Boundary Conditions for Logarithmic Superconformal Models

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Abstract

New Logarithmic Superconformal Models $\mathcal{LSM}(p, p')$ are constructed as loop models on the square lattice starting with the Logarithmic Minimal models $\mathcal{LM}(p, p')$. Specifically, the face weights of $\mathcal{LSM}(p, p')$ are obtained by fusing $2 \times 2$ blocks of the $\mathcal{LM}(p, p')$ face weights. The fused face weights satisfy the Yang-Baxter Equation (YBE) so these models are Yang-Baxter integrable. Algebraically, they are described by a fused Temperley-Lieb loop algebra. This braid-monoid or tangle algebra was introduced in the context of knot theory to study 2-d projections of knots through over-crossings and under-crossings. In this project, fusion is used to construct infinite families of solutions to the Boundary Yang-Baxter Equation (BYBE). These boundary conditions fall into Neveu-Schwarz and Ramond sectors and are organized into infinitely extended Kac tables related to the Conformal Field Theory obtained in the continuum scaling limit.
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1 Introduction

Exactly solvable or integrable models are of great importance in mathematical physics and allow one to go beyond standard perturbation techniques and provide a rigorous method for testing hypothesis. In particular, the study of integrable lattice models provides a framework in which seemingly unrelated areas of mathematics and physics are deeply interrelated. A notable example of this is the connection between conformal field theory and critical lattice models. Conformal field theory was introduced in 1984 by Belavin, Polyakov and Zamolodchikov [1] and has lead to the emergence of a fertile and diverse area of study which is still very active today particularly in string theory.

This thesis is concerned with integrable boundary conditions of loop models on the square lattice. Of particular interest is the superconformal case in which the compound face weights are constructed using $2 \times 2$ fusion of elementary face weights of the Temperley-Lieb algebra. The basic idea is to extend general fusion methods [2–4] to the loop model setting. Ultimately, the goal is to study the superconformal properties of the conformal field theory obtained in the continuum scaling limit.

Section 1 of this thesis aims to contextualize and motivate the results in the subsequent sections and offers a self contained introduction to the field of statistical mechanics, defining key constructs and properties relevant to conformal field theory. The concepts of critical exponents and phase transitions are outlined and explicitly demonstrated on the square lattice 2-$d$ Ising model. The connections between the Virasoro algebra and conformal field theory are also made concrete. The scaling hypothesis and Renormalization Group (RG) are briefly introduced in order to define scaling relations and scaling fields. The algebraic underpinnings behind conformal symmetry are extended to superconformal symmetry, which describes the superconformal minimal models resulting from the $2 \times 2$ fusion.

Sections 2 introduces the definition of integrability of a system in terms of the Yang-Baxter Equation YBE). The boundary Yang-Baxter equation is introduced. It is shown that satisfaction of the YBE, Boundary Yang-Baxter Equation (BYBE) and an inversion identity implies that the double row transfer matrices commute.

Section 3 examines particular integrable loop models which are constructed using elements of the Temperley-Lieb algebra. In the conformal limit, these loop models yield logarithmic minimal models $\mathcal{LM}(p, p')$. Integrable $(r, s)$ type boundary conditions are constructed in a similar manner and it is demonstrated that these boundary conditions satisfy the boundary Yang-Baxter equation.

Section 4 reviews the relevant superconformal spectra and extends the fusion procedure outlined in [3] to logarithmic minimal models. A general construction of $(r, s)$ type solutions to the boundary Yang-Baxter equation is also given using $2 \times 2$ fused weights. Lastly, it is argued that these results are relevant to the logarithmic superconformal models $\mathcal{LSM}(p, p')$ in the continuum limit.

1.1 Lattice models in statistical mechanics

1.1.1 Partition functions in statistical mechanics

Statistical mechanics is concerned with describing physical systems which have a large number of constituents or degrees of freedom. Such systems include the atomic behaviour of magnets, atoms in a boiling liquid or even the atmosphere in a room. In principal, if one knows the forces acting on each particle within such a system, then it should be possible to calculate any physical property of the system of interest. In practice, however, this is usually impractical as the number
of particles making up a system is of the order of $6 \times 10^{23}$ — so instead, a statistical description of the system is sought after. This constitutes the study of statistical mechanics. The holy grail for analyzing a physical system using statistical mechanics is the so called partition function — if one can evaluate a partition function, then the system is said to be solved and it is possible to obtain the physical quantities of interest, such as internal energy, entropy, and so on by simple differentiation.

Working in the canonical ensemble and assuming a discrete set of allowed configurations $\sigma$, the Gibbs partition function \([5]\) is

$$Z = \sum_{\sigma} \exp(-\beta H(\sigma)) \quad (1.1)$$

where $\beta = \frac{1}{k_B T}$, and $k_B$ is the Boltzmann constant \([6]\). The partition function gives the probability of finding the system in the state $\sigma$;

$$P(\sigma) = Z^{-1} \exp(-\beta E(\sigma)) \quad (1.2)$$

### 1.1.2 The 2-d Ising model on a square lattice

The class of physical systems which are of interest in this thesis are lattice models in two dimensions. Such lattice models are of great interest in statistical mechanics, where the physics imposes a crystalline structure on a discrete (regular) lattice in a plane — rather than in continuous space. Lattice models can be defined in $N$ dimensions, but due to the great complexity of lattice models in $N \geq 3$ and their triviality for $N = 1, N = 2$ is the playground for many mathematical physicists. Miraculously, lattice models in two dimensions can be solved exactly using Yang-Baxter techniques \([7]\). Specifically, the solvability of a lattice model is established by exhibiting a solution to the famous Yang-Baxter Equation (YBE) \([7]\), which if satisfied, guarantees that the transfer matrices commute and integrability of the model. The first example of an exactly solvable two-dimensional lattice model is the Ising model \([8]\).

The 2-d Ising \([8]\) model on a square lattice was first introduced by Ising in 1925 as a mathematical model of a ferromagnetic and solved exactly by Onsager \([9]\) in 1944. The model is constructed by placing a spin $\sigma_j = \pm 1$ at each vertex or site of the square lattice. The spin represents an elementary magnetic dipole moment of an atom in the two-dimensional crystal structure. It is constrained to be up ($\sigma_j = +1$) or down ($\sigma_j = -1$) where the up direction is the direction of the external magnetic field $h$. Figure 1 shows a configuration of the 2-d Ising model on a square lattice. Assuming that the spins interact only with their nearest neighbours, the energy of a given configuration of the spins on the lattice is given by the Hamiltonian

$$H(\sigma) = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - h \sum_{j=1}^{N} \sigma_j \quad (1.3)$$

where $J > 0$ for ferromagnetic model and $N = n \times n$ is the number of lattice sites on the $n \times n$ lattice in two dimensions and the first sum over $\langle i, j \rangle$ indicates summing over nearest neighbours only as described in Figure 2 below. Thus the partition function becomes

$$Z_N = \sum_{\sigma} \exp\left(\beta J \sum_{\langle i,j \rangle} \sigma_i \sigma_j + \beta h \sum_j \sigma_j\right) \quad (1.4)$$
1.1 Lattice models in statistical mechanics

Figure 1: A typical configuration of the 2D Ising model. Only a part of the square lattice is shown. The spins \( \sigma_j \in \{-1, 1\} \) sit on sites, shown as red dots. One is interested in the case when the lattice becomes very large.

![Figure 1: A typical configuration of the 2D Ising model.](image)

Figure 2: The black central spin is surrounded by four nearest-neighbour spins shown in red. The blue lines indicate the bonds of the square lattice with coordination number four.

