' - 1 \).

\[ \lambda := \frac{(m' - m) \pi}{m'}, \quad 2 \leq m \leq m', \quad m, m' \text{ coprime} \]  

(1.5.7)

where \( m' = h \) is the Coxeter number of the A Dynkin diagram. The associated loop fugacity \( \beta \) assigned to every closed loop in the Temperley-Lieb algebra is defined as

\[ \beta := 2 \cos \lambda \]  

(1.5.8)

Our first aim here is to show that Temperley-Lieb algebra and hence the Yang-Baxter equation holds for the A-D-E models. The terms, \( h_{ab} \), in (1.5.3) should be thought of as three-spin interactions (i.e. \( h_{ab} = h_{abd} \delta(b, d) \) - The Kronecker delta makes the diagonally opposite spins \( b \) & \( d \) the same). These spin interactions in general are not symmetric (i.e. \( h_{ab} \neq h_{ba} \)) and can be visualised as seen in (1.5.9).

\[ A_{ab} = \]  

(1.5.9)

By rotating the lattice by 45°, we define the local face operators as a linear combination of elementary diamonds as seen in (1.5.10).

\[ X_j(u) = d_j + s_1(-u) d_j + s_0(u) d_j \]  

(1.5.10)

Where the first diamond is the identity, \( I \), and the second diamond (monoid) denotes the operators \( e_j \) (where \( j = 1, \ldots, n - 1 \) - for some \( n \in \mathbb{Z}_{\geq 0} \)) that satisfy the relations (1.7.2) - (1.7.4). Owczarek and Baxter [21] have shown that if the following conditions are satisfied

\[ h'_{ba} h_{ba} h_{ab} = 1, \quad a \sim b \]  

(1.5.11)

\[ \sum h_{ab} A_{ab} \beta = \beta \]  

(1.5.12)

then the monoid operators \( e_j \) satisfy a Temperley-Lieb algebra (1.7.1) - (1.7.4). Furthermore, it is known [10] that any representation of the Temperley-Lieb algebra \( \{e_j\} \) gives rise to face operators

\[ X_j(u) = s_1(-u) I + s_0(u) e_j \]  

(1.5.13)

satisfying the Yang-Baxter Equation (YBE)

\[ X_j(u) X_{j+1}(u + v) X_j(v) = X_{j+1}(v) X_j(u + v) X_{j+1}(u) \]  

(1.5.14)
The YBE can be represented diagrammatically as

\[ j j + 1 j + 2 = j j + 1 j + 2 \]

In terms of the face weights the YBE is

\[ \sum_g W(\begin{array}{ccc} a & g & b \\ b & e & c \end{array} | u) W(\begin{array}{ccc} f & e & g \\ a & d & c \end{array} | v) W(\begin{array}{ccc} e & d & g \\ g & b & c \end{array} | v - u) = \sum_g W(\begin{array}{ccc} f & d & g \\ a & e & b \end{array} | v - u) W(\begin{array}{ccc} g & d & c \\ b & e & a \end{array} | v) W(\begin{array}{ccc} f & e & d \\ a & d & b \end{array} | u) \]

which can be represented diagrammatically as

\[ \]

where the solid circle in the center indicates that the height \( g \) is summed over all allowed values.

### 1.6 Verification of Owzarek-Baxter Criteria for A-D-E Models

Consider a lattice model of A, D or E-type. Let \( \lambda \) be any eigenvalue of the adjacency matrix \( A \) corresponding to a unitary or non-unitary model with corresponding eigenvector \( S \) with entries \( S_a \).

It follows that

\[ \sum_b A_{ab} S_b = \Lambda S_a \]  

with \( \Lambda = \beta = 2 \cos \lambda \). If \( g_a \) are arbitrary gauge factors, set

\[ h'_{ab} = g_a \frac{S_a}{S_b}, \quad h_{ab} = \frac{1}{g_a} \]

It is then trivial to see that (1.5.11) is satisfied

\[ h'_{ba} h_{ba} h'_{ab} h_{ab} = \frac{g_a g_b S_a S_b}{g_a g_b S_a S_b} = 1, \quad a \sim b \]

Similarly, for (1.5.12),

\[ \sum_b A_{ab} h'_{ba} h_{ba} = \sum_b A_{ab} \frac{g_b S_b}{g_a S_a} = \frac{\Lambda S_a}{S_a} = \Lambda = 2 \cos \lambda = \beta \]

The general face weights for A-D-E models are thus given by

\[ W(\begin{array}{cc} d & c \\ a & b \end{array} | u) = s_1(u) \delta(a, c) + s_0(u) \frac{g_c S_d}{g_a S_b} \delta(b, d) \]
where $g_a$ are arbitrary gauge factors. For convenience in performing fusion, it is common to fix the gauge factors to sign factors $\epsilon_a = \frac{1}{\epsilon_a} = \pm 1$ determined by

$$\epsilon_1 = \epsilon_2 = +1, \quad \epsilon_a\epsilon_c = -1, \quad a, c \text{ second neighbours}$$

(1.6.5)

For the A-type models it follows that

$$\epsilon_a = (-1)^{\left\lfloor \frac{a-1}{2} \right\rfloor}$$

(1.6.6)

### 1.7 Temperley-Lieb Algebra

The Yang-Baxter equation determines whether a lattice model is exactly solvable. We will show in detail that it holds for A-type, and D-type models. But before we look at these, we need to introduce an algebra that will be used throughout this thesis, called the Temperley-Lieb algebra.

The Temperley-Lieb (TL) algebra is generated by operators $I$ and $e_j$ which satisfy the relations

1. $Ig = gI = g$ (1.7.1)
2. $e_j^2 = \beta e_j$ (1.7.2)
3. $e_je_{j \pm 1}e_j = e_j$ (1.7.3)
4. $e_i e_j = e_j e_i, \quad |i - j| \geq 2$ (1.7.4)

where $g$ is any element of the TL algebra. The TL algebra can be represented the TL graphically by monoids $e_j$ acting on $n$ strings

$$I = \begin{array}{cccccc}
1 & 2 & \cdots & j-1 & j & j+1 & \cdots & n-1 & n \\
\end{array}$$

(1.7.5)

$$e_j = \begin{array}{cccccc}
1 & 2 & \cdots & j-1 & j & j+1 & \cdots & n-1 & n \\
\end{array}$$

(1.7.6)

These diagrams can be used to give a graphical representation of relations (1.7.2) - (1.7.4)

$$Ie_j = \begin{array}{c}
\bigcup_j \bigcup_{j+1} = \bigcup_j \bigcup_{j+1} = e_j \\
\end{array}$$

(1.7.7)

$$e_j^2 = \begin{array}{c}
\bigcup_j \bigcup_{j+1} = \beta \bigcup_j \bigcup_{j+1} = \beta e_j \\
\end{array}$$

(1.7.8)
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1.8 Temperley-Lieb Relations for A-D-E Models

1.8.1 $e_j^2 = \beta e_j$

Using the local face operators given in (1.5.10), this relation can be re-expressed as

\[
e_j^2 = \beta e_j
\]  

(1.8.1)

It can be seen that, for the equality in (1.8.1) to hold, the weight of the central square with the sum over $d$ must be equal to $\beta$. This gives

\[
e_j^2 = \sum_d A_{bd} h'_{db} h_{db} = \sum_d A_{bd} h'_{db} = \sum_d A_{bd} \frac{S_d}{S_b} = \beta
\]  

(1.8.2)

Thus the equality in (1.8.1) holds.

1.8.2 $e_j e_{j+1} e_j = e_j$

Likewise, in this case, we can use the local face operators given in (1.5.10) to show this identity. However, there are now two cases to consider (i.e. $e_j e_{j+1} e_j = e_j$ & $e_j e_{j-1} e_j = e_j$). Since both cases are similar, we just consider the first case.

\[
e_j e_{j+1} e_j = e_j
\]  

(1.8.3)
It is easily seen that equality holds since

\[ h'_{cb} h_{bc} h'_{bc} A_{bc} = h'_{cb} h'_{bc} A_{bc} = \frac{S_c}{S_b} \frac{S_b}{S_e} A_{bc} = A_{bc} = 1, \quad b \sim c \quad (1.8.4) \]
Chapter 2

A-Type Models

In this chapter we consider the A-type or Forrester-Baxter models. We consider the gauge and crossing symmetries and how they effect the YBE. We also show that the YBE holds within connectivity classes as given by the loops included in the drawing of the faces. For A-type models, the lattice heights sit on the graph

\[ 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow \cdots \rightarrow m' - 2 \rightarrow m' - 1 \]

Figure 2.1: \( A_{m' - 1} \)

For the A-type models, the eigenvector corresponding to the eigenvalue \( 2 \cos \lambda \) takes the form

\[
S = (S_1, S_2, \ldots, S_{m' - 1})
\]

(2.0.1)

where \( S_a = \sin(a\lambda), \ a \in \{1, 2, 3, \ldots, m' - 1\} \).

2.1 Unitary Minimal Weights (\( m' - m = 1 \))

In the unitary case, the crossing parameter given in (1.5.7) becomes

\[
\lambda = \frac{\pi}{m'}
\]

(2.1.1)

Here the spin interactions, \( h_{ab} \) \& \( h'_{ab} \), will be set

\[
h'_{ab} = h_{ab} := \sqrt{\frac{S_a}{S_b}}
\]

(2.1.2)

where \( S_a := \sin(a\lambda), \ a \in \{1, 2, 3, \ldots, m' - 1\} \). It can be seen that \( 0 < S_a < 1 \), since it’s argument is \( 0 < a\lambda < \pi \). Therefore there are no issues with taking the square root. Thus, the face weights given in (1.5.3) now become

\[
W\left( \begin{array}{c|c} d & c \\ \hline a & b \end{array} | u \right) = s_1(-u)\delta(a, c) + s_0(u)\sqrt{\frac{S_aS_c}{S_bS_d}}\delta(b, d)
\]

(2.1.3)

2.2 Non-Unitary Minimal Weights (\( m' - m \neq 1 \))

2.2.1 Broken \( \mathbb{Z}_2 \) Symmetry and Gauge Factors

In the non-unitary case, the crossing parameter given in (1.5.7) with \( m' - m > 1 \). As a result, the interaction terms given in (2.1.2) are not suitable since, for \( a \in \{1, 2, \ldots, m' - 1\} \), \( S_a \) can be
negative. To avoid complex valued face weights, we will redefine $h_{ab}'$, $h_{ab}$

$$h_{ab}' = \frac{S_a}{S_b}$$  \hspace{1cm} (2.2.1)  $$

$$h_{ab} = 1$$  \hspace{1cm} (2.2.2)  $$

Andrews, Baxter and Forrester showed in [19], that for some arbitrary $g_a$, the following transformation leaves the partition function unchanged.

$$W\left(\begin{array}{cc} d & c \\ a & b \end{array} \bigg| u \right) \rightarrow \frac{g_ag_b}{g_cg_d} W\left(\begin{array}{cc} d & c \\ a & b \end{array} \bigg| u \right)$$  \hspace{1cm} (2.2.3)  $$

$\mathbb{Z}_2$ symmetry of the face weights can be retained by carefully choosing $g_a$. First consider the new face weights.

$$W\left(\begin{array}{cc} d & c \\ a & b \end{array} \bigg| u \right) = s_1(-u)\delta(a,c) + \frac{S_c}{S_b}s_0(u)\delta(b,d)$$  \hspace{1cm} (2.2.4)  $$

$$g_ah_b W\left(\begin{array}{cc} d & c \\ a & b \end{array} \bigg| u \right) = \frac{g_bh_a}{g_dS_b}s_1(-u)\delta(a,c) + \frac{g_aS_c}{S_b}s_0(u)\delta(b,d)$$  \hspace{1cm} (2.2.5)  $$

It can be seen that setting $g_b = g_d = 1$ retains $\mathbb{Z}_2$ symmetry for the identity operator (i.e. the weight assigned to the identity operator is unchanged under $b \leftrightarrow d$). Now consider the $\epsilon_j$ operator, it can be seen that the gauge factors $g_a$ can be used to obtain the face weights for the unitary case $m' - m = 1$ given in (2.1.3) by picking $g_a = \sqrt{S_a}$, $g_c = \sqrt{S_c}$.

We can now use one expression to give a general face weight for all coprime $m, m'$.

$$W\left(\begin{array}{cc} d & c \\ a & b \end{array} \bigg| u \right) = \frac{g_aS_c}{g_cS_b}s_1(-u)\delta(a,c) + \frac{S_c}{S_b}s_0(u)\delta(b,d)$$  \hspace{1cm} (2.2.5)  $$

where $g_a$ are arbitrary gauge factors. These weights are no longer symmetric under interchange of $a$ and $c$. To deal with the non-unitary cases we will choose $g_a = \epsilon_a = \pm 1$ with appropriately chosen sign factors $\epsilon_a$. In this case the face weights are all real but some are negative. If $a \in$
\{1,2,\ldots,m' - 1\}, we have the following six possible configurations

\[ u_{a}, a_{a+1} = s_{1}(-u) \]  \hspace{1cm} (2.2.6)

\[ u_{a}, a_{a-1} = s_{1}(-u) \]  \hspace{1cm} (2.2.7)

\[ u_{a}, a_{a+1} = \frac{g_{a-1}}{g_{a+1}} S_{a+1} s_{0}(u) \]  \hspace{1cm} (2.2.8)

\[ u_{a}, a_{a-1} = \frac{g_{a+1}}{g_{a-1}} S_{a-1} s_{0}(u) \]  \hspace{1cm} (2.2.9)

\[ u_{a}, a_{a+1} = s_{1}(-u) \]  \hspace{1cm} (2.2.10)

\[ u_{a}, a_{a-1} = s_{1}(-u) \]  \hspace{1cm} (2.2.11)

### 2.2.2 Verification of Owczarek-Baxter Criteria for Non-Unitary Minimal Weights

Again it is trivial to see that (1.5.11) is satisfied.

\[ h'_{ba} h_{ba} h'_{ab} h_{ab} = S_{b} S_{a} \times 1 \times S_{a} S_{b} = 1 \]  \hspace{1cm} (2.2.12)

Now to show (1.5.12) holds

\[ \sum_{b} A_{ab} h'_{ba} h_{ba} = \sum_{b} A_{ab} S_{b} S_{a} = \beta \]  \hspace{1cm} (2.2.13)

where the adjacency matrix eigenvector equation was used.

### 2.2.3 Temperley-Lieb Proofs for Unitary Minimal Weights:

#### 2.2.4 \( e_{j}^{2} = \beta e_{j} \)

Using the local face operators given in (1.5.10), this relation can be re-expressed as follows in (2.2.14).

\[ c_{j}^{2} = \beta c_{j} \]  \hspace{1cm} (2.2.14)
It can be seen that for the equality in (2.2.14) to hold, the weight of the shaded region summed over \( d \) must be equal to \( \beta \). This region contains two \( h_{db} \), thus

\[
\sum_d A_{bd} h_{db}' = \sum_d A_{bd} S_d \frac{S_d}{S_b} = \sum_d A_{bd} \frac{S_d}{S_b} = \beta \quad (2.2.15)
\]

Where the adjacency matrix eigenvector equation was used.

### 2.2.5 \( e_j e_{j+1} e_j = e_j \)

Likewise in the above case, we can use the local face operators given in (1.5.10) to show this identity. However, there is now two cases to consider (i.e. \( e_j e_{j-1} e_j = e_j \) & \( e_j e_{j+1} e_j = e_j \)).

\[
e_j e_{j+1} e_j = e_j
\]

\[
ge_j e_{j-1} e_j = e_j \quad (2.2.16)
\]

It can be seen that for the equality in (2.2.16) to hold, it is required that

\[
1 = h_{db}' h_{db} h_{bd}' h_{bd} A_{bd} = S_d S_b \frac{S_d}{S_b} A_{bd} = \frac{S_d S_b}{S_b S_d} A_{bd} = A_{bd} = 1 \quad (2.2.17)
\]

### 2.2.6 Local Crossing Relations

Claim that the local crossing relation for an arbitrary gauge (\( g_a = 1 \) or \( g_a = \sqrt{S_a} \)) is

\[
\begin{align*}
\begin{array}{ccc}
\text{d} & \text{c} & \text{e} \\
\text{a} & \text{b} & \text{u}
\end{array} & = & \begin{array}{ccc}
\text{b} & \text{c} & \text{d} \\
\text{a} & \text{u} & \text{b}
\end{array} & \text{Relation for interchanging} & \text{b} & \text{and} & \text{d} \\
\begin{array}{ccc}
\text{d} & \text{c} & \text{e} \\
\text{a} & \text{b} & \text{u}
\end{array} & = & \begin{array}{ccc}
\text{b} & \text{c} & \text{d} \\
\text{a} & \text{b} & \text{u}
\end{array} & \frac{g_a S_c}{g_b S_d} & \frac{S_c}{S_d} & \frac{g_a}{g_b} & \lambda - u \\
\begin{array}{ccc}
\text{d} & \text{c} & \text{e} \\
\text{a} & \text{b} & \text{u}
\end{array} & = & \begin{array}{ccc}
\text{d} & \text{c} & \text{e} \\
\text{a} & \text{b} & \text{u}
\end{array} & \frac{g_a g_b S_c}{g_c S_a} & \lambda - u & \text{Relation for rotating the marked corner by 180°} \\
\begin{array}{ccc}
\text{d} & \text{c} & \text{e} \\
\text{a} & \text{b} & \text{u}
\end{array} & = & \begin{array}{ccc}
\text{d} & \text{c} & \text{e} \\
\text{a} & \text{b} & \text{u}
\end{array} & \frac{g_a g_b S_c}{g_c S_a} & \lambda - u & \text{Relation for rotating the marked corner by 270°}
\end{align*}
\]
We’ll now show that these relations hold. First consider the relation given in (2.2.18), the RHS can be expressed as,

\[
\begin{align*}
\lambda - u & = s(\lambda - u)\delta(a,c) \\
\frac{g_c S_c}{g_2 S_a} & = \frac{g_c S_c}{g_2 S_a} s(u)\delta(b,d) \\
\end{align*}
\]

Now let’s interchange \( b \) and \( d \). It can be seen that the identity term is unaffected by this interchange, and then looking at the generator term, even though \( S_b \rightarrow S_d \), the \( \delta(b,d) \) forces \( S_b = S_d \), and thus, the generator term is unaffected as well. Thus, we have established that the relation (2.2.18) holds.