![Figure 2: The black central spin is surrounded by four nearest-neighbour spins shown in red.](image)

The magnetization is defined by

\[
m = \frac{1}{N} \left\langle \sum_{j=1}^{N} \sigma_j \right\rangle \tag{1.5}
\]

where the canonical ensemble average is

\[
\langle \ldots \rangle = \frac{\sum_{\sigma} \ldots \exp(-\beta H(\sigma))}{\sum_{\sigma} \exp(-\beta H(\sigma))} \tag{1.6}
\]

The square lattice Ising model has been solved exactly in 1944 by Onsager [9] for zero external field \( h = 0 \). The magnetization was obtained by Yang [10] in 1952. It has not been solved in non-zero external field \( h \neq 0 \). Although the model can only be solved in zero-field, this includes the most interesting point in the thermodynamic space, namely, the critical point at \( h = 0 \) and \( T = T_c \). At the critical point the correlation length \( \xi \) diverges and the model is scale invariant. For \( h = 0 \) and \( T > T_c \), the spins are disordered and the magnetization vanishes \( m = 0 \). For \( h = 0 \) and \( T < T_c \), there is phase coexistence between a phase with positive magnetization \( m > 0 \) and a phase with negative magnetization \( m < 0 \). This is in accord with the existence of spontaneous magnetization in zero field and the manifestation of an order-disorder phase transition. Specifically, there is a first order phase transition for \( h = 0 \) and \( T < T_c \) and second order phase transition for \( h = 0 \) and \( T = T_c \).

Although the Ising model is deceptively simple, it exhibits remarkable behaviour. Physically, at low temperatures there are small thermal fluctuations of the spins, the system is ordered and the magnetization is close to either \( m = +1 \) or \( -1 \) depending on the boundary conditions. However, at high temperatures, thermal fluctuations dominate and the system becomes disordered so averaging over all configurations gives \( m = 0 \).
1.1 Lattice models in statistical mechanics

![Phase Diagram of the Ising Model](image_url)

**Figure 3:** Phase diagram of the Ising model. There is a critical point at \((h, T) = (0, T_c)\) and a first-order phase coexistence line at \(h = 0\) for \(T < T_c\).

![Plot of the Ising Model Magnetization](image_url)

**Figure 4:** Plot of the Ising model magnetization.

1.1.3 Boundary conditions

So far, the spins on the outer edges of the finite lattice have not been discussed. Due to the nature of the Ising model there are several different possible configurations.
1.1 Lattice models in statistical mechanics

Figure 5: A typical configuration of the 2D Ising model on a square lattice with boundaries indicated by the solid blue lines and boundary spins by the black dots.

In general the boundary conditions for the 2D Ising model are as follows

- **Periodic boundary conditions**
  - If the spins on the lattice are labelled $\sigma_{i,j}$, then $\sigma_{n+1,j} = \sigma_{1,j}$ and $\sigma_{i,n+1} = \sigma_{i,1}$ for $i, j \in n$ on an $n \times n$ lattice. These conditions can be implemented in one or both directions of the lattice, corresponding to the 2D Ising model on a cylinder or torus respectively.

- **Free boundary conditions**
  - There are no constraints placed on the boundary spins $\sigma_{n,j}$ and $\sigma_{i,n}$. Thus $\sigma_{n,j}, \sigma_{i,n} \in \{-1, +1\}$ for $i, j \in n$.

- **Fixed boundary conditions**
  - The boundary spins are fixed to either all up (+1) or all down (−1) or a combination of both.

These boundary conditions can be formalized in the A-D-E classification of certain lattice models in terms of A-type Dynkin diagrams [11].

1.1.4 Phase transitions and critical phenomena

The phase transition of the square lattice Ising model is an order-disorder transition. It is a consequence of the underlying symmetry being broken. Consider the model in zero field $h = 0$, then the Hamiltonian $H(\sigma)$ is invariant under spin reversal $\sigma_j \mapsto -\sigma_j$. In the high temperature non-magnetic phase, the spins are randomly oriented due to thermal fluctuations, so the phase is spin-reversal symmetric. However, in the low temperature phase, the spins are not strongly thermally disrupted, so they tend to align either all up or all down. There are thus two equivalent groundstates and the system chooses one or the other depending on the boundary conditions. This breaks the $\mathbb{Z}_2$ spin-reversal symmetry and there is spontaneous magnetization even in the absence of an aligning external field. To distinguish between the two phases of the Ising model, one requires a quantity which is not invariant under spin reversal — the magnetization which is a measure of the common alignment of the spins. The phase is thus determined by the sign of the magnetization. The magnetization is therefore an example of an order parameter.

The bulk free energy $f$ [12] for the square lattice Ising model

$$F(h, T) = -kT \ln Z(h, T)$$

(1.7)
1.1 Lattice models in statistical mechanics

acts as a generating function for the order parameter and other such thermodynamic quantities of interest. The thermodynamic limit of \( F(h, T) \) is defined as

\[
f(h, T) = \lim_{N \to \infty} \frac{1}{N} F(h, T)
\]  

which describes the intensive bulk free energy per site. Since the free energy is extensive, that is grows linearly with the system size, this limit will exist under mild conditions and will certainly exist for nearest-neighbour interactions. The magnetization \( m \) can in fact be expressed in terms of the bulk free energy as

\[
m(h, T) = -\frac{\partial f}{\partial h}(h, T)
\]  

Hence the spontaneous magnetization \( m_0 \) is

\[
m_0 = \lim_{h \to 0^+} m(h, T)
\]  

There are a few other quantities that are useful in the description of the phase transition. The magnetic susceptibility [9], which is given by

\[
\chi(h, T) = \left. \frac{\partial m}{\partial h} \right|_{h=0}
\]  

The specific heat is defined as

\[
C(h, T) = -T^2 \frac{\partial^2 F}{\partial T^2}(h, T)
\]  

which diverges logarithmically at the critical point [9]. Finally, the correlation length \( \xi \) gives a measure of how two spins at different points on the lattice are correlated. As \( T \to T_c \) it behaves as

\[
\xi \sim |T - T_c|^{-\nu}
\]  

where \( \nu \) is the correlation length critical exponent. This divergence implies that, at criticality, the spins are correlated over all length scales. Thus the theory becomes scale invariant at this point which is a precursor for the conformal invariance associated with conformal field theories. Another useful measure is the pair correlation function

\[
g(r) = \langle \sigma_j \sigma_{j+r} \rangle - \langle \sigma_j \rangle \langle \sigma_{j+r} \rangle
\]  

where \( r \) indexes the separation between the two spin locations. A \( T = T_c \) this behaves as

\[
g(r) \sim \frac{e^{-r/\xi}}{r^{d-2+\eta}}
\]  

where \( d = 2 \) is the number of dimensions of the square lattice Ising model

1.1.5 Critical exponents

The aim is to obtain a measure of disorder in the system as well as quantifying the behaviour of the above thermodynamic functions for different values of \( T \). To do so, it is necessary to introduce the notion of critical exponents [13], which describe the power law behaviour of certain
thermodynamic functions for different phases. For an arbitrary thermodynamic function $G(t)$, the critical exponent $\lambda$ is defined to be

$$
\lambda = \lim_{t \to 0} \frac{\ln G(t)}{\ln t}
$$

where $t = \frac{T-T_c}{T_c}$ is called the reduced temperature. See below for a summary of the critical exponents for the square lattice Ising model.

<table>
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<tr>
<th>Critical Exponent</th>
<th>Thermodynamic Function</th>
<th>Value</th>
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<tr>
<td>$\alpha$</td>
<td>$C \sim</td>
<td>t</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$m \sim</td>
<td>t</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$\chi \sim</td>
<td>t</td>
</tr>
<tr>
<td>$\nu$</td>
<td>$\xi \sim</td>
<td>t</td>
</tr>
<tr>
<td>$\eta$</td>
<td>$g(r) \sim r^{-\eta}$</td>
<td>1/4</td>
</tr>
<tr>
<td>$\delta$</td>
<td>$m(T_c) \sim h^{\frac{\delta}{2}}$</td>
<td>15</td>
</tr>
</tbody>
</table>

The critical exponents $\eta$ and $\delta$ are defined at the critical point $t = 0$. $\delta$ describes the behaviour of the magnetization as $h \to 0$. Critical exponents play a fundamental role in classifying universality classes. Universality suggests that many different models share the same critical exponents and that the classification of such models depends only on the dimension of the system and characteristics of the order parameters [14]. It is widely accepted (although never proven) that only two of the critical exponents are independent — this is an artifact of the scaling hypothesis [15]. The notions of scaling and universality are intimately related by the renormalization group [16].

## 1.2 The scaling hypothesis and renormalization group

The Renormalization Group (RG) offers a framework for the analysis of scale invariant critical points. These scale invariant points are fixed points under the action of the RG. See [17], [18], [19] for a more in depth discussion and analysis of the RG. The scaling hypothesis for the discrete lattice is the statement that as $T \to T_c$, the singular part of the free energy per site is a homogeneous function of $h$ and $t$ [20]. That is, as $T \to T_c$ one can write

$$
f(\lambda^b h, \lambda^a t) = \lambda f(h, t)
$$

for some exponents $a, b \in \mathbb{R}$ and $\lambda > 0$. By judicious choice of the scaling parameter $\lambda$, one can make the combination $\lambda^b h$ or $\lambda^a t$ constant. This in turn places constraints on the critical exponents of the thermodynamic functions, obtained by taking derivatives. Explicitly, these scaling relations are [12]

- Rushbrooke’s law: $\alpha + \beta + \gamma = 2$
- Widom’s law: $\gamma = \beta(\delta - 1)$
- Fisher’s law: $\gamma = \nu(2 - \eta)$
- Josephson’s law: $\nu d = 2 - \alpha$
Where, once again \( d = 2 \) for the square lattice Ising model. Thus, one can use the scaling hypothesis to write all six critical exponents in terms of two independents exponents. Although this procedure allows one to implement relations between critical exponents — it does not allow one to explicitly determine their numerical values.

### 1.3 Continuum scaling limit and conformal transformations

In order to realize a CFT from a lattice model, one needs to define the continuum or conformal scaling limit. In this limit the spacing between adjacent nodes on the lattice \( a \) is taken \( a \to 0 \), while at the same time take the number of lattice nodes \( N \to \infty \) all while keeping the quantity \( V = na \) fixed [7].

#### 1.3.1 Conformal invariance

The scale invariance of the lattice in the continuum limit at the critical point is not a full description of the symmetries at that point — it is also conformally invariant. For a system with short range interactions which is in addition to being scale invariant, is translationally and rotationally invariant then it is said to be conformally invariant. Therefore, it is natural to study such phenomena using a theory which respects these symmetries — Conformal Field Theory.

From an abstract point of view, CFTs are Euclidean field theories with an additional symmetry - local conformal symmetry. CFTs are invariant under transformations which are angle preserving. In two dimensions \((x^1, x^2)\), the requirement is that under an infinitesimal coordinate transformation \( x^\mu \mapsto x^\mu + \epsilon^\mu(x) \) is

\[
\frac{\partial \epsilon_1}{\partial x^1} = \frac{\partial \epsilon_2}{\partial x^2} \quad \text{and} \quad \frac{\partial \epsilon_2}{\partial x^1} = -\frac{\partial \epsilon_1}{\partial x^2}
\]

which are exactly the Cauchy-Riemann equations [21]. It is natural to work in the context of complex analysis and hence use complex coordinates \((z, \bar{z})\) where \( z, \bar{z} = x^1 \pm ix^2 \). Hence any function \( z \mapsto f(z) \) which is analytic, defines a conformal transformation. Since there are an infinite number of analytic functions on the complex plane, this means that there are an infinite number of conformal transformations. Therefore any conformally invariant theory has an infinite number of symmetries and associated charges. Thus, in the language of statistical mechanics, such a theory can be exactly solved by symmetry considerations alone. Compared with usual field theories, which require the analysis of the associated action, conformal field theories can be solved by investigating the behaviour of certain objects under conformal transformations. These properties are encoded in the energy momentum tensor.

#### 1.3.2 Conformal transformations

The generators and relations of the local conformal algebra associated with these infinitesimal transformations are [22]

\[
[l_m, l_n] = (m - n)l_{m+n}
\]

where \( n = -z^{n+1} \partial_z \) and an analogous expressions and relations for \( \bar{l}_n \). Since the holomorphic \((z)\) and anti-holomorphic \(\bar{z})\) sectors of the conformal algebra are isomorphic, it is convenient to do the analysis for the holomorphic sector only, with the implication that the same result holds for the anti-holomorphic part. The global conformal algebra is generated by \( \{l_{-1}, l_0, l_1\} \) subject to the same relations. This distinction is needed since only for these specific values of \( l_n \) are the conformal transformations globally defined.
1.4 Conformal field theory

1.4.1 Primary fields

The fields \( \phi(z_1, \bar{z}_1), \ldots \phi(z_n, \bar{z}_n) \) can transform in arbitrary ways under conformal mappings. A field which transforms in the following way plays a central role in the theory; given a conformal map \( z \rightarrow w(z), \bar{z} \rightarrow \bar{w}(\bar{z}) \), then a primary field \([23]\) \( \phi(z, \bar{z}) \) transforms as

\[
\phi(z, \bar{z}) \mapsto \phi'(z, \bar{z}) = \left( \frac{dw}{dz} \right)^h \left( \frac{d\bar{w}}{d\bar{z}} \right)^{\bar{h}} \phi(w(z), \bar{w}(\bar{z}))
\]

where \( h \) and \( \bar{h} \) are called the holomorphic and anti-holomorphic conformal dimension of the field respectively, \( \Delta \equiv h + \bar{h} \) is the scaling dimension and \( s \equiv h - \bar{h} \) is the spin of the field. Primary fields are of great importance, since correlation functions which involve any field in the theory can be reduced to correlation functions only involving primary fields. Note that a field which satisfies the above transformation property for global conformal transformations is called quasi-primary.

1.4.2 Correlation functions

The main quantities of interest of a conformal field theory are the correlation functions between the fields \( \phi_1(z_1, \bar{z}_1), \ldots \phi_n(z_n, \bar{z}_n) \)

\[
\langle \prod_j \phi_j(z_j, \bar{z}_j) \rangle \equiv \langle \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n) \rangle
\]

(1.21)

where \( z_i \) are the complex coordinates of the plane, and \( \bar{z}_i \) the complex conjugate of \( z \). In general (non-conformal theories), calculating an \( n \)-point correlation function is very difficult, if not impossible. However, due to the infinite symmetry group of the theory, correlation functions in Conformal Field Theory can be calculated using the so-called Ward Identities \([23]\). Under conformal transformations, correlation functions transform in the following way \([24]\)

\[
\langle \prod_j \phi_j(z_j, \bar{z}_j) \rangle \mapsto \prod_j w'(z_j)^{h_j} \bar{w}'(\bar{z}_j)^{\bar{h}_j} \langle \prod_j \phi_j(w(z_j), \bar{w}(\bar{z}_j)) \rangle
\]

(1.22)

where the local dilation factor is \( |w'(z)|^{-1} \) and the local rotation is \( \arg w'(z) \).