To show the relation given in (2.2.19) holds, consider the RHS

\[
\begin{align*}
\frac{g_a g_d S_c}{g_b g_c S_d} & = \frac{g_a g_d S_c}{g_b g_c S_d} \left( s(u)\delta(b,d) + \frac{g_b S_d}{g_d S_a} s(\lambda - u)\delta(a,c) \right) \\
& = \frac{g_a S_c}{g_c S_a} s(\lambda - u)\delta(a,c) + \frac{g_a g_d S_c}{g_b g_c S_a} s(u)\delta(b,d) \\
& = s(\lambda - u)\delta(a,c) + g_a S_c s(u)\delta(b,d) \\
& = s(\lambda - u)\delta(a,c) + g_a S_c s(u)\delta(b,d) \\
& = s(\lambda - u)\delta(a,c)
\end{align*}
\]

To show the relation given in (2.2.20) holds, consider the RHS.

\[
\begin{align*}
\frac{g_2^2 S_a}{g_c^2 S_c} & = \frac{g_2^2 S_a}{g_c^2 S_c} \left( s(\lambda - u)\delta(a,c) + \frac{g_c S_c}{g_a S_b} s(u)\delta(b,d) \right) \\
& = \frac{g_2^2 S_a}{g_c^2 S_c} s(\lambda - u)\delta(a,c) + \frac{g_c S_c}{g_a S_b} s(u)\delta(b,d) \\
& = s(\lambda - u)\delta(a,c) + \frac{g_c S_c}{g_a S_b} s(u)\delta(b,d) \\
& = s(\lambda - u)\delta(a,c)
\end{align*}
\]

To show the relation given in (2.2.21) holds, it can be done by first rotating the marked corner by 180°, then by 90°

\[
\begin{align*}
\frac{g_2^2 S_c}{g_2^2 S_a} & = \frac{g_2^2 S_c}{g_2^2 S_a} \left( s(\lambda - u)\delta(a,c) + \frac{g_c S_c}{g_a S_b} s(u)\delta(b,d) \right) \\
& = \frac{g_2^2 S_c}{g_2^2 S_a} s(\lambda - u)\delta(a,c) + \frac{g_c S_c}{g_a S_b} s(u)\delta(b,d) \\
& = s(\lambda - u)\delta(a,c)
\end{align*}
\]

So it can be seen that the relations given in (2.2.18) - (2.2.21) hold.
Now in the symmetric gauge, these relations become

\[
\begin{align*}
\begin{array}{c}
\text{Relation for interchanging } b \text{ and } d \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Relation for a } 90^\circ \text{ rotation} \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Relation for rotating the marked corner by } 180^\circ \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Relation for rotating the marked corner by } 270^\circ \\
\end{array}
\end{align*}
\]

The relations given for a \(90^\circ\) rotation and a \(270^\circ\) rotation are expected, since we want \(\mathbb{Z}_2\) symmetry to be retained in the symmetric gauge.

In the non-symmetric gauge, these relations become

\[
\begin{align*}
\begin{array}{c}
\text{Relation for interchanging } b \text{ and } d \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Relation for a } 90^\circ \text{ rotation} \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Relation for rotating the marked corner by } 180^\circ \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Relation for rotating the marked corner by } 270^\circ \\
\end{array}
\end{align*}
\]

In the non-symmetric gauge, it can be seen that \(\mathbb{Z}_2\) symmetry isn’t retained.

\subsection{2.3 Yang-Baxter Equation}

The Yang-Baxter equation for the hybrid case is

\[
\begin{align*}
\begin{array}{c}
\text{(2.3.1)} \\
\end{array}
\end{align*}
\]

When the faces are incorporated into the Yang-Baxter Equation, they become distorted. I.e.
The angle denoted by $\theta_u$ is called the *anisotropy angle*, which can be expressed as,

$$\theta_u = \frac{\pi u}{\lambda} \quad (2.3.2)$$

There is also a complementary angle, $\bar{\theta}_u$, such that

$$\bar{\theta}_u = \frac{\pi(\lambda - u)}{\lambda} = \pi - \theta_u \quad (2.3.3)$$

In the Yang-Baxter equation given in (1.5.17), it can be seen that summing each of the angles of the red arcs gives

$$\theta_u + \theta_v + \bar{\theta}_{v-u} = \left( \pi - \frac{\pi u}{\lambda} \right) + \frac{\pi v}{\lambda} + \left( \pi - \frac{\pi(v - u)}{\lambda} \right)$$

$$= 2\pi \quad (2.3.4)$$

### 2.3.1 Symmetric Yang-Baxter Equation

Now our aim here is to show that the Yang-Baxter Equation holds for all possible height sets for \{a, b, c, d, e, f\}. The central height, $g$, must be such that for the LHS of (1.5.17): $|a - g| = |g - e| = |g - c| = 1$, and for the RHS: $|f - g| = |b - g| = |d - g| = 1$ (so the value of $g$ will not be the same value for the RHS and LHS of the Yang-Baxter equation). But going through each of these cases is rather exhaustive. It would be much simpler to use symmetry to go through this problem. Let us suppose that $a = \min\{a, b, c, d, e, f\}$, then by converting the Yang-Baxter equation given in (1.5.17) into a form that admits rotational symmetry. i.e.

$$f \quad e \quad v \quad \lambda - u \quad d \quad \rightarrow \quad f \quad e \quad \lambda - v \quad u \quad d \quad (2.3.5)$$

$$f \quad e \quad v \quad \lambda - u \quad d \quad \rightarrow \quad f \quad e \quad \lambda - v \quad u \quad d \quad (2.3.6)$$

We can reduce the amount of cases that need to be looked at. To obtain the rotationally symmetric planar tangles, we can use the local crossing relations given in (2.2.18) - (2.2.21) to equate each face.
LHS:

Thus, we can equate the LHS of the Yang-Baxter equation in standard form to the LHS of the Yang-Baxter equation in symmetric form.

\[
\Rightarrow \begin{array}{ccc}
(a, u) & (f, v) & (d, \lambda - v) \\
(b, g) & (e, g) & (c, \lambda - v) \\
\end{array}
= \left( \frac{g_b^2 S_g}{g_g^2 S_b} \right) \left( \frac{g_a g_f S_c}{g_e g_g S_f} \right) \left( \frac{g_b g_c S_d}{g_d g_g S_c} \right)
\]

(2.3.7)

RHS:

Thus, we can equate the RHS of the Yang-Baxter equation in standard form to the RHS of the Yang-Baxter equation in symmetric form.

\[
\Rightarrow \begin{array}{ccc}
(a, u) & (d, \lambda - v) \\
(b, g) & (c, \lambda - v) \\
\end{array}
= \left( \frac{g_b^2 S_g}{g_g^2 S_b} \right) \left( \frac{g_a g_e S_c}{g_e g_g S_f} \right) \left( \frac{g_b g_c S_d}{g_d g_g S_c} \right)
\]

(2.3.8)

Now supposing the Yang-Baxter equation given in (1.5.17) holds, it can be seen that

\[
\left( \frac{g_b^2 S_g}{g_g^2 S_b} \right) \left( \frac{g_a g_f S_c}{g_e g_g S_f} \right) \left( \frac{g_b g_c S_d}{g_d g_g S_c} \right) = \left( \frac{g_b^2 S_g}{g_g^2 S_b} \right) \left( \frac{g_a g_e S_c}{g_e g_g S_f} \right) \left( \frac{g_b g_c S_d}{g_d g_g S_c} \right)
\]
Now we can make some cancellations. Note: we can’t cancel out $S_g$ and $g_g$, since they’re being summed over.

\[
\sum \Rightarrow \frac{g_ag_f S_c S_g}{g_e g_d} \Rightarrow S_c S_g \Rightarrow S_c S_g
\]

Now we can make some cancellations. Note: we can’t cancel out $S_g$ and $g_g$, since they’re being summed over.

For simplicity, we’ll relabel the spectral parameters as follows: $u = w_1$, $\lambda - v = w_2$, $w_3 = v - u$.

\[
\frac{S_a S_c S_g S_g}{g_a g_c g_e g_g} \Rightarrow S_a S_c S_g S_g \Rightarrow S_a S_c S_g S_g
\]

2.3.2 Symmetries of the Symmetric Yang-Baxter Equation

Now observe that the coefficients out the front of the two planar tangles are invariant under rotating the heights by $120^\circ$, $240^\circ$, and $180^\circ$ (and compositions such as rotating by $180^\circ$ then by $120^\circ$ or rotating by $180^\circ$ then by $240^\circ$):

- A $120^\circ$ rotation does the following: $a \mapsto c$, $b \mapsto d$, $c \mapsto e$, $d \mapsto f$, $e \mapsto a$, $f \mapsto b$, $g \mapsto g$. This merely permutes the products, but doesn’t change the value.

- A $240^\circ$ rotation does the following: $a \mapsto e$, $b \mapsto f$, $c \mapsto a$, $d \mapsto b$, $e \mapsto c$, $f \mapsto d$, $g \mapsto g$. Again this merely permutes the products, but doesn’t change the value.
A $180^\circ$ rotation does the following: $a \mapsto d$, $b \mapsto e$, $c \mapsto f$, $d \mapsto a$, $e \mapsto b$, $f \mapsto c$, $g \mapsto g$. This interchanges the coefficients from the LHS and RHS, but it can also be seen that this rotation has actually transformed the LHS of the Yang-Baxter equation to the RHS, and vice-versa. Hence, the two planar tangles are invariant under $180^\circ$ rotations.

A rotation by $180^\circ$ then by $120^\circ$ does the following: $a \mapsto d \mapsto f$, $b \mapsto e \mapsto a$, $c \mapsto f \mapsto b$, $d \mapsto a \mapsto e$, $e \mapsto b \mapsto d$, $f \mapsto c \mapsto e$, $g \mapsto g \mapsto g$. Again it can be seen that all this does is permutes the products, but doesn’t change the value.

A rotation by $180^\circ$ then by $240^\circ$ does the following: $a \mapsto d \mapsto b$, $b \mapsto e \mapsto c$, $c \mapsto f \mapsto d$, $d \mapsto a \mapsto e$, $e \mapsto b \mapsto f$, $f \mapsto c \mapsto a$, $g \mapsto g \mapsto g$. Again it can be seen that all this does is permutes the products, but doesn’t change the value.

2.3.3 Height Paths of the Symmetric Yang-Baxter Equation

These rotational invariances are extremely useful, since without loss of generality we can let $a = \min\{a, b, c, d, e, f\}$, and then use rotations to cover the cases when $b, c, d, e, or f$ is the minimal value. To see this, there are four different height paths when $a = \min\{a, b, c, d, e, f\}$. They’re
• \{a, a + 1, a, a + 1, a, a + 1, a\}

It can be seen that this path is invariant under rotations of 120° and 240°, but rotating by
180° gives the following path \{a + 1, a, a + 1, a, a + 1, a\}. The first path has minimal height
assigned to \(a, c\) and \(e\), and the second path has the minimal height \(a\) assigned to \(b, d\) and \(f\). Thus, every vertex has the minimal and maximal height assigned to it.

• \{a, a + 1, a + 2, a + 1, a, a + 1, a\}

For this path we have the minimal height assigned to \(a\) and \(e\), and the maximal height is assigned to \(c\).

  – Rotating by 120° gives the following path: \{a, a + 1, a, a + 1, a + 2, a + 1, a\}. For this path we have the minimal height assigned to \(a\) and \(c\), and the maximal height is assigned to \(e\).

  – Rotating by 240° gives the following path: \{a + 2, a + 1, a, a + 1, a, a + 1, a + 2\}. For this path we have the minimal height assigned to \(c\) and \(e\), and the maximal height assigned to \(a\).

  – Rotating by 180° then by 120° gives the following path: \{a + 1, a + 2, a + 1, a, a + 1, a, a + 1\}. For this path we have the minimal height assigned to \(d\) and \(f\), and the maximal height assigned to \(b\).

Thus, we have been able to use the rotations to ensure we can cover the cases such that every vertex has the minimal and maximal height assigned to it for this particular path.

• \{a, a + 1, a + 2, a + 1, a + 2, a + 1, a\}:

For this path we have the minimal height assigned to \(a\), and the maximal height is assigned to \(c\) and \(e\).

  – Rotating by 120° gives the following path: \{a + 2, a + 1, a, a + 1, a + 2, a + 1, a + 2\}. For this path we have the minimal height assigned to \(c\) and \(e\), and the maximal height is assigned to \(a\).

  – Rotating by 240° gives the following path: \{a + 2, a + 1, a + 2, a + 1, a, a + 1, a + 2\}. For this path we have the minimal height assigned to \(d\) and \(f\), and the maximal height is assigned to \(a\) and \(c\).

  – Rotating by 180° gives the following path: \{a + 1, a + 2, a + 1, a, a + 1, a + 2, a + 1\}. For this path we have the minimal height assigned to \(d\), and the maximal height is assigned to \(b\) and \(f\).

  – Rotating by 180° then by 120° gives the following path: \{a + 1, a + 2, a + 1, a + 2, a + 1, a, a + 1\}. For this path we have the minimal height assigned to \(f\), and the maximal height is assigned to \(b\) and \(d\).
Rotating by $180^\circ$ then by $240^\circ$ gives the following path: \{a + 1, a, a + 1, a + 2, a + 1, a + 2, a + 1\}. For this path we have the minimal height assigned to $b$, and the maximal height is assigned to $d$ and $f$.

Thus, we have been able to use the rotations to ensure we can cover the cases such that every vertex has the minimal and maximal height assigned to it for this particular path.

- $\{a, a + 1, a + 2, a + 3, a + 2, a + 1, a\}$:

For this path we have the minimal height assigned to $a$, and the maximal height assigned to $d$.

- Rotating by $120^\circ$ gives the following path: \{a + 2, a + 1, a, a + 1, a + 2, a + 3, a + 2\}. For this path we have the minimal height assigned to $c$, and the maximal height assigned to $f$.

- Rotating by $240^\circ$ gives the following path: \{a + 2, a + 3, a + 2, a + 1, a, a + 1, a + 2\}. For this path we have the minimal height assigned to $e$, and the maximal height assigned to $b$.

- Rotating by $180^\circ$ gives the following path: \{a + 3, a + 2, a + 1, a + 1, a + 2, a + 3\}. For this path we have the minimal height assigned to $d$, and the maximal height assigned to $a$.

- Rotating by $180^\circ$ then by $120^\circ$ gives the following path: \{a + 1, a + 2, a + 3, a + 2, a + 1, a + 1\}. For this path we have the minimal height assigned to $f$, and the maximal height assigned to $c$.

Thus, we have been able to use the rotations to ensure we can cover the cases such that every vertex has the minimal and maximal height assigned to it for this particular path.

### 2.4 Symmetric Yang-Baxter Equation: $a > 1$

Now that we have established that if we let $a = \min\{a, b, c, d, e, f\}$, then we obtain four different height paths and use rotations of $120^\circ$, $240^\circ$, $180^\circ$ and compositions to assign this minimal (and maximal) height to any of the 6 vertices on the tangles in the Symmetric Yang-Baxter Equation. We will now show that the Yang-Baxter equation holds for all paths by looking at the four main paths: \{a, a + 1, a, a + 1, a, a + 1, a\}, \{a, a + 1, a + 2, a + 1, a + 1, a\}, \{a, a + 1, a + 2, a + 1, a + 2, a + 1, a\} and \{a, a + 1, a + 2, a + 3, a + 2, a + 1, a\}.

#### 2.4.1 Path: \{a, a + 1, a, a + 1, a, a + 1, a\}

Here the Symmetric Yang-Baxter Equation becomes

\[
\frac{S_a^3 \sigma_g}{g_a^3 g_g^3} \alpha + 1 = \frac{S_{a+1}^3 \sigma_g}{g_{a+1}^3 g_g^3} \alpha + 1
\]

(2.4.1)
For the LHS of (2.4.1), the only valid values of \( g \) are \( g = a - 1, a + 1 \), thus expanding the LHS gives

\[
\frac{S^3 a g}{g^3 a g} \frac{a + 1}{a} w_2 w_3 \frac{a}{a} \frac{a}{a} = \frac{S^3 a g}{g^3 a g} \frac{a + 1}{a} w_2 w_3 \frac{a}{a} \frac{a}{a} + \frac{S^3 a g}{g^3 a g} \frac{a + 1}{a} w_2 w_3 \frac{a}{a} \frac{a}{a} (2.4.2)
\]

Now look at the first tangle on the RHS of (2.4.2), it can be seen that each of the faces have the configuration given in (2.2.8), thus we obtain the following equality

\[
\frac{S^3 a g}{g^3 a g} \frac{a + 1}{a} w_2 w_3 \frac{a}{a} \frac{a}{a} = \frac{S^3 a g}{g^3 a g} \frac{a + 1}{a} w_2 w_3 \frac{a}{a} \frac{a}{a} + \frac{S^3 a g}{g^3 a g} \frac{a + 1}{a} w_2 w_3 \frac{a}{a} \frac{a}{a}
\]

Now consider the second tangle on the RHS of (2.4.2), here each face has the configuration given in (2.2.11). Now since this configuration has non-zero weights assigned to both the identity and \( e_j \) operator, we can construct tangles from combinations of them. The total number of possible tangle configurations is \( 2^3 \), as the tangles are constructed from three lattice faces that can either act as an identity or \( e_j \) operator. Thus we obtain the following equality

\[
\frac{S^3 a g}{g^3 a g} \frac{a + 1}{a} w_2 w_3 \frac{a}{a} \frac{a}{a} = \frac{S^3 a g}{g^3 a g} \frac{a + 1}{a} w_2 w_3 \frac{a}{a} \frac{a}{a} + \frac{S^3 a g}{g^3 a g} \frac{a + 1}{a} w_2 w_3 \frac{a}{a} \frac{a}{a}
\]
Now let’s consider the RHS of (2.4.1), it can be seen that the only valid values of \( g \) are \( g = a, a + 2 \). So let’s expand the RHS

\[
S^3_{a+1}S^2_a \frac{g_a}{g_a+1} g_a^2 = S^3_{a+1}S^2_a \frac{g_a}{g_a+1} g_a^2 + S^3_{a+1}S^3_a \frac{g_a^3}{g_a+1} g_a^3 + S^3_{a+1}S^3_{a+2} \frac{g_a^3}{g_a+1} g_a^3 + \ldots
\]

First consider the second tangle on the RHS of (2.4.3), it can be seen that each face has the configuration given in (2.2.9), thus we obtain the following equality

\[
S^3_{a+1}S^2_a \frac{g_a}{g_a+1} g_a^2 = S^3_{a+1}S^3_a g_a^3 + S^3_{a+1}S^3_{a+2} g_a^3 + \ldots
\]

Now consider the first tangle given in the RHS of (2.4.3), it can be seen that each face has the configuration given in (2.2.11). Thus, we see that exactly \( 2^3 \) different tangle configurations occur as given below

Next, the tangles are equivalent under isotopy (i.e. two tangles are equivalent if the loop arrangement in one can be continuously deformed to look like the loop arrangement in the other), so we can just look at the external nodes and use that to see which tangles are equivalent. A group of equivalent tangles is called a connectivity class. Here we’ll see that there are 5 connectivity classes:
the first three are tautologies,

**Class I:**

\[
s_0(w_1)s_1(-w_2)s_1(-w_3) a + 1 = s_0(w_1)s_1(-w_2)s_1(-w_3) a + 1 \quad (2.4.4)
\]

**Class II:**

\[
s_1(-w_1)s_0(w_2)s_1(-w_3) a + 1 = s_1(-w_1)s_0(w_2)s_1(-w_3) a + 1 \quad (2.4.5)
\]

**Class III:**

\[
s_1(-w_1)s_1(-w_2)s_0(w_3) a + 1 = s_1(-w_1)s_1(-w_2)s_0(w_3) a + 1 \quad (2.4.6)
\]

Using the relation for \( \beta \) given in (??) and the identity shown in (A.2.1), we obtain the next two connectivity classes

**Class IV:**

\[
s_1(-w_1)s_1(-w_2)s_1(-w_3) a + 1 = s_0(w_1)s_0(w_2)s_1(-w_3) a + 1
\]

\[
+ s_0(w_1)s_1(-w_2)s_0(w_3) a + 1 = s_1(-w_1)s_0(w_2)s_0(w_3) a + 1
\]

\[
+ \frac{S_a}{S_{a+1}} s_0(w_1)s_0(w_2)s_0(w_3) a + 1 + \frac{S_{a+2}}{S_{a+1}} s_0(w_1)s_0(w_2)s_0(w_3) a + 1
\]

(2.4.7)
Here the Symmetric Yang-Baxter Equation becomes

\[ w \text{ parameter} \]

For the LHS of (2.4.9), the only valid value for \( g \) is \( g = a + 1 \). It can be seen that the face with parameter \( w_1 \) has the configuration given in (2.2.6), the face with parameter \( w_2 \) has the configuration given in (2.2.11), and the face with parameter \( w_3 \) has the configuration given in (2.2.6), thus we obtain the following expansion for the LHS of (2.4.9)

For the RHS of (2.4.9), the only valid value for \( g \) is \( g = a + 1 \). It can be seen that the only valid values of \( g \) are \( g = a, a + 2 \). So let's expand the RHS

\[ 2.4.2 \text{ Path:} \{a, a + 1, a + 2, a + 1, a, a + 1, a\} \]
Now for the first tangle it can be seen that the face with parameter $w_1$ has the configuration given in (2.2.11), the face with parameter $w_2$ has the configuration given in (2.2.8), and the face with parameter $w_3$ has the configuration given in (2.2.11). Thus we obtain the following expansion for the first tangle of the RHS of (2.4.9)

$$S_{a+3} S_{a+1} S_a \frac{s_3}{g_{a+1} g_a} a + 1 =$$

Now looking at the second tangle, it can be seen that the face with parameter $w_1$ has the configuration given by (2.2.9), the face with parameter $w_2$ has the configuration given by (2.2.10), and the face with parameter $w_3$ has the configuration given by (2.2.9). Thus we obtain the following expansion for the second tangle of the RHS of (2.4.9)

$$S_{a+3} S_{a+1} S_a \frac{s_2}{g_{a+1} g_a} s_1(-w_1) s_0(w_2) s_1(-w_3) a + 1 + S_{a+3} S_{a+1} S_a \frac{s_1}{g_{a} g_a} s_0(w_1) s_0(w_2) s_0(w_3) a + 1 =$$

Now by looking at the external nodes of the expansions, we can see which connectivity classes we obtain. From (??) and the identity given in (A.2.1), we obtain connectivity class II and connectivity class IV

**Class II:**

$$s_1(-w_1) s_0(w_2) s_1(-w_3) a + 1 = s_1(-w_1) s_0(w_2) s_1(-w_3) a + 1 \quad (2.4.12)$$
Class IV:

\[ s_1(-w_1)s_1(-w_2)s_1(-w_3) = s_0(w_1)s_0(w_2)s_1(-w_3) \]

\[ s_1(-w_1)s_0(w_2)s_0(w_3) \]

\[ + \frac{S_a}{S_{a+1}} s_0(w_1)s_0(w_2)s_0(w_3) \]

2.4.3 Path: \( \{a, a + 1, a + 2, a + 1, a + 2, a + 1, a\} \)

Here the symmetric Yang-Baxter Equation becomes

\[ \frac{S_a S_a^2 S_a^3}{g_a g_{a+2} g_{a+3}^2} a + 1 = \frac{S_{a+1} S_{a+2} S_{a+3}}{g_{a+1} g_{a+2} g_{a+3}^2} a + 1 \] (2.4.13)

For the LHS of (2.4.13), the only valid value for \( g \) is \( g = a + 1 \). The face with parameter \( w_1 \) has the configuration given in (2.2.11), the face with parameter \( w_2 \) has the configuration given in (2.2.9), and the face with parameter \( w_3 \) has the configuration given in (2.2.11). Thus we obtain the following expansion for the LHS of (2.4.13):

\[ \frac{S_a S_a^2 S_a^3}{g_a g_{a+2} g_{a+3}^2} a + 1 = \frac{S_{a+1} S_{a+2} S_{a+3}}{g_{a+1} g_{a+2} g_{a+3}^2} s_1(-w_1)s_1(-w_2)s_1(-w_3) a + 1 \]

(2.4.14)

Now let’s consider the RHS of (2.4.13), it can be seen that the only valid values of \( g \) are \( g = a, a + 2 \). So let’s expand the RHS.

\[ \frac{S_{a+1} S_a}{g_{a+1} g_a} a + 1 = \frac{S_{a+1} S_a}{g_{a+1} g_a} a + 1 \] (2.4.15)
Now for the first tangle it can be seen that the face with parameter $w_1$ has the configuration given in (2.2.8), the face with parameter $w_2$ has the configuration given in (2.2.8), and the face with parameter $w_3$ has the configuration given in (2.2.11). Thus it can be seen that the first tangle of (2.4.15) expands as follows,

\[
S_{a+1}S_3 S_2 g_{a+1}g_{a+2}^3 \\
\]

\[
\frac{S_{a+1}S_3 S_2 g_{a+1}g_{a+2}^2}{S_0(w_1)s_0(w_2)s_1(w_3)}
\]

\[
+ \frac{S_{a+1}S_3 S_2 g_{a+1}g_{a+2}^2}{S_0(w_1)s_0(w_2)s_0(w_3)}
\]

Now for the second tangle it can be seen that the face with parameter $w_1$ has the configuration given in (2.2.10), the face with parameter $w_2$ has the configuration given in (2.2.10), and the face with parameter $w_3$ has the configuration given in (2.2.9). From this it can be seen that the second tangle of (2.4.15) can be expanded as follows,

\[
S_{a+1}S_2 S_3 g_{a+1}g_{a+2}^3 \\
\]

\[
\frac{S_{a+1}S_2 S_3 g_{a+1}g_{a+2}^2}{S_0(w_1)s_1(w_2)s_1(w_3)}
\]

\[
+ \frac{S_{a+1}S_2 S_3 g_{a+1}g_{a+2}^2}{S_0(w_1)s_2(w_2)s_1(w_3)}
\]

From (2.2.10) and (A.2.1), we obtain the following connectivity classes,

**Class III:**

\[
s_1(w_1)s_1(w_2)s_0(w_3) = s_1(w_1)s_1(w_2)s_0(w_3)
\]
Class IV:

\[ s_1(-w_1)s_1(-w_2)s_1(-w_3) = s_0(w_1)s_0(w_2)s_0(w_3) \]

\[ + s_1(-w_1)s_0(w_2)s_0(w_3) + \frac{S_a}{S_{a+1}}s_0(w_1)s_0(w_2)s_0(w_3) \]

(2.4.17)

2.4.4 Path: \( \{a, a + 1, a + 2, a + 1, a + 2, a + 1, a\} \)

Here the Symmetric Yang-Baxter Equation becomes

\[ \frac{S_a S_{a+2} S_g}{g_a g_{a+2} g_3^3} a + 1 + 2 = \frac{S_a S_{a+2} S_g}{g_a g_{a+2} g_3^3} a + 3 \]

(2.4.18)

For the LHS of (2.4.18), the only valid value for \( g \) is \( g = a + 1 \). The face with parameter \( w_1 \) has the configuration given by (2.2.7), the face with parameter \( w_2 \) has the configuration given by (2.2.6), and the face with parameter \( w_3 \) has the configuration given by (2.2.8). Thus the LHS of (2.4.18) simply expands to

\[ \frac{S_a S_{a+2} S_g}{g_a g_{a+2} g_3^3} a + 1 + 2 = \frac{S_a S_{a+2} S_g}{g_a g_{a+2} g_3^3} a + 3 \]

(2.4.19)

For the RHS of (2.4.18), the only valid value for \( g \) is \( g = a + 2 \). The face with parameter \( w_1 \) has the configuration given in (2.2.6), the face with parameter \( w_2 \) has the configuration in (2.2.6), and the face with parameter \( w_3 \) has the configuration in (2.2.9). Thus the RHS of (2.4.18) simply expands to

\[ \frac{S_a S_{a+2} S_g}{g_a g_{a+2} g_3^3} a + 1 + 2 = \frac{S_a S_{a+2} S_g}{g_a g_{a+2} g_3^3} a + 3 \]

(2.4.20)

Thus, we obtain the following connectivity class.
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Class III:

\[ s_1(-w_1)s_1(-w_2)s_0(w_3) = s_1(-w_1)s_1(-w_2)s_0(w_3) \quad (2.4.21) \]

Now it can be seen that the Yang-Baxter Equations previously shown for paths \{a, a + 1, a + 2, a + 1, a, a + 1, a\}, \{a, a + 1, a + 2, a + 1, a + 2, a + 1, a\} and \{a, a + 1, a + 2, a + 3, a + 2, a + 1, a\} hold for even \(a = 1\), whereas the Yang-Baxter previously shown for path \{a, a + 1, a, a + 1, a, a + 1, a\}, has only been shown to hold for \(a > 1\). Thus, we will now show that the Yang-Baxter Equation holds for \(a = 1\).

2.4.5 Path: \{1, 2, 1, 2, 1, 2, 1\}

For the LHS of (2.4.22) it can be seen that the only valid value for \(g\) is \(g = 2\). The face with parameter \(w_1\) has the configuration given by (2.2.10), the face with parameter \(w_2\) has the configuration given by (2.2.10), and the face with parameter \(w_3\) has the configuration given by (2.2.10). Thus, the tangle on the LHS of (2.4.22) expands as follows,
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Now let’s consider the RHS of (2.4.22). It can be seen that the valid values for \( g \) is \( g = 2, 3 \), thus expanding gives

\[
\frac{S^3_S}{g_2^2 g_0^3} 1 \begin{array}{c}
2 \\
1 \\
w_3
\end{array} 2 = \frac{S^3_S}{g_2^3 g_0^1} 1 \begin{array}{c}
2 \\
1 \\
w_3
\end{array} 2 + \frac{S^3_S}{g_2^2 g_0^3} 1 \begin{array}{c}
2 \\
1 \\
w_3
\end{array} 2 \tag{2.4.23}
\]

Looking at the second tangle, it can be seen that the face with parameter \( w_1 \) has the configuration given by (2.2.9), the face with parameter \( w_2 \) has the configuration given by (2.2.9), the face with parameter \( w_3 \) has the configuration given by (2.2.9). Thus the second tangle expands as follows,

\[
\frac{S^3_S}{g_2^2 g_3^3} 1 \begin{array}{c}
w_3 \\
w_2
\end{array} 2 = \frac{S^3_S}{g_1^2 g_2^3} s_0(w_1) s_0(w_2) s_0(w_3) 1 \begin{array}{c}
w_1 \\
w_3 \\
w_2
\end{array} 2
\]

Now consider the first tangle, it can be seen that the face with parameter \( w_1 \) has the configuration given by (2.2.11), the face with parameter \( w_2 \) has the configuration given by (2.2.11), and the face with parameter \( w_3 \) has the configuration given by (2.2.11). Thus the first tangle expands as follows,

\[
\frac{S^3_S}{g_2^2 g_1^3} 1 \begin{array}{c}
w_3 \\
w_2
\end{array} 2 = \frac{S^3_S}{g_2^3 g_1^1} s_1(-w_1) s_1(-w_2) s_1(-w_3) 1 \begin{array}{c}
w_1 \\
w_3 \\
w_2
\end{array} 2 + \frac{S^3_S}{g_2^2 g_1^3} s_0(w_1) s_1(-w_2) s_1(-w_3) 1 \begin{array}{c}
w_1 \\
w_3 \\
w_2
\end{array} 2 + \frac{S^3_S}{g_2^3 g_1^1} s_1(-w_1) s_1(-w_2) s_0(w_3) 1 \begin{array}{c}
w_1 \\
w_3 \\
w_2
\end{array} 2 + \frac{S^3_S}{g_2^2 g_1^3} s_0(w_1) s_1(-w_2) s_0(w_3) 1 \begin{array}{c}
w_1 \\
w_3 \\
w_2
\end{array} 2
\]

\[
+ \frac{S^2_S}{g_2^1 g_1^3} s_1(-w_1) s_0(w_2) s_1(-w_3) 1 \begin{array}{c}
w_1 \\
w_3 \\
w_2
\end{array} 2 + \frac{S^2_S}{g_2^1 g_1^3} s_1(-w_1) s_1(-w_2) s_0(w_3) 1 \begin{array}{c}
w_1 \\
w_3 \\
w_2
\end{array} 2 + \frac{S^2_S}{g_2^2 g_1^1} s_0(w_1) s_1(-w_2) s_0(w_3) 1 \begin{array}{c}
w_1 \\
w_3 \\
w_2
\end{array} 2
\]

\[
+ \frac{S^2_S}{g_2^1 g_1^3} s_1(-w_1) s_0(w_2) s_0(w_3) 1 \begin{array}{c}
w_1 \\
w_3 \\
w_2
\end{array} 2 + \frac{S^4_S}{g_2^2 g_1^1} s_0(w_1) s_0(w_2) s_0(w_3) 1 \begin{array}{c}
w_1 \\
w_3 \\
w_2
\end{array} 2
\]

From (??) and (A.2.1), we obtain the following connectivity classes.
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Class I:

\[ s_0(w_1) s_1(-w_2) s_1(-w_3) = s_0(w_1) s_1(-w_2) s_1(-w_3) \] (2.4.24)

Class II:

\[ s_1(-w_1) s_0(w_2) s_1(-w_3) = s_1(-w_1) s_0(w_2) s_1(-w_3) \] (2.4.25)

Class III:

\[ s_1(-w_1) s_1(-w_2) s_0(w_3) = s_1(-w_1) s_1(-w_2) s_0(w_3) \] (2.4.26)

Class IV:

\[ s_1(-w_1) s_1(-w_2) s_1(-w_3) = s_0(w_1) s_0(w_2) s_1(-w_3) + s_0(w_1) s_1(-w_2) s_0(w_3) + \frac{S_1}{S_2} s_0(w_1) s_0(w_2) s_0(w_3) \] (2.4.27)
Class V:

\[ s_1(-w_1)s_1(-w_2)s_1(-w_3) \equiv s_0(w_1)s_0(w_2)s_1(-w_3) + s_0(w_1)s_1(-w_2)s_0(w_3) + \beta s_0(w_1)s_0(w_2)s_0(w_3) \]  

(2.4.28)
Chapter 3

D-Type Models

For D-Type models, the lattice heights sit on the graph

\[ \begin{align*}
\begin{array}{cccccccc}
3' & 2 & 3 & 4 & 5 & \cdots & L-2 & L-1 \\
\end{array}
\end{align*} \]

Figure 3.1: \( D_{L-1} \)

If we order the nodes as \( \{1, 3', 2, 3, 4, \ldots, L-1\} \), the D-Type model has the following eigenvectors

\[ S = (1, 1, 2 \cos \lambda, 2 \cos 2\lambda, \ldots, 2 \cos((L-1)\lambda)), \quad m_j = 1, 3, 5, \ldots, 2L-3 \] (3.0.1)

and

\[ S = (0, 0, \ldots, 0, -1, 1), \quad m_j = L-1 \] (3.0.2)

Here the crossing parameter \( \lambda \) is given by,

\[ \lambda = \frac{m_j \pi}{2(L-1)}, \quad m_j = 1, 3, 5, \ldots, 2L-3 \] (3.0.3)

The entries \( S_a \) of the eigenvectors are given by

\[ S_a = \begin{cases} 
1 & a \in \{1, 3'\} \\
2 \cos((a-1)\lambda) & a \in \{2, 3, \ldots, L-1\}
\end{cases} \] (3.0.4)

It is expected that D-type logarithmic minimal (loop) models are obtained in the logarithmic limit

\[ m_j \to \infty, \quad L \to \infty, \quad \frac{m_j}{2(L-1)} \to \frac{p}{2p'}, \quad p \text{ and } 2p' \text{ coprime} \] (3.0.5)

3.1 Loop Fugacities

Here we move from the height representation to a loop representation for D-type models and discuss loop fugacities. We use a bold black loop as follows

\[ \beta = a \begin{array}{c}
\circ \\
\oplus
\end{array} b = \begin{cases} 
\frac{S_2}{S_a}, & a = 1, 3' \\
\frac{S_{a-1} + S_{a+1}}{S_a}, & a \geq 3
\end{cases} \] (3.1.1)
The solid blue loops carry a spin-$\frac{1}{2}$ charge. The solid red loops carry a spin-1 charge. Using the previous elementary trigonometric identities, we conclude that the various D-type loop fugacities are given by

$$
= \beta = 2 \cos \lambda, \quad = \beta^2 - \frac{1}{\beta}, \quad = \frac{1}{\beta}
$$

There is an equality of the fugacities of solid red and blue loops.