Consider the two point correlation function \( \langle \phi_1(z_1) \phi_2(z_2) \rangle \) of two spinless fields (\( h = \bar{h} \)). Then invariance under translations and rotations forces the correlation function to only depend on the difference \( |z_1 - z_2| \) and the Jacobian term \( w'(z) \) is unity for these transformations. Invariance under dilations \( z \mapsto \lambda z \) and special conformal transformations

\[
|z_1 - z_2|^2 \mapsto \frac{|z_1 - z_2|^2}{(1 + 2b \cdot z_1 + b^2 z_1^2)(1 + 2b \cdot z_2 + b^2 z_2^2)}
\]

implies that

\[
\langle \phi_1(z_1) \phi_2(z_2) \rangle = \begin{cases} 
\frac{C_{12}}{|z_1 - z_2|^{2|\Delta|}} & \Delta \equiv \Delta_1 = \Delta_2 \\
0 & \Delta_1 \neq \Delta_2 
\end{cases}
\]

(1.23)

where \( C_{12} \) is a constant which is fixed by the normalization of the fields. The two point correlation functions allow one to identify the scaling dimensions of the operators of a given theory, which is often sufficient to identify the relevant representation of the Virasoro algebra \([22]\). An analogous argument for the three point correlation functions results in a similar expression \([25]\).
1.4.3 Operator-state correspondence

The notion of fields in a Conformal Field Theory (CFT) are formalized in the construction of a Hilbert space and associated operators — this is called the operator state correspondence. This procedure frames the theory in a form which is subject to the powerful tools of representation and group theoretical analysis. This correspondence is in fact a bijection between states and local operators

\[ \phi(z, \bar{z}) \mapsto |\phi\rangle = \lim_{z, \bar{z} \to 0} \phi(z, \bar{z})|0\rangle \]  

(1.24)

Where as usual, the assumption of the existence of the vacuum \(|0\rangle\) is necessary in order to build the Fock space using creation operators.

1.4.4 The energy momentum tensor

The energy momentum tensor is a fundamental object of a CFT. The energy momentum tensor arises as a conserved current from Noether’s Theorem [26] in a general field theory. Using the fact that in (flat space) CFTs, the energy momentum tensor is traceless [22], one can write the two non-vanishing components of \(T_{\mu\nu}\) in complex coordinates as

\[ T(z) \equiv T_{zz}(z) \quad \text{and} \quad \bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z}) \]  

(1.25)

It is useful to introduce mode expansions for these expressions in terms of operators \(L_n\)

\[ T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n \quad \text{and} \quad \bar{T}(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{z}^{-n-2} \bar{L}_n \]

These expressions can be inverted to find

\[ L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z) \quad \text{and} \quad \bar{L}_n = \oint \frac{d\bar{z}}{2\pi i} \bar{z}^{n+1} \bar{T}(\bar{z}) \]

From this, one can calculate the commutation relations [27]

\[ [L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} m (m^2 - 1) \delta_{m+n,0} \quad \text{for} \quad m, n \in \mathbb{Z} \]

where \(c\) is a real number called the central charge. Similar relations also hold for the antiholomorphic modes \(\bar{L}_n\) (other chiral half of the theory). Together, these commutation relations define the Virasoro Algebra associated with a bulk physical system on a torus. On a strip with a specified boundary conditions, there is only one copy of the Virasoro algebra describing this sector of the theory.

1.5 Virasoro algebra

The Virasoro algebra is an infinite dimensional Lie algebra defined by elements \(\{L_n, c\}\) and relations

- \([L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} m (m^2 - 1) \delta_{m+n,0} \quad \text{for} \quad m, n \in \mathbb{Z} \]
- \([L_n, c] = 0 \quad \forall n \in \mathbb{Z} \]

where \(c\) is called the central charge. The Virasoro algebra occurs as the algebra of two commuting copies of the conformal group in two dimensions [28] (one for the holomorphic and one for the antiholomorphic sectors or left-right chiral halves of the theory). The study of CFTs thus reduces to the study of representations of the Virasoro algebra. The simplest representations in applications are highest weight state representations.
1.6 Highest weight state representations of the Virasoro algebra

Consider a sector of the theory and suppose there is a representation acting on a vector space $V$, then a highest weight state $|h\rangle \in V$ is a non-zero state satisfying

$$L_0|h\rangle = h|h\rangle \quad \text{and} \quad L_n|h\rangle = 0 \quad \forall n > 0 , \ h \in \mathbb{C}$$

(1.26)

Where $h$ is known as the conformal dimension or conformal weight. The modes $L_{-n}$ and $L_n$ act like raising and lowering operators respectively. Under the operator state correspondence, if $\phi(z)$ is a primary field with conformal dimension $h$, then its corresponding state is a highest weight state. For the representation to correspond to an action on physical states, it is required that there does not exist any negative energy states — this manifests itself in the requirement that the representation be unitary [29], which places the restriction on $L_n$

$$L_n^\dagger = L_{-n}$$

(1.27)

It turns out that although $c$ and $h$ uniquely define this representation [29], (1.27) constrains the allowed values for the representation to be unitary. These restrictions are embodied in the following theorem [17].

Theorem 1 (Unitarity of representations) For a highest weight representation of the Virasoro algebra to be unitary, it is necessary that either:

a) $c \geq 1$ and $h \geq 0$  

b)

$$c = 1 - \frac{6}{m(m+1)} \quad \text{and} \quad h = h_{r,s} = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}$$

where $m = 3, 4, \ldots$  

$r = 1, 2, \ldots, m - 1$  

and  

$s = 1, 2, \ldots, m$.

The proof of this can be found in [30] which relies on the use of the Kac determinant formula [31]. The lattice models which are associated with the second case are called unitary minimal models —which are of great interest, since for a given value of $c$, there are only a finite number of conformal weights $h$. CFTs with only a finite number of conformal weights are called rational. The CFT associated with the 2D Ising model is both rational and unitary and will be discussed in subsequent sections.

1.6.1 Descendant states and Verma modules

Additional states can be obtained from a highest weight state $|h\rangle$ by the action of $L_{-n}$’s for $n > 0$, such states are called descendant states. By the successive creation of descendant states, one builds a hierarchy of states called a Verma module, denoted $V_{h,c}$. Note that not all of the above states are necessarily linearly independent; a linear combination of states which vanishes is called a null state or null vector. Verma modules are important in the context of rational CFTs since the irreducible representations of the Virasoro algebra with highest weight $|h\rangle$ are constructed from Verma modules by removing all null states and their descendants [30]. In general, the Fock space decomposes as

$$\bigoplus_{h,c} V_{h,c} \otimes V_{h,c}^\dagger$$
1.7 Operator content of the Ising model

<table>
<thead>
<tr>
<th>Level</th>
<th>Conformal Dimension</th>
<th>State</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( h )</td>
<td>(</td>
</tr>
<tr>
<td>1</td>
<td>( h + 1 )</td>
<td>( L_{-1}</td>
</tr>
<tr>
<td>2</td>
<td>( h + 2 )</td>
<td>( L_{-2}</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( N )</td>
<td>( h + N )</td>
<td>( P(N) ) states</td>
</tr>
</tbody>
</table>

Table 2: Structure of an arbitrary Verma module where \( P(N) \) is the number of partitions of the integer \( N \).

### 1.6.2 Virasoro characters

In order to eventually evaluate the partition function of a given lattice model, one introduces the notion of a character \( \chi_{h,c} \), which is associated to each Verma module \( V_{h,c} \)

\[
\chi_{h,c} = \text{tr} q^{L_0 - c/24}
\]

where the trace is taken over \( V_{h,c} \). The \( \text{tr} q^n \) term indicates the number of states \( P(N) \) which can occur at level \( n \) and \( q = e^{2\pi i \tau} \) is associated with a modular parameter \( \tau \). The modular parameter ensures modular invariance of the partition function [32] — a requirement of conformal field theories describing lattice models. Thus the character of a generic Verma module can be written

\[
\chi_{h,c} = \sum_{n=0}^{\infty} P(N)q^n
\]

or

\[
\chi_{h,c} = \frac{q^{h+(1-c)/24}}{\eta(\tau)}
\]  

(1.28)

where \( \eta(\tau) \equiv q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \) is the Dedekind function. The characters are thus the generating functions of the dimensions of the eigenspaces of a given energy.

### 1.7 Operator content of the Ising model

Consider the \( c = 1/2 \) (\( m = 3 \)) minimal model with corresponding values of \( h_{r,s} \in \{0, 1/16, 1/2\} \). Modular invariance requires that the left and right chiral halves of the theory be identical such that the fields \( \Phi_{r,s}(z, \bar{z}) = \phi_{r,s}(z)\bar{\phi}_{r,s}(\bar{z}) \). Thus the fields and corresponding conformal weights \((h, \bar{h})\) are

\[
\begin{align*}
\phi_{1,1} &= \phi_{2,3} & (h, \bar{h}) &= (0, 0) \\
\phi_{2,2} &= \phi_{1,2} & (h, \bar{h}) &= (\frac{1}{16}, \frac{1}{16}) \\
\phi_{2,1} &= \phi_{1,3} & (h, \bar{h}) &= (\frac{1}{2}, \frac{1}{2})
\end{align*}
\]

(1.29)  

(1.30)  

(1.31)

These symmetries are summarized in the following Kac table.
The $\phi_{1,1}$ field with $(h, \bar{h}) = (0, 0)$ is present in every theory and is the identity operator. Using the results from Section (1.4.2) and Table 1, then

$$\langle \sigma_j \sigma_{j+r} \rangle \sim \frac{1}{r^{\Delta_\sigma}} \sim \frac{1}{r^{2\Delta_\sigma}}$$

Thus $\Delta_\sigma = h_\sigma + \bar{h}_\sigma = \frac{1}{8}$ and since the fields under consideration are spinless $(h_\sigma - \bar{h}_\sigma = 0)$, then $(h, \bar{h}) = (\frac{1}{16}, \frac{1}{16})$ for the field $\phi_{1,2}$. This should be identified as the continuum version of the lattice spin $\sigma_j$ of the Ising model. A similar analysis can be done for $\phi_{1,3}$ by considering the two point correlation function

$$\langle \varepsilon_j \varepsilon_{j+r} \rangle \sim \frac{1}{r^{2\Delta_\varepsilon}} \sim \langle (\sigma_j \sigma_{j+1}) (\sigma_{j+r} \sigma_{j+r+1}) \rangle \sim \frac{1}{r^{4-\nu}}$$

Again, using the fact that $\nu = 1$ and that a two point correlation function in a CFT transforms as described in Section (1.4.2), one can see that $\Delta_\varepsilon = h_\varepsilon + \bar{h}_\varepsilon = 1$. Since these fields are also spinless, then $\phi_{1,3}$ has conformal weights $(h, \bar{h}) = (\frac{1}{2}, \frac{1}{2})$. The $\phi_{1,3}$ field should be identified as the energy associated with nearest pairs of Ising spins $\varepsilon_j = \sigma_j \sigma_{j+1}$. It is now possible to express the correspondence between operators and fields in this theory.

$$I \leftrightarrow \phi_{1,1} \quad \text{or} \quad \phi_{2,3} \quad (1.34)$$
$$\sigma \leftrightarrow \phi_{1,2} \quad \text{or} \quad \phi_{2,2} \quad (1.35)$$
$$\varepsilon \leftrightarrow \phi_{1,3} \quad \text{or} \quad \phi_{2,1} \quad (1.36)$$

By inspecting the Hamiltonian (1.3) one sees that the energy density is coupled to the inverse temperature, similarly the spin field is coupled to $h$. These fields are called conjugate thermodynamic fields to the operators in the associated CFT.

### 1.7.1 Boundary conditions

Consider the Ising lattice pictured below and note that due to the $A_3$ adjacency conditions, there is a frozen sub lattice with vertices indicated by the 0.
Here, the left hand boundary has been fixed and the right boundary is free. Such a configuration makes it possible to identify a connection between the operator content of the underlying CFT and boundary conditions of the lattice model [33]. The boundary conditions can be produced from the $A_3$ adjacency graph

$$A_3 = \begin{array}{ccc}
1 & 2 & 3 \\
+ & F & -
\end{array}$$

such that for a boundary triangle

where the values $b \in \{1, 2, 3\}$ from $A_3$ determine $c$ and hence the boundary conditions. Consider $b = 1$ or 3 then, by the adjacency conditions $c = 2$ and this boundary triangle corresponds to a free boundary. Similarly, If $b = 2$ then $c = 1$ or 3 which corresponds to fixed boundary conditions of $+$ or $-$ respectively. Each type of (conformally invariant) boundary condition has a corresponding operator in the CFT. This correspondence and associated characters are listed below [33, 34]

$$+ \leftrightarrow \mathcal{I} \rightarrow \chi_0$$

$$F \leftrightarrow \sigma \rightarrow \chi_{\frac{1}{16}}$$

$$- \leftrightarrow \varepsilon \rightarrow \chi_{\frac{1}{2}}$$
where

\[ q^{1/48} \chi_0 = \frac{1}{2} \left\{ \prod_{k=1}^{\infty} (1 + q^{k-1/2}) + \prod_{k=1}^{\infty} (1 - q^{k-1/2}) \right\}, \]  

(1.41)

\[ q^{-23/48} \chi_{1/2} = \frac{q^{-1/2}}{2} \left\{ \prod_{k=1}^{\infty} (1 + q^{k-1/2}) - \prod_{k=1}^{\infty} (1 - q^{k-1/2}) \right\} \]  

and (1.42)

\[ q^{-1/24} \chi_{1/16} = \prod_{k=1}^{\infty} (1 + q^k) \]  

(1.43)

These values are well known and are explicitly calculated in [35].