### 3.2 Boltzmann Weights

To establish the Boltzmann weights, first consider the case when $a, b, c, d \neq 3'$. Then the weights become

$$
\begin{align*}
\begin{array}{c}
\bar{d} \\
\bar{c}
\end{array} & \begin{array}{c}
\bar{d} \\
\bar{c}
\end{array} = W\left(\begin{array}{c}
\bar{d} \\
\bar{c}
\end{array} \begin{array}{c}
\bar{a} \\
\bar{b}
\end{array} | u \right) = s_1(-u) \delta(a, c) + \frac{S_d S_c g_c}{S_b S_d g_a} s_0(u) \delta(b, d)
\end{align*}
$$

(3.2.1)

Where $g_a = \epsilon_a \sqrt{S_a}$ and the sign factor $\epsilon_a$ is such that $\epsilon_1 = \epsilon_2 = 1$, $\epsilon_3 = \epsilon_4 = -1$, $\epsilon_5 = \epsilon_6 = 1, \ldots, \epsilon_{L-2} = \epsilon_{L-1} = \pm 1$. This gives the following relation $\epsilon_a = -\epsilon_{a \pm 2}$, and we can now calculate the value for $\epsilon_a$ as follows.

$$
\epsilon_a = (-1)^{\frac{a-1}{2}}, \quad a \neq 3'
$$

(3.2.2)

Thus, (3.2.1) can be re-written as follows,

$$
\begin{align*}
\begin{array}{c}
\bar{d} \\
\bar{c}
\end{array} & \begin{array}{c}
\bar{d} \\
\bar{c}
\end{array} = W\left(\begin{array}{c}
\bar{d} \\
\bar{c}
\end{array} \begin{array}{c}
\bar{a} \\
\bar{b}
\end{array} | u \right) = s_1(-u) \delta(a, c) + \frac{S_d S_c \epsilon_c \sqrt{S_c}}{S_b S_d \epsilon_a \sqrt{S_a}} s_0(u) \delta(b, d)
\end{align*}
$$

$$
= s_1(-u) \delta(a, c) + \frac{\epsilon_a S_c}{\epsilon_c S_b} s_0(u) \delta(b, d)
$$

Now let’s consider when $a, b, c, d = 3'$. In order to distinguish loops contributed by $3'$, let us introduce some new loop notation:

$$
\begin{align*}
\begin{array}{c}
1 \\
2
\end{array} & \begin{array}{c}
1 \\
2
\end{array} \quad \begin{array}{c}
1 \\
2
\end{array} & \begin{array}{c}
1 \\
2
\end{array} \quad \begin{array}{c}
1 \\
2
\end{array} & \begin{array}{c}
1 \\
2
\end{array} 
\end{align*}
$$

(3.2.3)

(3.2.4)

(3.2.5)

(3.2.6)

(3.2.7)
Picking $\epsilon_3 = \epsilon_1 = 1$ gives $\mathbb{Z}_2$ symmetry under interchanging the heights $3'$ and 1 in Figure 3.1. Thus, this should also be present in the Boltzmann weights, hence we obtain the following relations,

\[ u_{3'}^2 = u_1^2 = s_1(-u) + \frac{s_0(u)}{S_2} \] (3.2.8)

\[ u_2 = u_1 = s_1(-u) + S_2s_0(u) \] (3.2.9)

\[ u_3' = u_2 = s_0(u) \] (3.2.10)

\[ u_2 = u_3' = S_2s_0(u) \] (3.2.11)

A general formula for the Boltzmann weights for $a, b, c, d \in \{3', 1, 2, 3, \ldots, L - 1\}$ is

\[ u_{abcd} = W(d \quad c \quad a \quad b \quad u) = s_1(-u)\delta(a, c) + \frac{\epsilon_a}{\epsilon_c} S_c s_0(u) \delta(b, d) \] (3.2.12)

where the bolder black loops incorporate this new loop notation, and the sign-changing factor for $a \in \{3', 1, 2, 3, \ldots, L - 1\}$ is

\[ \epsilon_a = \begin{cases} (-1)^{\frac{a-1}{2}}, & a \neq 3' \\ 1, & a = 3' \end{cases} \] (3.2.13)

### 3.3 Yang-Baxter Equation

We now wish to show the Yang-Baxter equation holds for all allowed heights in D-type models.

\[ u_{f} = u_{d} = u_{v} = u_{g} = \] (3.3.1)

To reduce the number of height paths that need to be considered, we’ll use the symmetric Yang-Baxter Equation given in (2.3.10), although since this was constructed by rotating the faces such that the small red arcs are in the centre by using the local crossing relations given in (2.2.13) - (2.2.21), we need to interchange the $g_a$ coefficients with $\epsilon_a$.

\[ \frac{S_d S_c S_e S_g}{\epsilon_a \epsilon_c \epsilon_e \epsilon_g} = \frac{S_b S_d S_f S_g}{\epsilon_b \epsilon_d \epsilon_f \epsilon_g} \] (3.3.2)
From this, we can set \( a = \min\{a, b, c, d, e, f\} \) and use rotational symmetries to see that the YBE holds for when \( b, c, d, e, \) or \( f \) is the minimal height. Now since, the heights for D-type models branch out at \( a = 2 \), we have to consider two cases for heights, \( a \in \{1, 3', 2\} \) and \( a \geq 3 \)

First consider the general case for \( a \geq 3 \). The height paths that need to be inspected here, are exactly the same as those for general \( a \) for A-type models, thus using the same analysis used in 2.3.3 we find that we need to consider the following paths: \( \{a, a + 1, a, a + 1, a, a + 1, a\} \), \( \{a, a + 1, a + 2, a + 1, a, a + 1, a\} \), \( \{a, a + 1, a + 2, a + 1, a, a + 1, a\} \), \( \{a, a + 1, a + 2, a + 1, a + 1, a + 1, a\} \), \( \{a, a + 1, a + 1, a + 2, a + 3, a + 2, a + 1, a\} \).

After going through those paths, we will need to look at the special cases. It was seen that for general \( a \) (3.3.3) was only shown for \( a \geq 3 \), thus we need to also inspect the paths \( \{1, 2, 1, 2, 1, 2\} \) and \( \{2, 3, 2, 3, 2, 3\} \) to show (3.3.3) holds for all \( a \in \{1, 3', 2, 3, 3, \ldots, L - 1\} \). The Yang-Baxter equations given in (3.3.6) - (3.3.17) were shown to only hold for \( a \geq 2 \), thus we need to inspect paths \( \{1, 2, 3, 2, 1, 2, 1\} \), \( \{1, 2, 3, 2, 3, 2, 1\} \), \( \{1, 2, 3, 4, 3, 2, 1\} \). Now \( 3' \) is also adjacent to \( 2 \), thus we also need to consider the paths \( \{1, 2, 3', 2, 1, 2\} \), \( \{1, 2, 3', 2, 3', 2, 1\} \) and \( \{1, 2, 3', 2, 3, 2, 1\} \). We are able to omit paths starting at \( 3' \), by rotational symmetries, and the fact that the Boltzmann weights are symmetric under interchanging \( 1 \leftrightarrow 3' \) by construction.

### 3.3.1 Path: \( \{a, a + 1, a, a + 1, a, a + 1, a\} \)

\[
\frac{S^3_a S_g}{e^3_a e^3_g} = a + 1 = \frac{S^3_{a+1} S_g}{e^3_{a+1} e^3_g} \tag{3.3.3}
\]

For the LHS of (3.3.3), it can be seen that the values \( g \) can take, is \( g = a - 1, a + 1 \). Thus we can express the LHS as shown in (3.3.4)

\[
\frac{S^3_a S_g}{e^3_a e^3_g} = a + 1 = \frac{S^3_{a-1} S_g}{e^3_{a-1} e^3_g} s_0(w_1)s_0(w_2)s_0(w_3) a + 1 + \frac{S^3_a S_{a+1}}{e^3_a e^3_{a+1}} \tag{3.3.4}
\]

Now consider the second tangle, each of the lattice faces can be expressed as a sum of faces where one has it’s vertices with heights \( a \) are matched and the other has it’s vertices with heights \( a + 1 \) matched.
For the RHS of (3.3.3), it can be seen that the values \( g \) can take is, \( g = a, a+2 \). Thus we can express the RHS as shown in (3.3.5)
Now consider the first tangle, each of the lattice faces can be expressed as a sum of faces where one has its vertices with heights \( a \) are matched and the other has its vertices with heights \( a + 1 \) matched.

\[
\begin{align*}
\frac{S^3_{a+1}}{a+1} \frac{S^3_a}{a+1} & \quad \frac{S^2_{a+1}}{a+1} \frac{S^2_a}{a+1} + \frac{S^2_{a+1}}{a+1} \frac{S^2_a}{a+1} s_1(-w_1)s_1(-w_2)s_1(-w_3) a + 1 \\
+ \frac{S^2_{a+1}}{a+1} \frac{S^2_a}{a+1} s_1(-w_1)s_0(w_2)s_1(-w_3) a + 1 & \quad + \frac{S^2_{a+1}}{a+1} \frac{S^2_a}{a+1} s_1(-w_1)s_1(-w_2)s_0(w_3) a + 1 \\
+ \frac{S^2_{a+1}}{a+1} \frac{S^2_a}{a+1} s_0(w_1)s_0(w_2)s_1(-w_3) a + 1 & \quad + \frac{S^2_{a+1}}{a+1} \frac{S^2_a}{a+1} s_0(w_1)s_1(-w_2)s_0(w_3) a + 1 \\
+ \frac{S^2_{a+1}}{a+1} \frac{S^2_a}{a+1} s_1(-w_1)s_0(w_2)s_0(w_3) a & \quad + \frac{S^2_{a+1}}{a+1} \frac{S^2_a}{a+1} s_0(w_1)s_0(w_2)s_0(w_3) a + 1
\end{align*}
\]

From this we obtain the following connectivity classes:

**Class I:**

\[
\frac{S^3_{a+1}}{a+1} \frac{S^3_a}{a+1} s_0(w_1)s_1(-w_2)s_1(-w_3) a + 1 = s_0(w_1)s_1(-w_2)s_1(-w_3) a + 1
\]

**Class II:**

\[
\frac{S^2_{a+1}}{a+1} \frac{S^2_a}{a+1} s_1(-w_1)s_0(w_2)s_1(-w_3) a + 1 = s_1(-w_1)s_0(w_2)s_1(-w_3) a + 1
\]

**Class III:**

\[
\frac{S^2_{a+1}}{a+1} \frac{S^2_a}{a+1} s_1(-w_1)s_1(-w_2)s_0(w_3) a + 1 = s_1(-w_1)s_1(-w_2)s_0(w_3) a + 1
\]
Class IV:

\[
s_1(-w_1)s_1(-w_2)s_1(-w_3) a^{a+1} = s_0(w_1)s_0(w_2)s_0(-w_3) a^{a+1}
\]

\[
+ s_0(w_1)s_1(-w_2)s_0(w_3) a^{a+1} + s_1(-w_1)s_0(w_2)s_0(w_3) a^{a+1}
\]

\[
+ \beta s_0(w_1)s_0(w_2)s_0(w_3) a^{a+1}
\]

Class V:

\[
s_1(-w_1)s_1(-w_2)s_1(-w_3) a^{a+1} = s_0(w_1)s_0(w_2)s_0(-w_3) a^{a+1}
\]

\[
+ s_0(w_1)s_1(-w_2)s_0(w_3) a^{a+1} + s_1(-w_1)s_0(w_2)s_0(w_3) a^{a+1}
\]

\[
+ \beta s_0(w_1)s_0(w_2)s_0(w_3) a^{a+1}
\]

The first three classes are tautologies, whereas class IV and V arise from the trigonometric identity given in (A.2.1) and the factor of \( \beta \) comes out from it’s relation to Perron-Frobenius eigenvector entries in (??)

### 3.3.2 Path: \( \{a, a + 1, a + 2, a + 1, a, a + 1, a\} \)

\[
\frac{S_a^2 S_{a+2} S_g}{\epsilon_a^2 \epsilon_{a+2} \epsilon_g^2} a^{a+1} + \frac{S_{a+1}^3 S_g}{\epsilon_{a+1} \epsilon_g^3} a^{a+1} = \frac{S_{a+1}^3 S_g}{\epsilon_{a+1} \epsilon_g^3} a^{a+1}
\]

(3.3.6)

First consider the LHS of (3.3.6), it can be seen that the only possible value \( g \) can take is \( g = a + 1 \). The face with parameter \( w_3 \) on the resulting tangle can be expressed as a sum of faces with one of the faces having vertices with heights \( a \) matched, and the other with heights \( a + 1 \) matched as
shown in (3.3.7).

Now consider the RHS of (3.3.6). The only possible values $g$ can take, is $g = a, a + 2$, thus, the RHS can be expanded as shown in (3.3.9).

Consider the first tangle, it can be seen that faces with parameters $w_1$ and $w_3$ can be written as a sum of faces with heights $a$ matched, and $a + 1$ matched.

Now consider the second tangle, it can be seen that face with parameter $w_2$ can be written as a
sum of faces with heights $a + 1$ matched, and $a + 2$ matched.

From this we obtain the following connectivity classes:

Class III:

$$s_1(-w_1)s_1(-w_2)s_0(w_3) a + 1 = s_1(-w_1)s_1(-w_2)s_0(w_3) a + 1$$  (3.3.10)

Class IV:

$$s_1(-w_1)s_1(-w_2)s_1(-w_3) a + 1 = s_0(w_1)s_0(w_2)s_1(-w_3) a + 1$$

$$+ s_0(w_1)s_1(-w_2)s_0(w_3) a + 1 + s_1(-w_1)s_0(w_2)s_0(w_3) a + 1$$

$$+ \beta s_0(w_1)s_0(w_2)s_0(w_3) a + 1$$

Where connectivity class III is a tautology, and connectivity class IV comes from the identity given in (A.2.1). Now for the LHS of (3.3.6), $g$ takes the value $g = a + 1$, and for the LHS of (3.3.6), $g = a, a + 2$, this implies that (3.3.6) holds for $a \geq 2$.