1.7.2 Operator fusion

The fusion rules for the Ising model are a specific example of fusion for more general A-type models [36]. They are written in terms of graph fusion matrices associated with model which are explicitly given in [37]. The fusion rules for the Ising model which are allowed by spin flip reversal \((\sigma_j \mapsto -\sigma_j)\) and duality \((\varepsilon \mapsto -\varepsilon)\) can be organized in the following Cayley table

<table>
<thead>
<tr>
<th>(\otimes)</th>
<th>(I)</th>
<th>(\sigma)</th>
<th>(\varepsilon)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I)</td>
<td>(I)</td>
<td>(\sigma)</td>
<td>(\varepsilon)</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>(I + \varepsilon)</td>
<td>(\sigma)</td>
<td></td>
</tr>
<tr>
<td>(\varepsilon)</td>
<td>(\sigma)</td>
<td>(I)</td>
<td></td>
</tr>
</tbody>
</table>

If this table is realized in a matrix, then it decomposes into a linear combination of the primary fields of the theory

\[
\begin{pmatrix}
I & \sigma & \varepsilon \\
\sigma & I + \varepsilon & \sigma \\
\varepsilon & \sigma & I
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} I + \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix} \sigma + \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix} \varepsilon
\]

One can use the Cayley table above to write the partition function in terms of the characters associated with the different boundary conditions. For example, \(I \otimes \varepsilon\) corresponds to having a fixed boundary of + on the left and – on the right which has associated partition function

\[ Z_{+,+}(q) = \chi_{1/2}(q) \]  

(1.44)

The partition functions and corresponding boundary conditions are listed below

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>Partition function</th>
</tr>
</thead>
<tbody>
<tr>
<td>+,+</td>
<td>(Z_{+,+} = \chi_0(q))</td>
</tr>
<tr>
<td>-,−</td>
<td>(Z_{-,−} = \chi_0(q))</td>
</tr>
<tr>
<td>+,−</td>
<td>(Z_{+,−} = \chi_{1/2})</td>
</tr>
<tr>
<td>(F_{−}) or (F_{+})</td>
<td>(Z_{F,−} = Z_{F,+} = \chi_{1/16})</td>
</tr>
</tbody>
</table>

Table 4: Types of boundary conditions and associated partition functions for the Ising model

Note that these are all of the possible partition functions since the fusion algebra for this theory is commutative. For the case of periodic conditions in both directions (or on a torus), the partition functions is given by [33]

\[ Z_{\text{torus}}(q) = \chi_0(q) + \chi_{1/2}(q) + \chi_{1/16}(q) \]  

(1.45)
1.8 Supersymmetry and superconformal algebra

Supersymmetry was initially introduced as an extension of the standard model [38, 39] which established relationships between bosons and fermions and unified spacetime and internal symmetries. In the field of statistical mechanics, supersymmetry is observed in the physical process of adsorbing helium on a krypton-graphite plate [40] which is described by the tricritical Ising model [41].

1.8.1 Super Virasoro algebra

A superconformal field theory arises from the supersymmetric generalization of conformal transformations and naturally leads to a supersymmetric generalization of the Virasoro algebra called the super Virasoro algebra. The super Virasoro algebra is generated by the super stress energy tensor

\[ T(z) = T_F + \theta T_B(z) \]  

where \( z = (z, \theta) \) is a super coordinate and \( \theta \) is a Grassmann variable such that \( \theta^2 = 0 \). \( T_B(z) \) is the standard stress energy tensor and \( T_F(z) \) is its superpartner. Since there is only one superpartner in this theory, it corresponds to \( \mathcal{N} = 1 \) supersymmetry. One can mode expand the super stress energy tensor as [41]

\[ T_F(z) + \theta T_B(z) = \sum_n \frac{1}{2} z^{-n-\frac{3}{2}} G_n + \theta z^{-n-2} L_n \]  

(1.47)

to obtain the following relations

- \([L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} m(m^2 - 1) \delta_{m+n,0}\)
- \(\{G_i, G_j\} = 2L_{i+j} + \frac{c}{4} i^2 - \frac{1}{4} j^2 \delta_{i+j,0}\)
- \([L_m, G_j] = (\frac{1}{2} m - j) G_{m+j}\)

Where if \( i \in \mathbb{Z} \) for \( G_i \), then the algebra is called the Ramond algebra, otherwise if \( i \in \mathbb{Z} + \frac{1}{2} \) for \( G_i \), then the algebra is called the Neveu-Schwarz algebra and \( c \) is the central charge. The elements \( \{L_n, G_i, c\} \) along with the above relations define the super Virasoro algebra.

2 Integrability and Yang-Baxter Algebra

This section focuses on techniques associated with solving specific classes of lattice models. There are three main types of lattice models which have been studied extensively;

- Heights Models [42]
  - Defined on a square lattice, where each vertex is assigned an integer \( l_i = 1, 2, \ldots, L \) subject to the constraint that \(|l_i - l_j| = \text{where } i \text{ and } j \text{ are nearest neighbour sites.}\)

- Vertex Models [7]
  - Each vertex of a lattice is assigned a Boltzmann weight and the energy associated with a vertex is given by the configuration of the bonds which connect it to adjacent vertices.

- Loop Models
2.1 Yang-Baxter equation

– Arise out of descriptions of non-local lattice models such as percolation [43] and polymers [44].

The focus in subsequent sections is on loop models and their construction.

2.1 Yang-Baxter equation

The Yang-Baxter equation was independently introduced in 1967 by Yang [45] and in 1972 by Baxter [46]. Since then, it has become a central tool in knot theory [47], quantum computing [48], theoretical physics and quantum groups [49].

If $V$ is a vector space and $R$ an invertible linear transformation

$$R : V \otimes V \mapsto V \otimes V$$

then $R$ is a solution to the Yang-Baxter equation (YBE) if

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R) \quad (2.1)$$

where $I$ is the identity map. In the proceeding sections, specific objects and algebras will be introduced in which one can construct solutions to the YBE. Such an object relevant to the study of lattice models is the transfer matrix — if it satisfies the YBE then it guarantees the model is solvable. Depending on the model and boundary conditions, various types of transfer matrices exist, such as corner, single-row and double-row transfer matrices. Although the bulk behaviour is unaffected by the behaviour at the boundaries in the limit as the lattice size grows to infinity, the associated CFT is very much so dependent on boundary conditions [50]. In order to incorporate boundary conditions into the transfer matrix formalism of a lattice model, one must use double row transfer matrices.

2.1.1 Transfer matrices

Transfer matrices are crucial in the study of lattice models as they encode the possible configurations of a given system. Consider the $n \times n$ 2-d lattice, one can associate each different configuration of a given row on the lattice as a vector in a Hilbert Space [11]. Then the transfer matrix $T$ is an operator which acts on this state and produces the state vector corresponding to the configuration of the next row. $T$ acts in such a way that the partition function on an $n \times n$ lattice can be expressed as [7]

$$Z_N = \text{tr} T^n$$

In taking the lattice size $n \to \infty$ one finds that

$$Z_N \sim \Lambda_{\text{max}}$$

where $\Lambda_{\text{max}}$ is the largest eigenvalue of $T$. The double row transfer matrix will be given an explicit definition in terms of the Temperley-Lieb algebra elements in the forthcoming sections.

2.2 Temperley-Lieb algebra

The Temperley-Lieb algebra $TL_n(\beta)$, where $n$ is a positive integer and $\beta$ the so called fugacity is of great interest to mathematicians and physicists alike. It was introduced by H. N. V. Temperley and E. H. Lieb in 1971 [51] as a complex associative algebra in the study of planar lattice models.
2.2 Temperley-Lieb algebra

In 1983 Jones [52] independently derived and used it in the context of knot theory to define the now infamous Jones polynomial. It has also been studied in great depth by representation theorists due to its deep connection with representations of symmetric groups as natural quotients of Hecke algebras [53]. More recent developments have used the Temperley-Lieb algebra as the mathematical framework for describing quantum computing [54] and even quantum teleportation [55].

The interest of thesis in the Temperley-Lieb algebra stems from its underlying role in integrable planar lattice models ([56] [57] [7]). The integrability of such models stems from the fact that the local interactions are built on the Temperley-Lieb generators. Thus they are subject to the relations of the algebra and as a consequence of this, satisfy the Yang-Baxter equation. In this context, a great deal of focus is on understanding the representations of the algebra.

The representations of interest are obtained as quotients from link representations called standard representations (see Section 3 of [58] for an explicit construction). These standard representations are irreducible for most values of $\beta$ and for such $\beta$ the finite-dimensional representations of $TL_n(\beta)$ are semisimple. It is the values of $\beta$ for which the standard representations are not irreducible that are generally of most interest.

$TL_n(\beta)$ [51] is the algebra generated by $e_i$ with $1 \leq i \leq n - 1$ such that

1. $e_i^2 = \beta e_i \quad \forall i,$
2. $e_i e_j = e_j e_i \quad \forall |i - j| \geq 2,$
3. $e_i e_j e_i = e_i \quad \forall |i - j| = 1$

The Temperley-Lieb algebra can be realized diagrammatically as follows

$$e_j = \begin{array}{cccccccc}
\cdots & \cdots & \cdots & \cdots \\
1 & j-1 & j & j+1 & j+2 & n
\end{array}$$

and the identity element $I$ is

$$I = \begin{array}{cccccccc}
\cdots & \cdots & \cdots & \cdots \\
1 & n
\end{array}$$

In this context, multiplication in the Temperley-Lieb algebra reduces to vertical concatenation of the relevant diagrams, for example the first relation is

$$e_j^2 = \begin{array}{cccccccc}
\cdots & \cdots & \cdots & \cdots \\
1 & j-1 & j & j+1 & j+2 & n
\end{array} = \begin{array}{cccccccc}
\cdots & \cdots & \cdots & \cdots \\
1 & j-1 & j & j+1 & j+2 & n
\end{array}$$

where $\beta$ is the associated weight given to a loop. Similarly, the third relation is
2.3 Temperley-Lieb construction of integrable lattice models

In order to define integrable lattice models which yield $\mathcal{LM}(p, p')$ in the continuum scaling limit, it is necessary to specialize the choice of algebra to the planar Temperley-Lieb (TL) algebra $\mathcal{T} = \mathcal{T}(\lambda)$. In the case of $\mathcal{LM}(p, p')$, the crossing parameter $\lambda \in \mathbb{R}$ is fixed as

$$\lambda = \frac{(p' - p) \pi}{p'}, \quad p, p' \text{ coprime} \quad (2.2)$$

Introducing a complex spectral parameter $u \in \mathbb{C}$ and setting

$$s_r(u) = \frac{\sin(u + r\lambda)}{\sin \lambda}, \quad r \in \mathbb{Z} \quad (2.3)$$

Using the elements of the Temperley-Lieb algebra, it is now possible to define local face operators as linear combinations of elementary 2-boxes

$$X(u) = \begin{array}{c} \ \ \\
\hline \ \\
\end{array} u = s_1(-u) \quad + \quad s_0(u) \quad (2.4)$$

and the local crossing relation

$$X(\lambda - u) = \begin{array}{c} \lambda - u \hline \ \\
\hline \end{array} = s_0(u) \quad + \quad s_1(-u) \quad (2.5)$$

Note that the 2 in 2-box refers to the fact that there are 2 connectivities in and 2 connectivities out of each box. The bottom left corner of each face is marked by convention to indicate which monoid receives the weight $s_1(-u)$ and which one gets the weight $s_0(u)$. These face operators can act in the four diagonal directions, acting on different vector spaces of link states. Once a direction of transfer is specified, then the linear Temperley-Lieb algebra as described in Section 2.2 is realized. From the diagonal reflection and crossing symmetries, there is the relation

$$X(u) = \begin{array}{c} \ \ \\
\hline \ \\
\end{array} u = \begin{array}{c} \ \ \\
\hline \ \\
\end{array} \lambda - u = \lambda - u \quad (2.6)$$

In order to incorporate boundaries into the Temperley-Lieb construction, 1-triangles are introduced

$$K(u) = K^{(1,1)}(u, \xi) = \begin{array}{c} \xi \hline \ \\
\hline \end{array} = \begin{array}{c} \xi \hline \ \\
\ \ \\
\end{array} \xi \quad (2.7)$$
2.4 Yang-Baxter Equation (YBE) and the Boundary Yang-Baxter Equation (BYBE)

The 1 in 1-triangle is analogous to the labelling of a 2-box and the 1-triangle is labeled by \((r, s) = (1, 1)\) and is often denoted the *vacuum* boundary condition. In order to define suitable boundary conditions, it is necessary to extend the planar TL algebra to a braid-monoid (tangle) algebra by adding braid 2-boxes. This can be achieved by defining such 2-boxes as the *braid limits* of the planar face operators

\[
b = k \lim_{u \to -i\infty} \frac{X(u)}{s_1(-u)} = \begin{array}{c}
\hline
1 \\
\hline
\end{array}, \quad b^{-1} = k^{-1} \lim_{u \to i\infty} \frac{X(u)}{s_1(-u)} = \begin{array}{c}
0 \\
0
\end{array}
\]  

(2.8)

2.3.1 Inversion identity in the Temperley-Lieb algebra

Consider the following configuration of Temperley-Lieb face weights

\[
\begin{array}{c}
\raisebox{-1.5cm}{
\begin{array}{c}
\hline
u \\
\hline
\end{array}
\end{array}

\begin{array}{c}
\hline
-u \\
\hline
\end{array}
\end{array}
\]  

(2.9)

where the dotted lines indicate associated edges. It is well known \([3, 59]\) that in the Temperley-Lieb algebra, this expression reduces to

\[
s_1(u)s_1(-u) = \begin{array}{c}
\hline
\hline
\end{array}
\]  

(2.10)

this is called the *inversion relation*.

2.4 Yang-Baxter Equation (YBE) and the Boundary Yang-Baxter Equation (BYBE)

It is now possible to obtain a solution to the YBE by taking the local face operators of the planar TL algebra and fixing a direction of transfer

\[
X_j(u) = \begin{array}{c}
\hline
u \\
\hline
\end{array} = s_1(-u) I + s_0(u) e_j
\]  

(2.11)

Where it can be shown that the \(X_j\) satisfy the operator form of the YBE

\[
X_j(u) X_{j+1}(u+v) X_j(v) = X_{j+1}(v) X_j(u+v) X_{j+1}(u)
\]  

(2.12)

depicted graphically by

\[
\begin{array}{c}
\begin{array}{c}
\hline
v \\
\hline
\end{array}

\begin{array}{c}
\hline
u + v \\
\hline
\end{array}
\end{array}

\begin{array}{c}
\begin{array}{c}
\hline
u \\
\hline
\end{array}

\begin{array}{c}
\hline
v \\
\hline
\end{array}
\end{array}
\]  

(2.13)
The local face operators also satisfy the single-site commutation relation

\[ X_j(u) X_j(v) = s_1(-u)s_1(-v)I + s_0(u + v)e_j = X_j(v) X_j(u) \]  

(2.14)

### 2.4.1 Boundary Yang-Baxter equation

The (vacuum) triangle boundary weights on the right

\[ K_j^{(1,1)}(u, \xi) = \triangle_{u, \xi} \]  

(2.15)

satisfy the **Boundary Yang-Baxter equation**

\[ X_j(u - v)K_{j+1}^{(1,1)}(u, \xi)X_j(u + v)K_{j+1}^{(1,1)}(v, \xi) = K_{j+1}^{(1,1)}(v, \xi)X_j(u + v)K_{j+1}^{(1,1)}(u, \xi)X_j(u - v) \]  