3.3.3 Path: $\{a, a+1, a+2, a+1, a+2, a+1, a\}$

First consider the LHS of (3.3.11), it can be seen that the only possible value $g$ can take is $g = a + 1$. Now consider the resulting tangle, it can be seen that the face with parameter $w_3$ can be written
as the sum of two faces where one face has vertices with heights \( a + 1 \) matched, and the other face has vertices with heights \( a + 2 \) as shown in (3.3.12).

\[
S_a S_{a+2} S_{a+1} \frac{\epsilon_a \epsilon_{a+2} \epsilon_{a+1}}{a^3} = \frac{S_a S_{a+2} S_{a+1}}{\epsilon_a \epsilon_{a+2} \epsilon_{a+1}} \epsilon_a \epsilon_{a+2} \epsilon_{a+1} S_a (w_1) s_0 (w_2) s_1 (w_3) a + 1 \ a + 2 \ (3.3.12)
\]

\[
\frac{-S_a S_{a+2} S_{a+1}}{\epsilon_a \epsilon_{a+2} \epsilon_{a+1}} \epsilon_a \epsilon_{a+2} \epsilon_{a+1} S_a (w_1) s_0 (w_2) s_1 (w_3) a + 1 \ a + 2 \ (3.3.13)
\]

Now consider the RHS of (3.3.11), it can be seen that the values \( g \) can take is \( g = a, a + 2 \). The RHS can then be expanded as shown in (3.3.14).

\[
\frac{S^3_a S_{a+1} S_g}{\epsilon_a \epsilon_{a+1} \epsilon_g} a + 1 \ a + 2 \ a + 1 \ = \ \frac{S^3_a S_{a+1} S_g}{\epsilon_a \epsilon_{a+1} \epsilon_g} a + 1 \ a + 2 \ a + 1 \ (3.3.14)
\]

We will first consider the first tangle. The lattice face with parameter \( w_3 \) can be written as a sum of faces where one face has vertices with heights \( a + 1 \) matched, and the other face with heights \( a + 1 \) matched, from this we get the following expansion.

\[
\frac{S^3_a S_{a+1} S_a}{\epsilon_a \epsilon_{a+1} \epsilon_a} a + 1 \ a + 2 \ a + 1 \ = \ \frac{S^3_a S_{a+1} S_a}{\epsilon_a \epsilon_{a+1} \epsilon_a} a + 1 \ a + 2 \ a + 1 \ (3.3.15)
\]

\[
\frac{S^2_a S_{a+2} S_{a+1}}{\epsilon_a \epsilon_{a+1} \epsilon_{a+2}} s_0 (w_1) s_0 (w_2) s_1 (w_3) a + 1 \ a + 2 \ (3.3.16)
\]

Now consider the second tangle. The faces with parameters \( w_1 \) and \( w_2 \) can be expressed as a sum of faces where one face has vertices with heights \( a + 1 \) matched, and the other face has vertices.
with heights \( a + 2 \) matched.

\[
S_{a+1}^3 S_{a+2}^3 \frac{a+2}{\epsilon_3^3 a + 1 \epsilon_3^3 a + 2} =
\]

\[
S_{a+1} S_{a+2} S_{a+2} \frac{a+2}{\epsilon_3 a \epsilon_3 a+1 \epsilon_3 a+2} s_1 (-w_1) s_1 (-w_2) s_0 (w_3) a + 1 +
\]

\[
S_{a} S_{a+1} S_{a+2} \frac{a+2}{\epsilon_3 a \epsilon_3 a+1 \epsilon_3 a+2} s_0 (w_1) s_1 (-w_2) s_0 (w_3) a + 1 +
\]

\[
S_{a} S_{a+1} S_{a+2} \frac{a+2}{\epsilon_3 a \epsilon_3 a+1 \epsilon_3 a+2} s_1 (-w_1) s_0 (w_2) s_0 (w_3) a + 1 +
\]

\[
S_{a} S_{a+1} S_{a+2} \frac{a+2}{\epsilon_3 a \epsilon_3 a+1 \epsilon_3 a+2} s_1 (-w_1) s_0 (w_2) s_0 (w_3) a + 1 +
\]

From this we obtain the following connectivity classes:

**Class III:**

\[
s_1 (-w_1) s_1 (-w_2) s_0 (w_3) a + 1 = s_1 (-w_1) s_1 (-w_2) s_0 (w_3) a + 1
\]

**Class IV:**

\[
s_1 (-w_1) s_1 (-w_2) s_1 (-w_3) a + 1 =
\]

\[
s_0 (w_1) s_0 (w_2) s_1 (-w_3) a + 1 + s_0 (w_1) s_1 (-w_2) s_0 (w_3) a + 1
\]

\[
s_1 (-w_1) s_0 (w_2) s_0 (w_3) a + 1 + \beta s_1 (-w_1) s_0 (w_2) s_0 (w_3) a + 1
\]

Where connectivity class III is a tautology, and connectivity class IV comes from the identity given in (A.2.1). Now for the LHS of (3.3.11), \( g \) takes the value \( g = a + 1 \), and for the LHS of (3.3.6), \( g = a, a + 2 \), this implies that (3.3.6) holds for \( a \geq 2 \).
3.3.4 Path: \( \{a, a+1, a+2, a+3, a+2, a+1, a\} \)

\[
\frac{S_aS_{a+2}S_g}{\epsilon_a\epsilon_{a+2}\epsilon_g} = \frac{S_{a+1}S_{a+3}S_g}{\epsilon_{a+1}\epsilon_{a+3}\epsilon_g} S_{a+1}(s_1(-w_1)s_1(-w_2)s_0(w_3) a \quad (3.3.17)
\]

First consider the LHS of (3.3.17), it can be seen that the only possible value \( g \) can take is \( g = a+1 \), thus the resulting tangle is given in (3.3.18)

\[
\frac{S_aS_{a+2}S_g}{\epsilon_a\epsilon_{a+2}\epsilon_g} = \frac{S_{a+1}S_{a+3}S_g}{\epsilon_{a+1}\epsilon_{a+3}\epsilon_g} s_1(-w_1)s_1(-w_2)s_0(w_3) a \quad (3.3.18)
\]

Now consider the RHS of (3.3.17), it can be seen that the only possible value \( g \) can take is \( g = a+1 \), thus the resulting tangle is given in (3.3.19)

\[
\frac{S_{a+1}S_{a+3}S_g}{\epsilon_{a+1}\epsilon_{a+3}\epsilon_g} = \frac{S_aS_{a+1}S_{a+2}S_{a+3}}{\epsilon_a\epsilon_{a+1}\epsilon_{a+2}\epsilon_{a+3}} s_1(-w_1)s_1(-w_2)s_0(w_3) a \quad (3.3.19)
\]

From this, we simply acquire the following connectivity class:

**Class III:**

\[
s_1(-w_1)s_1(-w_2)s_0(w_3) a = s_1(-w_1)s_1(-w_2)s_0(w_3) a \quad (3.3.20)
\]

Now for the LHS of (3.3.17), \( g \) takes the value \( g = a + 1 \), and for the LHS of (3.3.6), \( g = a + 2 \), this implies that (3.3.6) holds for \( a \geq 2 \).

3.3.5 Path: \( \{1, 2, 1, 2, 1, 2, 1\} \)

\[
\frac{S_1S_2S_g}{\epsilon_1^2} = \frac{S_1S_2S_g}{\epsilon_1^2} s_1(-w_1)s_1(-w_2)s_0(w_3) a \quad (3.3.21)
\]

From the graph given in Figure 3.1, it can be seen that for the LHS (3.3.21), the only possible value for \( g \) is \( g = 2 \), and for the RHS the possible values for \( g \) are \( g = 1, 3' \) or 3. Expanding the
LHS gives

\[
S_2 1 \quad \begin{array}{c}
1 \\
2 \\
1 \\
2 \\
1 \\
2
\end{array} = \quad \begin{array}{c}
1 \\
2 \\
1 \\
2 \\
1 \\
2
\end{array}
\]

\[
S_2 s_1(-w_1)s_1(-w_2)s_1(-w_3) \quad \begin{array}{c}
1 \\
2 \\
1 \\
2 \\
1 \\
2
\end{array} + S_2 s_0(w_1)s_1(-w_2)s_1(-w_3) \quad \begin{array}{c}
1 \\
2 \\
1 \\
2 \\
1 \\
2
\end{array}
\]

\[
S_2^2 s_1(-w_1)s_0(w_2)s_1(-w_3) \quad \begin{array}{c}
1 \\
2 \\
1 \\
2 \\
1 \\
2
\end{array} + S_2^2 s_1(-w_1)s_1(-w_2)s_0(w_3) \quad \begin{array}{c}
1 \\
2 \\
1 \\
2 \\
1 \\
2
\end{array}
\]

\[
S_2^3 s_0(w_1)s_0(w_2)s_1(-w_3) \quad \begin{array}{c}
1 \\
2 \\
1 \\
2 \\
1 \\
2
\end{array} + S_2^3 s_0(w_1)s_1(-w_2)s_0(w_3) \quad \begin{array}{c}
1 \\
2 \\
1 \\
2 \\
1 \\
2
\end{array}
\]

\[
S_2^3 s_1(-w_1)s_0(w_2)s_0(w_3) \quad \begin{array}{c}
1 \\
2 \\
1 \\
2 \\
1 \\
2
\end{array} + S_2^4 s_0(w_1)s_0(w_2)s_0(w_3) \quad \begin{array}{c}
1 \\
2 \\
1 \\
2 \\
1 \\
2
\end{array}
\]

Expanding the RHS gives

\[
\frac{S_3^3 S_2}{e_2^3} \quad \begin{array}{c}
1 \\
2 \\
1 \\
2 \\
1 \\
2
\end{array} = \quad \begin{array}{c}
1 \\
2 \\
1 \\
2 \\
1 \\
2
\end{array}
\]

\[
S_2^3 1 \quad \begin{array}{c}
1 \\
2 \\
1 \\
2 \\
1 \\
2
\end{array} + s_0(w_1)s_0(w_2)s_0(w_3) \quad \begin{array}{c}
1 \\
2 \\
1 \\
2 \\
1 \\
2
\end{array}
\]

\[
+ s_3 s_0(w_1)s_0(w_2)s_0(w_3) \quad \begin{array}{c}
1 \\
2 \\
1 \\
2 \\
1 \\
2
\end{array}
\]

(3.3.22)

Now for the first tangle it can be seen that all three faces can be decomposed into sums of faces where one face has it’s diagonal matching vertices with heights 1, and the other face with it’s
diagonal matching vertices with heights 2.

\[ S^2_3 = S^3_2 S^2_1 = S^3_2 S^2_0 \]

\[ S^3_2 s_1(-w_1)s_1(-w_2)s_1(-w_3)\]

\[ = S^3_2 s_0(w_1)s_1(-w_2)s_1(-w_3) \]

\[ S^2_2 s_1(-w_1)s_0(w_2)s_1(-w_3)\]

\[ = S^2_2 s_1(-w_1)s_0(w_2)s_1(-w_3) \]

\[ S_2 s_0(w_1)s_0(w_2)s_1(-w_3)\]

\[ = S_2 s_0(w_1)s_0(w_2)s_1(-w_3) \]

\[ S_2 s_1(-w_1)s_0(w_2)s_0(w_3)\]

\[ = S_2 s_1(-w_1)s_0(w_2)s_0(w_3) \]

From this, we acquire the following connectivity classes:

**Class I:**

\[ s_0(w_1)s_1(-w_2)s_1(-w_3) = s_0(w_1)s_1(-w_2)s_1(-w_3) \] (3.3.23)

**Class II:**

\[ s_1(-w_1)s_0(w_2)s_1(-w_3) = s_1(-w_1)s_0(w_2)s_1(-w_3) \] (3.3.24)

**Class III:**

\[ s_1(-w_1)s_1(-w_2)s_0(w_3) = s_1(-w_1)s_1(-w_2)s_0(w_3) \] (3.3.25)
Now from the definition of the loop fugacities, we find that:

\[ s_0(w_1)s_0(w_2)s_0(w_3) \]

\[
\left( \begin{array}{c}
2 & 1 & 2 \\
2 & 1 & 2 \\
2 & 1 & 2 \\
\end{array} \right) + \left( \begin{array}{c}
2 & 1 & 2 \\
2 & 1 & 2 \\
2 & 1 & 2 \\
\end{array} \right) + S_3 \\
\left( \begin{array}{c}
2 & 1 & 2 \\
2 & 1 & 2 \\
2 & 1 & 2 \\
\end{array} \right) = \beta s_0(w_1)s_0(w_2)s_0(w_3) \]

From this and the identity given in (A.2.1), we obtain the following connectivity classes.

**Class IV:**

\[ s_1(-w_1)s_1(-w_2)s_1(-w_3) = s_0(w_1)s_0(w_2)s_0(w_3) \]

\[ + s_0(w_1)s_1(-w_2)s_0(w_3) + s_1(-w_1)s_0(w_2)s_0(w_3) + \beta s_0(w_1)s_0(w_2)s_0(w_3) \]

where \( \alpha \sim 2 \)  

(3.3.26)

**Class V:**

\[ s_1(-w_1)s_1(-w_2)s_1(-w_3) = s_0(w_1)s_0(w_2)s_0(w_3) \]

\[ + s_0(w_1)s_1(-w_2)s_0(w_3) + s_1(-w_1)s_0(w_2)s_0(w_3) + \beta s_0(w_1)s_0(w_2)s_0(w_3) \]

(3.3.27)
3.3.6 Path: \( \{2, 3, 2, 3, 2, 3, 2\} \)

\[
\frac{S^3 g}{e_g^3} = -\frac{S^3 g}{e_g^3} \quad (3.3.28)
\]

First consider the LHS of (3.3.28). It can be seen that the only possible value of \( g \) is \( g = 1, 3', 3 \), thus, expanding the LHS gives

\[
S^3 g = -S^3 g \quad (3.3.29)
\]

Expanding the third tangle gives

\[
-S^3 g = \quad (3.3.30)
\]

\[
-S^3 g S^3 = \quad (3.3.31)
\]

\[
-S^3 g S^3 = \quad (3.3.32)
\]

\[
-S^3 g S^3 = \quad (3.3.33)
\]

\[
-S^2 g S^3 = \quad (3.3.34)
\]
Now consider expanding the RHS of (3.3.28). It can be seen that the only possible values of $g$ is $g = 2, 4$, thus expanding the RHS gives

$$-\frac{S^3_3 S_2}{g} = -S^3_3 S_2$$

Expanding the first tangle gives

$$-S^3_2 S_3$$

From this, we acquire the following connectivity classes:

**Class I:**

$$s_0(w_1)s_1(-w_2)s_1(-w_3) = s_0(w_1)s_1(-w_2)s_1(-w_3)$$
Class II:

\[ s_1(-w_1)s_0(w_2)s_1(-w_3) = s_1(-w_1)s_0(w_2)s_1(-w_3) \quad (3.3.42) \]

Class III:

\[ s_1(-w_1)s_1(-w_2)s_0(w_3) = s_1(-w_1)s_1(-w_2)s_0(w_3) \quad (3.3.43) \]

Now from the definitions of the loop fugacities, we find that:

\[-S_2^3s_0(w_1)s_0(w_2)s_0(w_3) \left( S_2^2 + S_4^2 \right) = -S_2^3\beta s_0(w_1)s_0(w_2)s_0(w_3) \quad (3.3.44) \]

From this and the identity given in (A.2.1), we obtain the following connectivity classes.

Class IV:

\[ s_1(-w_1)s_1(-w_2)s_1(-w_3) = s_0(w_1)s_0(w_2)s_1(-w_3) \quad (3.3.45) \]

\[ + s_0(w_1)s_1(-w_2)s_0(w_3) + s_1(-w_1)s_0(w_2)s_0(w_3) \quad (3.3.46) \]

\[ + \beta s_0(w_1)s_0(w_2)s_0(w_3) \quad \text{where } \alpha \sim 2 \quad (3.3.47) \]
Class V:

\[ s_1(-w_1)s_1(-w_2)s_1(-w_3)2 \]

\[ + s_0(w_1)s_1(-w_2)s_0(w_3)2 \]

\[ + \beta s_0(w_1)s_0(w_2)s_0(w_3)2 \]

where \( \alpha \sim 2 \) (3.3.45)

### 3.3.7 Path: \{1, 2, 3, 1, 2, 1\}

First consider the LHS of (3.3.46). It can be seen that the only possible value of \( g \) is \( g = 2 \), thus expanding the LHS gives

\[ -S_3S_2 \]

\[ \begin{array}{c}
\begin{array}{c}
2 \\
1 \\
0 \\
3 \\
\end{array}
\end{array} = \]

\[ -S_3S_2s_1(-w_1)s_1(-w_2)s_1(-w_3)1 \]

\[ -S_3S_2 = -S_3S_2s_1(-w_1)s_1(-w_2)s_1(-w_3)1 \]

(3.3.47)

Now consider expanding the RHS of (3.3.46). It can be seen that the only possible values of \( g \) is \( g = 1, 3', 3 \), thus expanding the RHS gives

\[ \frac{S_3^3S_0}{e_g^3} 1 \]

\[ \begin{array}{c}
\begin{array}{c}
2 \\
1 \\
0 \\
3 \\
\end{array}
\end{array} = \]

\[ \frac{S_3^3S_0}{e_g^3} 1 \]

\[ \begin{array}{c}
\begin{array}{c}
2 \\
1 \\
0 \\
3 \\
\end{array}
\end{array} = \]

\[ \frac{S_3^3S_0}{e_g^3} 1 \]

\[ \begin{array}{c}
\begin{array}{c}
2 \\
1 \\
0 \\
3 \\
\end{array}
\end{array} = \]

\[ -S_3^3S_3 1 \]

(3.3.48)
Expanding the first tangle gives

\[ S_2^3 1 = \]

\[-S_2^2 S_3 s_1(-w_1) s_0(w_2) s_1(-w_3) = -S_2 S_3 s_0(w_1) s_0(w_2) s_1(-w_3) \]

\[-S_2 S_3 s_1(-w_1) s_0(w_2) s_0(w_3) = -S_3 s_0(w_1) s_0(w_2) s_0(w_3) \]

Expanding the third tangle gives

\[-S_2^3 S_3 1 = S_2 S_3 s_0(w_1) s_1(-w_2) s_0(w_3) \]

\[-S_3^2 s_0(w_1) s_0(w_2) s_0(w_3) \]

From this, we acquire the following connectivity classes:

**Class II:**

\[ s_1(-w_1) s_0(w_2) s_1(-w_3) = s_1(-w_1) s_0(w_2) s_1(-w_3) \] (3.3.49)
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Class IV:

\[ s_1(-w_1)s_1(-w_2)s_1(-w_3) \]

\[ = s_0(w_1)s_0(w_2)s_1(-w_3) \]

\[ + s_0(w_1)s_1(-w_2)s_0(w_3) \]

\[ + s_1(-w_1)s_0(w_2)s_0(w_3) \]

\[ + \beta s_0(w_1)s_0(w_2)s_0(w_3) \]

where \( \alpha \sim 2 \) (3.3.50)

Where connectivity class III is a tautology, and connectivity class IV comes from the identity given in (A.2.1).

3.3.8 Path: \{1, 2, 3, 2, 3, 1\}

\[ \frac{S_2^2 S_g}{e_g} = \frac{S_3^3 S_g}{e_g} \] (3.3.51)

First consider the LHS of (3.3.51). It can be seen that the only possible value of \( g \) is \( g = 2 \), thus, expanding the LHS gives

\[ S_3^2 S_2 \]

\[ = S_3^2 S_2 s_1(-w_1)s_1(-w_2)s_1(-w_3) \]

\[ + S_3^2 S_2 s_1(-w_1)s_1(-w_2)s_1(-w_3) \]

(3.3.52)

(3.3.53)

Now consider expanding the RHS of (3.3.51). It can be seen that the only possible values of \( g \) is \( g = 1, 3', 3 \), thus expanding the RHS gives
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\[
\frac{S_2^3 S_3}{c_2^3} = S_2^3 + S_2^3 s_0(w_1) s_0(w_2) s_0(w_3)
\]

\[\ldots - S_3 S_2^3 \quad (3.3.54)\]

Expanding the first tangle gives

\[
S_2^3 = S_2 S_3^2 s_0(w_1) s_0(w_2) s_1(-w_3) + S_2^2 s_0(w_1) s_0(w_2) s_0(w_3)
\]

Expanding the third tangle gives

\[
- S_2^3 S_3 = S_2 S_3 s_1(-w_1) s_1(-w_2) s_0(w_3) + S_2 S_3^2 s_0(w_1) s_1(-w_2) s_0(w_3)
\]

From this, we acquire the following connectivity classes:

Class III:

\[
s_1(-w_1) s_1(-w_2) s_1(-w_3) = s_1(-w_1) s_1(-w_2) s_0(w_3) \quad (3.3.55)
\]
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Class IV:

\[ s_1(-w_1)s_1(-w_2)s_1(-w_3) = s_0(w_1)s_0(w_2)s_1(-w_3) \]

\[ + s_0(w_1)s_1(-w_2)s_0(w_3) \]

\[ + s_1(-w_1)s_0(w_2)s_0(w_3) \]

\[ + \beta s_0(w_1)s_0(w_2)s_0(w_3) \]

Where connectivity class III is a tautology, and connectivity class IV comes from the identity given in (A.2.1).

3.3.9 Path: \{1, 2, 3, 4, 3, 2, 1\}

First consider the LHS of (3.3.57). It can be seen that the only possible value of \( g \) is \( g = 2 \), thus, expanding the LHS gives

\[ S_2 S_3^2 \]

\[ = - S_2 S_3 S_4 s_1(-w_1)s_1(-w_2)s_0(w_3) \]

Now consider expanding the RHS of (3.3.57). It can be seen that the only possible value of \( g \) is \( g = 3 \), thus expanding the RHS gives

\[ - \frac{S_2^3 S_4}{c_g^3} \]

From this, we acquire the following connectivity class:

Class III:

\[ s_1(-w_1)s_1(-w_2)s_1(-w_3) \]

\[ = s_1(-w_1)s_1(-w_2)s_0(w_3) \]

(3.3.60)
3.3.10 Path: \{1, 2, 3', 2, 1, 2, 1\}

\[ S_2 \frac{c_3^3}{c_3^3} = S_2^3 S_2 \frac{c_3^3}{c_3^3}, \quad (3.3.61) \]

First consider the LHS of (3.3.61). It can be seen that the only possible value of \(g\) is \(g = 2\), thus, expanding the LHS gives

\[ S_2 \frac{c_3^3}{c_3^3} = S_2 s_1(-w_1)s_1(-w_2)s_1(-w_3) \quad (3.3.62) \]

Now consider expanding the RHS of (3.3.61). It can be seen that the only possible values of \(g\) is \(g = 1, 3', 3\), thus expanding the RHS gives

\[ \frac{S_2^3 S_2}{c_3^3} \quad (3.3.63) \]

Expanding the first tangle gives

\[ S_2^3 = \]

\[ + S_2^3 s_1(-w_1)s_0(-w_2)s_0(-w_3) \quad (3.3.64) \]
Expanding the second tangle gives

\[
S_3^2 \left( \begin{array}{c}
2 \\
1 \\
\end{array} \right) = S_2 s_0 (w_1) s_1 (-w_2) s_0 (w_3) + s_0 (w_1) s_0 (w_2) s_0 (w_3)
\]

From this, we acquire the following connectivity classes:

**Class II:**

\[
S_2^2 s_1 (-w_1) s_0 (-w_2) s_1 (-w_3) + s_0 (w_1) s_0 (w_2) s_1 (-w_3) = S_2^2 s_1 (-w_1) s_0 (w_2) s_1 (-w_3) + s_0 (w_1) s_0 (w_2) s_0 (w_3)
\]  \hspace{1cm} (3.3.64)

**Class IV:**

\[
s_1 (-w_1) s_1 (-w_2) s_1 (-w_3) + s_0 (w_1) s_0 (w_2) s_1 (-w_3) + \beta s_0 (w_1) s_0 (w_2) s_0 (w_3) = s_0 (w_1) s_0 (w_2) s_1 (-w_3)
\]  \hspace{1cm} (3.3.65)

\[
+ s_1 (-w_1) s_0 (w_2) s_0 (w_3) + s_0 (w_1) s_1 (-w_2) s_0 (w_3) + \beta s_0 (w_1) s_0 (w_2) s_0 (w_3) = s_0 (w_1) s_0 (w_2) s_0 (w_3)
\]  \hspace{1cm} (3.3.66)

\[
+ \beta s_0 (w_1) s_0 (w_2) s_0 (w_3) = s_0 (w_1) s_0 (w_2) s_0 (w_3)
\]  \hspace{1cm} (3.3.67)

Where connectivity class II is a tautology, and connectivity class IV comes from the identity given in \[A.2.1\].
3.3.11 Path: \{1, 2, 3', 2, 3', 2, 1\}

\[
S_3 S_g = \frac{S_3^3 S_g}{\epsilon_g^3}
\]

(3.3.68)

First consider the LHS of (3.3.68). It can be seen that the only possible value of \(g\) is \(g = 2\), thus, expanding the LHS gives

\[
S_2 S_1 = S_2 s_1(-w_1)s_1(-w_2)s_1(-w_3) \tag{3.3.69}
\]

\[
+S_3^2 s_1(-w_1)s_1(-w_2)s_0(w_3) \tag{3.3.70}
\]

Now consider expanding the RHS of (3.3.68). It can be seen that the only possible values of \(g\) is \(g = 1, 3', 3\), thus expanding the RHS gives

\[
\frac{S_3^3 S_g}{\epsilon_g^3}
\]

Expanding the first tangle gives

\[
S_2 S_1 = S_2 s_0(w_1)s_0(w_2)s_1(-w_3) \tag{3.3.71}
\]

\[
+S_0(w_1)s_0(w_2)s_0(w_3) \tag{3.3.72}
\]
Expanding the second tangle gives

\[ S^3_2 w_1 w_2 w_3 \]

From this, we acquire the following connectivity classes:

**Class III:**

\[ S^2_2 s_1(-w_1)s_1(-w_2)s_0(w_3) + S^2_2 s_0(w_1)s_1(-w_2)s_0(w_3) \]

**Class IV:**

\[ S^2_2 s_1(-w_1)s_0(w_2)s_0(w_3) + s_0(w_1)s_0(w_2)s_0(w_3) \]

Where connectivity class III is a tautology, and connectivity class IV comes from the identity given in \( (A.2.1) \).
3.3.12 Path: \{1, 2, 3, 2', 3'\}

\(- \frac{S_3 S_g}{v^3} \ell_3 \) 2 3 2' 2 1

\(= \frac{S_3^3 S_g}{v^3} \ell_3 \) 1 2 3 2' 2 1

(3.3.76)

First consider the LHS of (3.3.76). It can be seen that the only possible value of \(g\) is \(g = 2\), thus, expanding the LHS gives

\[- \frac{S_3 S_g}{v^3} \ell_3 \] 2 3 2' 2 1

\(= - \frac{S_2 S_3 s_1 (- w_1) s_1 (- w_2) s_1 (- w_3)}{v^3} \) 1 2 3 2' 2 1

(3.3.77)

Now consider expanding the RHS of (3.3.76). It can be seen that the only possible values of \(g\) is \(g = 1, 3', 3\), thus expanding the RHS gives

\[\frac{S_3^3 S_g}{v^3} \ell_3 \] 2 3 2' 2 1

\(= S_2^3 \) 1 2 3 2' 2 1

\(+ S_2^3 \) 2 3 2' 2 1

\(- S_2^3 S_3 \) 1 2 3 2' 2 1

(3.3.78)

Expanding the first tangle gives

\[S_2^3 \] 2 3 2' 2 1

\(= - \frac{S_2 S_3 s_0 (w_1) s_0 (w_2) s_1 (- w_3)}{v^3} \) 1 2 3 2' 2 1

(3.3.79)

\(- S_3 s_0 (w_1) s_0 (w_2) s_0 (w_3) \) 1 2 3 2' 2 1

(3.3.80)

Expanding the second tangle gives

\[S_2^3 \] 2 3 2' 2 1

\(= - \frac{S_2 S_3 s_1 (- w_1) s_0 (w_2) s_0 (w_3)}{v^3} \) 1 2 3 2' 2 1

(3.3.81)

\(- S_3 s_0 (w_1) s_0 (w_2) s_0 (w_3) \) 1 2 3 2' 2 1

(3.3.82)
Expanding the third tangle gives

\[-S_2^3S_3 \begin{pmatrix} 2 & 3 \\ 2 & 3' \end{pmatrix} = -S_2S_3s_0(w_1)s_1(-w_2)s_0(w_3) \]

(3.3.83)

\[-S_2S_3^2s_0(w_2)s_0(w_3) \]

(3.3.84)

From this and the identity given in (A.2.1), we acquire the following connectivity class

\[-S_2S_3s_1(-w_1)s_1(-w_2)s_1(-w_3) \]

(3.3.85)

\[-S_2S_3s_1(-w_1)s_0(w_2)s_0(w_3) \]

(3.3.86)

\[-S_2S_3\beta s_0(w_1)s_0(w_2)s_0(w_3) \]

(3.3.87)
Chapter 4

Action on Link States

4.1 Link States for A-Type Loop Models

4.1.1 Constructing Link States from Temperley-Lieb Operators

Following [23], matrix representatives of the Temperley-Lieb generators are obtained by acting on a vector space of link states. To obtain the link states, consider the action of the TL generators on themselves. For example, let’s look at the A-type models with $n = 4$ so that $T(4, \lambda) = \{I, e_1, e_2, e_3\}$. We would like to calculate the number of individual words (combinations of $e_j \in T$) by multiplying $T(4, \lambda)$ by combinations of elements in $T$, we obtain the following:

$$I\{I, e_1, e_2, e_3\} = \{I, e_1, e_2, e_3\}$$

$$e_1\{I, e_1, e_2, e_3\} = \{e_1, e_1^2, e_1 e_2, e_1 e_3\} = \{e_1, \beta e_1, e_1 e_2, e_1 e_3\}$$

$$e_2\{I, e_1, e_2, e_3\} = \{e_2, e_2 e_1, e_2^2, e_2 e_3\} = \{e_2, e_2 e_1, \beta e_2, e_2 e_3\}$$

$$e_3\{I, e_1, e_2, e_3\} = \{e_3, e_3 e_1, e_3 e_2, e_3^2\} = \{e_3, e_3 e_1, e_3 e_2, \beta e_3\}$$

$$e_1 e_2\{I, e_1, e_2, e_3\} = \{e_1 e_2, e_1 e_2 e_1, e_1 e_2 e_3\} = \{e_1 e_2, e_1, \beta e_1 e_2, e_1 e_2 e_3\}$$

$$e_2 e_1\{I, e_1, e_2, e_3\} = \{e_2 e_1, e_2 e_1^2 e_1, e_2 e_1 e_2, e_2 e_1 e_3\} = \{e_2 e_1, \beta e_2 e_1, e_2, e_2 e_1 e_3\}$$

$$e_1 e_3\{I, e_1, e_2, e_3\} = \{e_1 e_3, e_1 e_3 e_1, e_1 e_3 e_2, e_1 e_3^2\} = \{e_1 e_3, \beta e_1 e_3, e_1 e_3 e_2, \beta e_1 e_3\}$$

$$e_2 e_3\{I, e_1, e_2, e_3\} = \{e_2 e_3, e_2 e_3 e_1, e_2 e_3 e_2, e_2 e_3^2\} = \{e_2 e_3, e_2 e_3 e_1, e_2, \beta e_2 e_3\}$$

$$e_3 e_2\{I, e_1, e_2, e_3\} = \{e_3 e_2, e_3 e_2 e_1, e_3 e_2 e_2, e_3 e_2 e_3\} = \{e_3 e_2, e_3 e_2 e_1, \beta e_3 e_2, e_3 e_2 e_3\}$$

It can be seen that the above combinations is exhaustive, since there’s only three distinct generators that aren’t the identity, and it can be seen that the only (distinct) word of length 4 is $e_2 e_1 e_3 e_2$, and words of length 5 degenerate into either $e_2 e_1 e_3 e_2 = e_2 e_3 e_1 e_2$ or words of length 3 of lower. Now,
So here we count 14 distinct words (including $I,e_1,e_2,e_3$). For general $n$, the number of distinct words is given by the Catalan numbers:

$$C_n = \frac{1}{n+1} \binom{2n}{n} \quad (4.1.1)$$

We can also group the words into sets $S_\ell$, by counting the number of strings, $\ell$, that pass from the bottom to the top, which will be called defects. For the $n=4$ case we obtain the following sets:

$\ell = 0:\quad S_0 = \{e_1e_3,e_1e_3e_2,e_2e_1e_3,e_2e_1e_3e_2\}$

$\ell = 2:\quad S_2 = \{e_1,e_2,e_3,e_1e_2,e_2e_1,e_2e_3,e_3e_2,e_1e_2e_3,e_3e_2e_1\}$

$\ell = 4:\quad S_4 = \{I\}$

Now for general $n$ it can be seen that for $n$ even, the number of defects must be some even number $\ell = 0, 2, 4, \ldots, n-2, n$. To see this, suppose we start with zero defects and then break a loop, this
will create two defects as seen in (4.1.2),

\[
\begin{align*}
  i & \quad \rightarrow \quad j \\
  i & \quad \rightarrow \quad j
\end{align*}
\]  \hspace{1cm} (4.1.2)

From this, it can be seen that it’s impossible to obtain an odd number of defects for even \( n \). Likewise for odd \( n \), the number of defects is some odd number \( \ell = 1, 3, \ldots, n-2, n \). To observe this, it is trivial to see that there is always going to be one site that’s a defect. Now it can be seen that the defect is either located at \( j = 1 \) or \( j = n \), in which removing it produces a smaller monoid with an even number of sites, and hence an even number of defects, thus the total number of defects for the monoid is odd. If the defect is located somewhere in-between, removing it will produce smaller monoids that either both contain an odd number of sites, or both contain an even number of sites. If the smaller monoids have an even number of sites, then we’re done, otherwise if the smaller monoids have an odd number of sites, we check whether they are the identity monoid, and if not we continue this argument until the smaller monoids have an even number of sites or are the identity monoid.

### 4.1.2 Link Diagrams

Link diagrams can be constructed from the resulting monoid representations of operator words, by looking at the bottom half of the monoid representation. To understand this, we will look back at the \( n = 4 \) case.

\[
\begin{align*}
e_1e_3 & = \begin{array}{|c|c|c|c|}
1 & 2 & 3 & 4 \\
\end{array} & e_1e_3e_2 & = \begin{array}{|c|c|c|c|}
1 & 2 & 3 & 4 \\
\end{array} \\
e_2e_1e_3 & = \begin{array}{|c|c|c|c|}
1 & 2 & 3 & 4 \\
\end{array} & e_2e_1e_3e_2 & = \begin{array}{|c|c|c|c|}
1 & 2 & 3 & 4 \\
\end{array} \\
e_1 & = \begin{array}{|c|c|c|c|}
1 & 2 & 3 & 4 \\
\end{array} & e_1e_2 & = \begin{array}{|c|c|c|c|}
1 & 2 & 3 & 4 \\
\end{array} & e_1e_2e_3 & = \begin{array}{|c|c|c|c|}
1 & 2 & 3 & 4 \\
\end{array} \\
e_2 & = \begin{array}{|c|c|c|c|}
1 & 2 & 3 & 4 \\
\end{array} & e_2e_1 & = \begin{array}{|c|c|c|c|}
1 & 2 & 3 & 4 \\
\end{array} & e_2e_3 & = \begin{array}{|c|c|c|c|}
1 & 2 & 3 & 4 \\
\end{array} \\
e_3 & = \begin{array}{|c|c|c|c|}
1 & 2 & 3 & 4 \\
\end{array} & e_3e_2 & = \begin{array}{|c|c|c|c|}
1 & 2 & 3 & 4 \\
\end{array} & e_3e_2e_1 & = \begin{array}{|c|c|c|c|}
1 & 2 & 3 & 4 \\
\end{array} \\
I & = \begin{array}{|c|c|c|c|}
1 & 2 & 3 & 4 \\
\end{array} \\
\end{align*}
\]

It can be seen that some of the words give the same link diagram, so we can group them together into sets as follows:
CHAPTER 4. ACTION ON LINK STATES

\[ \ell = 0 : \{e_1 e_3, e_1 e_2 e_3\} = 1 \quad 2 \quad 3 \quad 4 \]

\[ \{e_2 e_1 e_3, e_2 e_1 e_3 e_2\} = 1 \quad 2 \quad 3 \quad 4 \]

\[ \ell = 2 : \{e_1, e_1 e_2, e_1 e_2 e_3\} = 1 \quad 2 \quad 3 \quad 4 \]

\[ \{e_2, e_2 e_1, e_2 e_3\} = 1 \quad 2 \quad 3 \quad 4 \]

\[ \{e_3, e_3 e_2, e_3 e_2, e_1\} = 1 \quad 2 \quad 3 \quad 4 \]

\[ \ell = 4 : \{I\} = 1 \quad 2 \quad 3 \quad 4 \]

From this it can be seen that there’s: two distinct link diagrams for \( \ell = 0 \), three distinct link diagrams for \( \ell = 2 \), and one distinct link diagrams for \( \ell = 4 \).

Now for general \( n \), we find that the distinct number of link diagrams, \( \chi_{\ell}^{(n)} \), is given by

\[
\chi_{\ell}^{(n)} = \left(\frac{n}{\ell}\right) - \left(\frac{n-\ell}{2}\right)
\]

(4.1.3)

where \( \ell \in \{0, 2, \ldots, n-2, n\} \) for \( n \) even, and \( \ell \in \{1, 3, \ldots, n-2, n\} \) for \( n \) odd. Now if we consider the case where \( \ell = 0 \),

\[
\chi_0^{(n)} = \left(\frac{n}{2}\right) - \left(\frac{n}{2}-1\right) = \left(\frac{n}{2}\right) - \left(\frac{n}{2} + 1\right) = C_\frac{n}{2} = 1, 2, 5, 14, 42, 132, \ldots \quad n = 2, 4, 6, 8, 10, 12, \ldots
\]

(4.1.4)

where \( \frac{n}{2} \) can be interpreted as the number of links in a link diagram of \( n \) sites. Since the cardinality of the link diagrams of defects \( \ell = 0 \) is given by the Catalan numbers, it can be seen that there exists a bijection between them and restricted solid-on-solid (RSOS or Dyck) Paths.

### 4.1.3 Bijection between Link Diagrams of defect \( \ell = 0 \) and RSOS Paths

RSOS paths, are paths of length \( n+1 \) that can be denoted as a string of numbers (or heights) \((a_0, a_1, \ldots, a_n)\), such that \( a_0 = a_n = 1 \), and \( |a_{i+1} - a_i| = 1 \). For example, if \( n = 8 \), then a possible RSOS path is:
CHAPTER 4. ACTION ON LINK STATES

From now on we will omit the axes and grid. We can construct RSOS paths from link diagrams by counting the number of half-loops above the sites \( j \) and \( j + 1 \) (where \( 1 \leq j \leq n - 1 \)), and assigning this number plus one to \( a_j \). For example, consider the link diagrams for \( n = 4 \):

Conversely, we can construct link diagrams from RSOS paths using the following algorithm:

Input: path given by string of numbers \((a_0, a_1, \ldots, a_n)\)

(i) For \( i = 1, \ldots, n \):
- Place site \( i \) in-between \( a_{i-1} \) and \( a_i \).
- Set \( j = 1 \)

(ii) While \( j \leq n \):
- Check if site \( j \) is linked: If true, then increment \( j \) by 1, otherwise,
  - Check if \( a_j > a_{j+1} \): If true, then link site \( j \) with \( j + 1 \), otherwise,
    - Keep incrementing by 1 until there exists \( a_k \) (\( 1 \leq j \leq k \leq n \)) such that \( a_j > a_k \), and then link site \( j \) with site \( k \).
- Increment \( j \) by 1.

Output: Corresponding link diagram.
Applying this algorithm to RSOS paths for $n = 6$ should give the following:

(1, 2, 1, 2, 1, 2, 1) $\mapsto \begin{array}{ccc} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{array}$

(1, 2, 3, 2, 1, 2, 1) $\mapsto \begin{array}{ccc} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{array}$

(1, 2, 1, 2, 3, 2, 1) $\mapsto \begin{array}{ccc} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{array}$

(1, 2, 3, 2, 1) $\mapsto \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{array}$

(1, 2, 3, 2, 1) $\mapsto \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{array}$

(1, 2, 3, 4, 2, 1) $\mapsto \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 1 & 2 & 3 \end{array}$

4.2 Link States for D-Type Loop Models

4.2.1 Using Bijections to Construct Link States from D-Type Height Paths

The D-type height graph is an extension of the one for the A-type graph, as we now have a height $3'$ included. For paths that dont go through this height $3'$, we can apply the previous algorithm to obtain the corresponding link diagram from the paths. But for paths that include the height $3'$, this algorithm does not apply, so we need to generalise the previous algorithm to allow for this new value.

Algorithm:

Input: RSOS path given by the string of numbers $(a_0, a_1, \ldots, a_n)$

(i) For $i = 1, \ldots, n$:

Place site $i$ in-between $a_{i-1}$ and $a_i$.

Set $j = 1$

(ii) While $j \leq n$:

Check if site $j$ is linked: If true, the increment $j$ by 1, otherwise,

Check if $a_{j+1} = 3'$: If true, then draw a dashed link between sites $j$ and $j + 1$, otherwise, increment $j$ by 1

Set $k = 1$

(iii) While $k \leq n$:

Check if site $k$ is linked: If true, then increment $k$ by 1, otherwise,
Check if $a_k > a_{k+1}$: If true, then link site $k$ with $k + 1$, otherwise,
Keep incrementing by 1 until there exists $a_l$ ($1 \leq k \leq l \leq n$) such that $a_k > a_l$, and then link site $k$ with site $l$.
Increment $k$ by 1.

Output: Corresponding link diagram.

Applying this algorithm to paths on the D-type graph for $n = 6$ should give the following

\[
\begin{align*}
\{1, 2, 3', 2, 1, 2, 1\} & \mapsto 1 \xrightarrow{\cdot} 2 3' 4 5 6 1 \\
\{1, 2, 1, 2, 3', 2, 1\} & \mapsto 1 \xrightarrow{\cdot} 2 3 4 5 6 1 \\
\{1, 2, 3', 2, 3', 2, 1\} & \mapsto 1 \xrightarrow{\cdot} 2 3' 4 5 6 1 \\
\{1, 2, 3', 2, 3, 1, 2\} & \mapsto 1 \xrightarrow{\cdot} 2 3 4 5 6 1 \\
\{1, 2, 3', 2, 3', 2, 1\} & \mapsto 1 \xrightarrow{\cdot} 2 3' 4 5 6 1
\end{align*}
\]

Conversely, we can construct the paths from link diagrams using the following algorithm:

Input: Link diagram.

path is a string $(a_0, a_1, \ldots, a_n)$, where $a_0 = a_n = 1$

(i) For $i = 1, \ldots, n$:
Place $a_i$ halfway in-between sites $i$ and $i + 1$

(ii) For $j = 1, \ldots, n$:
Check if the loop above $a_j$ is a dashed loop: If true then $a_j = 3'$, otherwise $a_j = \#\text{half-loops above} + 1$

Output: path

We would now like to enumerate the total number of paths that exist for general even $n$. This can be done two ways experimentally using Mathematica by making the transformation:

\[
a \mapsto \begin{cases} 
m' - a & a = 1, \ldots, m' - 1 \\
m' & a = 3'
\end{cases}
\] (4.2.2)

This transformation does the following:
Now in order to enumerate the distinct paths of length \( n \) from 1 to 1 (for the second graph it’s from \( m' - 1 \) to \( m' - 1 \)), we require that \( n \leq 2(m' - 2) \), since the largest value \( a_i \) (\( i \in \{1, \ldots, n\} \)) can take is \( \frac{m'}{2} + 1 \), and we require that \( m' - 1 \) is at least as large as this. There are two ways this can be done using Mathematica:

(i) Generate all the paths, and count them. This is very inefficient, and can only be done for values of \( n \leq 14 \).

(ii) The second method was to find the generating function of all steps up to \( 2(m' - 2) \). To understand, let’s consider the second graph \( D_m \) of Figure 4.2, the graph has the corresponding adjacency matrix:

\[
A_{ij} = \begin{cases} 
1, & |i - j| = 1 \\
1, & i = m' - 2 \text{ and } j \in \{m' - 1, m'\} \\
1, & i \in \{m' - 1, m'\} \text{ and } j = m' - 2 \\
0, & \text{otherwise}
\end{cases} \tag{4.2.3}
\]

The entries \( A_{ij} \), can be understood to be the number of one step walks from \( i \) to \( j \), and \( (A^n)_{ij} \) can be understood to be the number of \( n \)-step walks from \( i \) to \( j \) (\( w_n(i \rightarrow j) \)). Now the generating function for all walks from \( i \) to \( j \) is given by,

\[
\Gamma (i \rightarrow j; z) = \sum_{n=0}^{\infty} (A^n)_{ij} z^n = \frac{1}{1 - z A_{ij}} = [(I - z A)^{-1}]_{ij} = \frac{\text{cofactor}_{ij}(I - z A)}{\det(I - z A)} \tag{4.2.4}
\]

\[
\Rightarrow \Gamma (m' - 1 \rightarrow m' - 1; z) = \frac{\text{cofactor}_{m' - 1,m' - 1}(I - z A)}{\det(I - z A)} \tag{4.2.5}
\]

Using Mathematica, we can express the generating function as a Taylor series as follows:

\[
\Gamma (i \rightarrow j; z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \left. \frac{d \Gamma_i}{dz} \right|_{z=0} = 1 + z^2 + 3z^4 + 10z^6 + 35z^8 + 126z^{10} + 462z^{12} + 1716z^{14} + 6435z^{16} + \cdots
\]

\[
= \sum_{k=0}^{m' - 2} \binom{2k - 1}{k} z^{2k} + O(z^{2(m' - 2)}) \rightarrow \sum_{k=0}^{\infty} \binom{2k - 1}{k} z^{2k}, \ m' \rightarrow \infty \tag{4.2.6}
\]

\[\text{This is due to the fact that the largest value for } a_i \text{ only occurs at } i = \frac{m'}{2} + 1\]
From this, we can deduce that the number of distinct paths of length $n$ from node 1 back to node 1, for $m'$ much larger than $n$, is given by the following formula:

$$\# \text{ of paths of length } n = \binom{n-1}{n/2} = 1, 3, 10, 35, 126, \ldots \quad n = 2, 4, 6, 8, 10, 12, \ldots$$

(4.2.7)

### 4.2.2 D-type TL Algebra

From the output in §4.3.1, we find the 10 link states for $n = 6$ are

![Diagrams showing 10 link states for n=6]

(4.2.8)

(4.2.9)

The action of the two-colored TL generators on link states gives the result zero unless the colors of the strands match. Moreover, the action of the two-color TL generators must map link states onto link states. In this way, the red loops can only enter as small closed circular loops. Although there is a symmetry between solid blue and solid red loops in the D-type TL algebra described below, we notice that this symmetry is broken by the choice of link states.

The D-type TL algebra is a two-color algebra \[24\]. Let

$$e_j = \begin{array}{c}
| \\
\hline
| \\
\hline
| \\
\hline
| \\
\hline
\end{array} \quad j = 1, 2, \ldots, n \quad (4.2.10)$$

be the TL generators. Also let $p^1_j, p^2_j$ be complementary orthogonal projectors onto color $\alpha = 1$ (blue) and color $\alpha = 2$ (red) respectively so that

$$p^1_j + p^2_j = I, \quad p^1_j p^2_j = 0 \quad (4.2.11)$$

The two-color TL generators are then

$$e_j^{11} = p^1_j e_j p^1_j = \begin{array}{c}
| \\
\hline
| \\
\hline
| \\
\hline
| \\
\hline
\end{array} \quad e_j^{12} = p^1_j e_j p^2_j = \begin{array}{c}
| \\
\hline
| \\
\hline
| \\
\hline
| \\
\hline
\end{array} \quad (4.2.12)$$

$$e_j^{21} = p^2_j e_j p^1_j = \begin{array}{c}
| \\
\hline
| \\
\hline
| \\
\hline
| \\
\hline
\end{array} \quad e_j^{22} = p^2_j e_j p^2_j = \begin{array}{c}
| \\
\hline
| \\
\hline
| \\
\hline
| \\
\hline
\end{array} \quad (4.2.13)$$

These satisfy the D-type TL relations

$$e_j^{\alpha \gamma} e_j^{\delta \eta} = \beta \delta(\gamma, \delta) e_j^{\alpha \eta} \quad (4.2.14)$$

as well as four relations such as

$$e_j^{12} e_{j+1}^{22} e_j^{21} = e_j^{11} p^2_{j+1} \quad (4.2.15)$$

obtained from \[1.8.3\] by coloring the strings in the two colors. There are four similar relations with $j + 1$ replaced by $j - 1$. 
Chapter 5

YBE for A-type Loop Representation

In this Chapter, we review the A-type loop models (logarithmic minimal models) following the work of Pearce, Rasmussen and Zuber [23]. These models are constructed by replacing the height representation of the TL algebra with the loop representation as given diagrammatically by the monoid diagrams.

The local face operators are defined as linear combinations of elementary two-faces by

\[
X(u) = \begin{bmatrix} u \end{bmatrix} = s_1(-u) \begin{bmatrix} -u \end{bmatrix} + s_0(u) \begin{bmatrix} u \end{bmatrix}
\] (5.0.1)

The lower left hand corner of the lattice face is marked by a small red arc to fix which monoid gets the weight \( s_1(-u) \). The two-in-two box refers to the fact that there are two connectivities in, and two connectivities out. For example, consider the identity two-in-two box given in Figure 5.1

![Figure 5.1](image)

5.1 Local Crossing Relation

The local crossing relation is given in (5.1.1)

\[
X(u) = \begin{bmatrix} u \end{bmatrix} = \begin{bmatrix} u \end{bmatrix} = \begin{bmatrix} \lambda - u \end{bmatrix} = \begin{bmatrix} \lambda - u \end{bmatrix}
\] (5.1.1)

Now by definition of the face weights, it can be seen that they’re symmetric under rotations about the two diagonals, i.e.

\[
\begin{bmatrix} u \end{bmatrix} = \begin{bmatrix} u \end{bmatrix}
\] (5.1.2)
So we now only need to consider the following,

\[
\lambda - u = s_0(\lambda - u) + s_1(-\lambda + u)
\]

\[
= \frac{\sin(\lambda - u)}{\sin\lambda} + \frac{\sin(u)}{\sin\lambda}
\]

\[
= s_1(-u) + s_0(u) = u
\]

Thus we can see that the local crossing relation holds.

### 5.2 Relations Between Elementary 2-boxes

Elementary 2-boxes satisfy the simple relations

\[
I_2^2 = I, \quad e_j^2 = \beta e_j
\]

The dashed lines indicate that the corners and associated edges are identified. For example, consider the first relation shown in Figure 5.2.

![Figure 5.2](image)

The corners and the associated edges are coloured in red. By gluing these edges together and pulling the loops, it can be seen that the RHS of the first identity in (5.2.1) is obtained. The second relation involves pulling out the closed loop and assigning it a value \(\beta\), again the corners and associated edges can be glued together to obtain RHS of (5.2.1). Viewed as acting horizontally, these are the standard relations \(I^2 = I\) and \(e_j^2 = \beta e_j\) of the Linear Temperley-Lieb Algebra. Another useful relation is that

\[
= e_j =
\]

This can be seen by pulling the loops towards the corners, and again gluing the corners and associated edges together.
5.3 Inversion Relation

Diagrammatically, the inversion relation is

\[
\begin{array}{c}
\begin{array}{c}
\text{u} \\
\text{u} \\
\end{array}
\end{array}
\quad = \quad
\begin{array}{c}
\begin{array}{c}
\text{s}_1(u) s_1(-u) \\
\end{array}
\end{array}
\quad (5.3.1)
\]

The inversion identity is proved by summing over all possible configurations.

\[
\begin{array}{c}
\begin{array}{c}
\text{u} \\
\text{u} \\
\end{array}
\end{array}
\quad = \quad
\begin{array}{c}
\begin{array}{c}
\text{s}_1(-u) s_1(u) \\
\end{array}
\end{array}
\quad + \quad
\begin{array}{c}
\begin{array}{c}
\text{s}_1(-u) s_0(-u) \\
\end{array}
\end{array}
\quad + \quad
\begin{array}{c}
\begin{array}{c}
\text{s}_0(u) s_1(u) \\
\end{array}
\end{array}
\quad + \quad
\begin{array}{c}
\begin{array}{c}
\text{s}_0(u) s_0(-u) \\
\end{array}
\end{array}
\quad + \quad
\begin{array}{c}
\begin{array}{c}
\text{s}_0(u) s_0(-u) \\
\end{array}
\end{array}
\quad + \quad
\begin{array}{c}
\begin{array}{c}
\text{βs}_0(u) s_0(-u) \\
\end{array}
\end{array}
\quad + \quad
\begin{array}{c}
\begin{array}{c}
\text{s}_1(-u) s_1(-u) \\
\end{array}
\end{array}
\quad + \quad
\begin{array}{c}
\begin{array}{c}
\text{s}_0(u) s_1(u) - s_1(-u) - βs_0(u) \\
\end{array}
\end{array}
\quad = \quad
\begin{array}{c}
\begin{array}{c}
\text{s}_1(u) s_1(-u) \\
\end{array}
\end{array}
\quad + \quad
\begin{array}{c}
\begin{array}{c}
\text{s}_1(-u) s_0(-u) + s_0(u) s_1(u) + βs_0(u) s_0(-u)) \\
\end{array}
\end{array}
\quad = \quad
\begin{array}{c}
\begin{array}{c}
\text{s}_1(u) s_1(-u) \\
\end{array}
\end{array}
\quad + \quad
\begin{array}{c}
\begin{array}{c}
\text{s}_0(u)(s_1(u) - s_1(-u) - βs_0(u)) \\
\end{array}
\end{array}
\quad = \quad
\begin{array}{c}
\begin{array}{c}
\text{s}_1(u) s_1(-u) \\
\end{array}
\end{array}
\quad + \quad
\begin{array}{c}
\begin{array}{c}
\text{s}_1(-u) s_1(-u) \quad + \quad βs_0(u) s_0(-u) \quad = \quad 0
\end{array}
\end{array}
\quad
\]

But in (A.1.1) we showed that,

\[
s_1(u) - s_1(-u) - βs_0(u) = 0
\]

Thus, we obtain the inversion relation

\[
\begin{array}{c}
\begin{array}{c}
\text{u} \\
\text{u} \\
\end{array}
\end{array}
\quad = \quad
\begin{array}{c}
\begin{array}{c}
\text{s}(λ + u) s(λ - u) \\
\end{array}
\end{array}
\quad (5.3.1)
\]

5.4 Yang-Baxter Equation

The Yang-Baxter equations express the equality between two planar tangles

\[
\begin{array}{c}
\begin{array}{c}
\text{v} \\
\text{v} \\
\text{v} \\
\text{v} \\
\text{v} \\
\end{array}
\end{array}
\quad = \quad
\begin{array}{c}
\begin{array}{c}
\text{v} \\
\text{v} \\
\text{v} \\
\text{v} \\
\text{v} \\
\end{array}
\end{array}
\quad (5.4.1)
\]
Let us now prove the YBE. Set \( w = v - u \). The LHS of (5.4.1) is simply a sum over all possible configurations.

\[
\begin{align*}
&= s_1(-u)s_0(v)s_1(-w) + s_0(u)s_0(v)s_1(-w) \\
&+ s_1(-u)s_1(-v)s_1(-w) + s_1(-u)s_0(v)s_0(w) \\
&+ s_0(u)s_1(-v)s_1(-w) + s_1(u)s_0(v)s_0(w) \\
&+ s_1(-u)s_1(-v)s_0(w) + s_0(u)s_1(-v)s_0(w) \\
&+ s_1(-u)s_1(-v)s_0(w) + s_0(u)s_1(-v)s_0(w)
\end{align*}
\]

Likewise, the RHS of (5.4.1) is

\[
\begin{align*}
&= s_1(-u)s_0(v)s_1(-w) + s_0(u)s_0(v)s_1(-w) \\
&+ s_1(-u)s_1(-v)s_1(-w) + s_1(-u)s_0(v)s_0(w) \\
&+ s_0(u)s_1(-v)s_1(-w) + s_1(u)s_0(v)s_0(w) \\
&+ s_1(-u)s_1(-v)s_0(w) + s_0(u)s_1(-v)s_0(w) \\
&+ s_1(-u)s_1(-v)s_0(w) + s_0(u)s_1(-v)s_0(w)
\end{align*}
\]
Now the tangles are equivalent under isotopy (i.e., two tangles are equivalent if the loop arrangement in one can be continuously deformed to look like the loop arrangement in the other), so we can just look at the external nodes and use that to see which tangles are equivalent. It can be seen that there are 5 connectivity classes:

Class I:

\[
s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \tag{5.4.4}
\]

Class II:

\[
s_1(-u)s_0(v)s_0(w) = s_1(-u)s_0(v)s_0(w) \tag{5.4.5}
\]

Class III:

\[
s_1(-u)s_1(-v)s_1(-w) = s_1(-u)s_1(-v)s_1(-w) \tag{5.4.6}
\]

Class IV:

\[
s_1(-u)s_0(v)s_1(-w) = s_0(u)s_1(-v)s_1(-w) + s_0(u)s_0(v)s_0(w) \tag{5.4.7}
\]

\[
+ s_1(-u)s_1(-v)s_0(w) + s_0(u)s_1(-v)s_0(w)
\]

Class V:

\[
s_1(-u)s_0(v)s_1(-w) = s_0(u)s_1(-v)s_1(-w) + s_0(u)s_0(v)s_0(w) \tag{5.4.8}
\]

\[
+ s_1(-u)s_1(-v)s_0(w) + s_0(u)s_1(-v)s_0(w)
\]
Connectivity Classes I - III are tautologies that related by 120° rotations, and connectivity classes IV - V come about from the identity

\[ s_1(-w_1)s_1(-w_2)s_1(-w_3) = s_0(w_1)s_0(w_2)s_1(-w_3) + s_0(w_1)s_1(-w_2)s_0(w_3) + s_1(-w_1)s_0(w_2)s_0(w_3) + \beta s_0(w_1)s_0(w_2)s_0(w_3) \]

which was shown in (A.2.1). What we have obtained from equality in the connectivity classes, is actually stronger than what we set out to prove. Not only do the 5 relations show the YBE holds, but it has given us relations between the tangles themselves.
Chapter 6

YBE for D-Type Loop Representation

The local face operators D-Type link states is a generalisation of the local face operators for the original link states

\[ X(u) = u = s_1(-u) + s_0(u) \]  

(6.0.1)

where:

\[ = , , , , \]  

(6.0.2)

\[ = , , , , \]  

(6.0.3)

We now wish to look at the Yang-Baxter equation for these local face operators.

6.1 Yang-Baxter Equation

The Yang-Baxter equations express the equality between two planar tangles

\[ = \]  

(6.1.1)

Let us now prove the YBE. Set \( w = v - u \). The LHS of (6.1.1) is simply a sum over all possible configurations.
\[
\begin{align*}
&u - v = s_1(-u)s_0(v)s_1(-w) + s_0(u)s_0(v)s_1(-w) \\
&\quad + s_1(-u)s_1(-v)s_1(-w) + s_1(-u)s_0(v)s_0(w) \\
&\quad + s_0(u)s_1(-v)s_1(-w) + s_1(u)s_0(v)s_0(w) \\
&\quad + s_1(-u)s_1(-v)s_0(w) + s_0(u)(-v)s_0(w) \tag{6.1.2}
\end{align*}
\]

Likewise, the RHS of (5.4.1) is

\[
\begin{align*}
&u - v = s_1(-u)s_0(v)s_1(-w) + s_0(u)s_0(v)s_1(-w) \\
&\quad + s_1(-u)s_1(-v)s_1(-w) + s_1(-u)s_0(v)s_0(w) \\
&\quad + s_0(u)s_1(-v)s_1(-w) + s_1(u)s_0(v)s_0(w) \\
&\quad + s_1(-u)s_1(-v)s_0(w) + s_0(u)(-v)s_0(w) \tag{6.1.3}
\end{align*}
\]
We can then decompose these faces into a sum of possible red/blue loop representations, i.e.

\[
\begin{aligned}
\text{Face 1} & = \text{Face 2} + \text{Face 3} + \text{Face 4} + \text{Face 5} \\
\text{Face 2} & = \text{Face 6} + \text{Face 7} + \text{Face 8} + \text{Face 9} \\
\text{Face 3} & = \text{Face 10} + \text{Face 11} + \text{Face 12} + \text{Face 13} \\
\text{Face 4} & = \text{Face 14} + \text{Face 15} + \text{Face 16} + \text{Face 17} \\
\text{Face 5} & = \text{Face 18} + \text{Face 19} + \text{Face 20} + \text{Face 21}
\end{aligned}
\]
\begin{equation}
= \begin{align*}
\begin{array}{cccc}
\text{Hexagon} & \text{Hexagon} & \text{Hexagon} & \text{Hexagon} \\
\text{Hexagon} & \text{Hexagon} & \text{Hexagon} & \text{Hexagon} \\
\end{array}
\end{align*}
\end{equation}

(6.1.8)
\[ (6.1.16) \]

\[ (6.1.17) \]

\[ (6.1.18) \]
Now that we've decomposed the tangles into a sum of possible red/blue loop representations, by looking at external nodes, and matching like-link states together, we obtain connectivity classes similar to those seen for the A-Type loop representation:

Class I:

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]
Class II:

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]
Class III:

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]

\[ s_0(u)s_0(v)s_1(-w) = s_0(u)s_0(v)s_1(-w) \]
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Class IV:

\[ s_1(-u)s_0(v)s_1(-w) = s_0(u)s_1(-v)s_1(-w) + s_0(u)s_0(v)s_0(w) \]

\[ + s_1(-u)s_1(-v)s_0(w) + s_0(u)s_1(-v)s_0(w) \]  

(6.1.19)

Class V:

\[ s_1(-u)s_0(v)s_1(-w) = s_0(u)s_1(-v)s_1(-w) + s_0(u)s_0(v)s_0(w) \]

\[ + s_1(-u)s_1(-v)s_0(w) + s_0(u)s_1(-v)s_0(w) \]  

(6.1.20)

Connectivity Classes I - III are tautologies that related by 120° rotations, and connectivity classes IV - V come about from the identity

\[ s_1(-w_1)s_1(-w_2)s_1(-w_3) = s_0(w_1)s_0(w_2)s_1(-w_3) + s_0(w_1)s_1(-w_2)s_0(w_3) \]

\[ + s_1(-w_1)s_0(w_2)s_0(w_3) + \beta s_0(w_1)s_0(w_2)s_0(w_3) \]

which was shown in (A.2.1). This result is very similar to what we saw for the dense loop representation of the A-type models. From this we see that the Yang-Baxter equation for D-type models in the dense loop representation holds.
Conclusion

This thesis started by looking at A-D-E lattice models, and showed that the Yang-Baxter equation holds connectivity class by connectivity class for A and D-type lattice models so that they are exactly solvable. We showed a bijection exists between Dyck paths and link states for the dense loop representation and then generalised this to allow for D-type height paths by introducing a dashed loop for the height $3'$. From this, we expect this generalisation to produce commuting transfer matrices for D-type logarithmic minimal models, as in is the case for A-type logarithmic minimal models. The next task to follow after this project would be to compute these transfer matrices and to verify their properties. It is not expected that there are logarithmic models associated with E-type lattice models.
Appendix A

Identities

A.1 Trigonometric Identity I

\[ s_1(u) - s_1(-u) - \beta s_0(u) = 0 \quad (A.1.1) \]

Proof

\[
\begin{align*}
  s_1(u) - s_1(-u) - \beta s_0(u) &= \frac{zx - z^{-1}x^{-1}}{x - x^{-1}} - \frac{z^{-1}x - x^{-1}}{x - x^{-1}} - \frac{(x + x^{-1})(z - z^{-1})}{x - x^{-1}} \\
  &= \frac{zx - z^{-1}x^{-1} - z^{-1}x - zx^{-1} - zx + z^{-1}x - zx^{-1} + z^{-1}x^{-1}}{x - x^{-1}} \\
  &= 0
\end{align*}
\]

□

A.2 Trigonometric Identity II

\[
\begin{align*}
  s_1(-w_1)s_1(-w_2)s_1(-w_3) &= s_0(w_1)s_0(w_2)s_1(-w_3) + s_0(w_1)s_1(-w_2)s_0(w_3) \\
  &\quad + s_1(-w_1)s_0(w_2)s_0(w_3) + \beta s_0(w_1)s_0(w_2)s_0(w_3) \\
  &= s_0(-w_1)s_0(-w_2)s_0(-w_3) + \beta s_0(w_1)s_0(w_2)s_0(w_3) \quad (A.2.1)
\end{align*}
\]

Where \( w_1 = u, w_2 = v \), \( w_3 = v - u \) and

\[ s_r(u) = \frac{\sin(u + r\lambda)}{\sin \lambda} \]

Proof:

For simplicity we'll write the trigonometric functions as Laurent polynomials as follows. Let

\[ z_1 = e^{iu}, \quad z_2 = e^{iv}, \quad z_3 = e^{i(v-u)} = e^{i\lambda}e^{-iu} = z_2z_1^{-1}, \quad x = e^{i\lambda} \]

Then,

\[
\begin{align*}
  \beta &= x + x^{-1} \\
  s_0(w_1) &= \frac{z_1 - z_1^{-1}}{x - x^{-1}} \\
  s_0(w_2) &= \frac{xz_2^{-1} - x^{-1}z_2}{x - x^{-1}} \\
  s_0(w_3) &= \frac{z_2z_1^{-1} - z_2^{-1}z_1}{x - x^{-1}} \\
  s_1(-w_1) &= \frac{xz_1^{-1} - x^{-1}z_1}{x - x^{-1}} \\
  s_1(-w_2) &= \frac{z_2 - z_2^{-1}}{x - x^{-1}} \\
  s_1(-w_3) &= \frac{xz_1z_2^{-1} - x^{-1}z_1z_2}{x - x^{-1}}
\end{align*}
\]
We will show \( A.2.1 \) holds, by showing the following

\[
s_1(-w_1)s_1(-w_2)s_1(-w_3) - s_0(w_1)s_0(w_2)s_1(-w_3) - s_0(w_1)s_1(-w_2)s_0(w_3) - s_1(-w_1)s_0(w_2)s_0(w_3) = 0
\]

Now the terms can be grouped as follows,

\[
s_1(-w_1)s_1(-w_2)s_1(-w_3) - s_0(w_1)s_0(w_2)s_1(-w_3) = s_1(-w_3) (s_1(-w_1)s_1(-w_2) - s_0(w_1)s_0(w_2))
\]

\[
- s_0(w_1)s_1(-w_2)s_0(w_3) - s_1(-w_1)s_0(w_2)s_0(w_3) - \beta s_0(w_1)s_0(w_2)s_0(w_3)
\]

\[
= -s_0(w_3) (s_0(w_1)s_1(-w_2) + s_1(-w_1)s_0(w_2) + \beta s_0(w_1)s_0(w_2))
\]

Now,

\[
s_1(-w_1)s_1(-w_2) - s_0(w_1)s_0(w_2) = \frac{(xz_1^{-1} - x^{-1}z_1)(z_2 - z_2^{-1}) - (z_1 - z_1^{-1})(xz_2^{-1} - x^{-1}z_2)}{(x - x^{-1})^2}
\]

\[
= \frac{xz_1^{-1}z_2 - xz_1^{-1}z_2^{-1} - x^{-1}z_1z_2 + x^{-1}z_1z_2^{-1}}{(x - x^{-1})^2}
\]

\[
+ \frac{-xz_1z_2^{-1} + x^{-1}z_1z_2 + xz_1^{-1}z_2^{-1} - x^{-1}z_1z_2}{(x - x^{-1})^2}
\]

\[
= \frac{(x - x^{-1})z_1^{-1}z_2 - (x - x^{-1})z_1z_2^{-1}}{(x - x^{-1})^2}
\]

\[
= s_0(w_3)
\]

And now we’ll simplify the other grouped terms

\[
s_0(w_1)s_1(-w_2) + s_1(-w_1)s_0(w_2) + \beta s_0(w_1)s_0(w_2) =
\]

\[
\frac{(z_1 - z_1^{-1})(z_2 - z_2^{-1}) + (xz_1^{-1} - x^{-1}z_1)(xz_2^{-1} - x^{-1}z_2) + (x + x^{-1})(z_1 - z_1^{-1})(xz_2^{-1} - x^{-1}z_2)}{(x - x^{-1})^2}
\]

\[
= \frac{z_1z_2 - z_1z_2^{-1} - z_1^{-1}z_2 + z_1^{-1}z_2^{-1} + x^2z_1^{-1}z_2^{-1} - z_1^{-1}z_2 - z_1z_2^{-1} + xz_1z_2}{(x - x^{-1})^2}
\]

\[
+ \frac{-z_1^{-1}z_2^{-1} - x^2z_1z_2 + z_1z_2^{-1} + x^2z_1z_2 + z_1^{-1}z_2 + x^{-2}z_1^{-1}z_2 - z_1z_2 - x^{-2}z_1z_2}{(x - x^{-1})^2}
\]

\[
= \frac{(x^2 - 1)z_1z_2^{-1} + (x^{-2} - 1)z_1^{-1}z_2}{(x - x^{-1})^2}
\]

\[
= \frac{(x^2 - 1)z_1z_2^{-1} - x^{-2}(x^2 - 1)z_1^{-1}z_2}{(x - x^{-1})^2}
\]

\[
= \frac{(x^2 - 1)x^{-1}(xz_1z_2^{-1} - x^{-1}z_1^{-1}z_2)}{(x - x^{-1})^2}
\]

\[
= \frac{(xz_1z_2^{-1} - x^{-1}z_1^{-1}z_2)}{x - x^{-1}}
\]

\[
= s_1(-w_3)
\]
Putting these back into the original equation yields

\[ s_0(w_3)s_1(-w_3) - s_0(w_3)s_1(-w_3) = 0 \]
Bibliography


[27] Doochul Kim and Paul A Pearce, *Scaling dimensions and conformal anomaly in anisotropic lattice spin models*, 1987