(2.16)

depicted graphically by

This is simply the statement of the equality of two boundary tangles. The proof of this follows from the following identities which state the equality of connectivities and weights [59] as shown below.

\[ \omega_1 = \omega_1, \omega_2 = \omega_2 \]  

(2.18)

\[ \omega_3 = \omega_3, \omega_4 = \omega_4 \]

where \( \omega_1 = s_1(v - u)s_0(u + v), \omega_2 = s_0(u - v)s_0(u + v), \omega_3 = s_1(v - u)s_1(-u - v), \omega_4 = s_0(u - v)s_1(-u - v). \)
2.4.2 Double row transfer matrix in the Temperley-Lieb algebra

The double row transfer matrix $D(u)$ for a loop model can be expressed in terms of the aforementioned face weights in the Temperley-Lieb algebra, it is depicted below

\[ D(u) = \begin{array}{ccc}
\lambda - u & \lambda - u & \lambda - u \\
u & u & u
\end{array} \]  \hspace{1cm} (2.19)

Where $u$ is the spectral parameter and $\lambda$ is the model dependent crossing parameter. Since the elementary face weights satisfy the inversion relation, the YBE and the BYBE then

\[ [D(u), D(v)] = 0 \] \hspace{1cm} (2.20)

Let $\eta(u, v)$ and $\tilde{\eta}(u, v)$ be the weights arising from use of the inversion relations in the following proof.
2.4 Yang-Baxter Equation (YBE) and the Boundary Yang-Baxter Equation (BYBE)

\[ D(u) D(v) \]

\[
\begin{array}{ccc}
\lambda - u & \lambda - v & v \\
\lambda - v & v & \\
\lambda - u & \lambda - u & u \\
u & u & u \\
\end{array}
\]

\[ = \frac{1}{\eta(u, v)} \]

\[
\begin{array}{ccc}
\lambda - v & \lambda - v & \lambda - v \\
\lambda - v & v & v \\
\lambda - u & \lambda - u & \lambda - u \\
u & u & u \\
\end{array}
\]

\[ = \frac{1}{\eta(u, v)} \]

\[
\begin{array}{ccc}
\lambda - v & \lambda - v & \lambda - v \\
\lambda - v & v & v \\
\lambda - u & \lambda - u & \lambda - u \\
u & u & u \\
\end{array}
\]

\[ = \frac{1}{\eta(u, v)} \]

\[
\begin{array}{ccc}
\lambda - v & \lambda - v & \lambda - v \\
\lambda - v & v & v \\
\lambda - u & \lambda - u & \lambda - u \\
u & u & u \\
\end{array}
\]

\[ = \frac{1}{\eta(u, v)} \]

\[
\begin{array}{ccc}
\lambda - v & \lambda - v & \lambda - v \\
\lambda - v & v & v \\
\lambda - u & \lambda - u & \lambda - u \\
u & u & u \\
\end{array}
\]

\[ = \frac{1}{\eta(u, v)} \]

\[
\begin{array}{ccc}
\lambda - v & \lambda - v & \lambda - v \\
\lambda - v & v & v \\
\lambda - u & \lambda - u & \lambda - u \\
u & u & u \\
\end{array}
\]

\[ = \frac{1}{\eta(u, v)} \]
Thus there is a family of commuting transfer matrices.

3 Logarithmic Minimal Models

The loop models considered in this section are neither rational nor unitary and it is proposed that they yield $\mathcal{LM}(p, p)$ in the conformal limit.
3.1 Construction of integrable and conformal boundary conditions

Starting with the \((r, s) = (1, 1)\) vacuum boundary condition, one is able to build an infinite family of solutions to the BYBE indexed by labels \((r, s)\) with \(r, s = 1, 2, \ldots\). In the continuum scaling limit, this integrable boundary condition \((r, s)\) leads to a conformal boundary condition. This is precisely the fusion procedure prescribed in [59]. Projectors \(P^r\) are defined below, they act on \(r - 1\) strings to ensure that none of them are connected.

The projectors satisfy \((P^r)^2 = P^r\). For \(\lambda/\pi\) irrational there are an infinite number of projectors labelled by \(r = 1, 2, 3\ldots\). For \(\lambda/\pi\) rational, the restriction of \(1 \leq r \leq m\) ensures the existence of the above projectors (see Section 2.1 of [59] for a more in depth discussion).

The \((r, s)\) boundary solution is constructed out of the fusion product \((r, 1) \otimes_f (1, s)\) of integrable seams acting on the \((1, 1)\) vacuum boundary condition. It is represented by the 1-triangle

\[
\begin{align*}
P^r & \propto \begin{array}{cccc}
-\lambda & -2\lambda & -\lambda & \\
-\lambda & -2\lambda & -\lambda & \\
-\lambda & -2\lambda & -\lambda & \\
(r-2)\lambda & & & \\
\end{array} = \begin{array}{cccc}
\lambda & 2\lambda & \lambda & \\
\lambda & 2\lambda & \lambda & \\
\lambda & 2\lambda & \lambda & \\
(r-2)\lambda & & & \\
\end{array} \\
\text{(3.1)}
\end{align*}
\]

Where there are \(r - 1\) double columns of faces and \(\xi_k\), the column inhomogeneities are defined as

\[
\xi_k = \xi + k\lambda \quad (3.3)
\]
and the solid dots indicate that a projector $P^r$ is applied along the bottom of the $r - 1$ columns.

### 3.1.1 The $(1, s)$ boundary condition

For $1 \leq s \leq m$, the $(1, s)$ boundary is given by the aforementioned braid limit of the $(s, 1)$ solution.

\[
(1, s) = \lim_{\xi \to -i\infty} u, \xi = (s, 1)
\]

This limit exists provided the face weights are normalized accordingly. Note that a similar expression would be obtained, but with underpasses replacing overpasses in the limit $\xi \to +i\infty$.

Applying (3.23) of [59] to (3.4), one obtains

\[
(1, s) = \pm i\infty \quad (3.5)
\]

### 3.1.2 The $(r, 1)$ boundary condition

By combining (2.29) and (2.30) of [37] and using the fact that the projectors eliminate any half-arcs, one defines the normalized $(r, 1)$ boundary tangle as

\[
(r, 1) = \frac{s_{r-1}(0)s_0(2u)}{s_0(\xi + u)s_r(\xi - u)}
\]

Where in the limit as $\xi \to \pm i\infty$, the coefficient of the second term goes to zero and hence an $s$-type boundary condition of type $(1, r)$ is recovered.

### 3.1.3 The $(r, s)$ boundary condition

The $(r, s)$ boundary triangles

\[
K_{j}^{(r,s)}(u, \xi) = (r, s)
\]

(3.7)
3.1 Construction of integrable and conformal boundary conditions

which are of the form of (3.2), satisfy the fused BYBE

\[
X_j(u-v)K_{j+1}^{(r,s)}(u,\xi)X_j(u+v)K_{j+1}^{(r,s)}(v,\xi) = K_{j+1}^{(r,s)}(v,\xi)X_j(u+v)K_{j+1}^{(r,s)}(u,\xi)X_j(u-v)
\] (3.8)
depicted graphically by

Note that the analogous relation holds for the left boundary. \( \xi \in \mathbb{C} \) is not fixed and considered a free parameter which physically, governs boundary interactions as a generalized boundary magnetic field. Since the face weights satisfy (2.14), (3.9) gives a fundamental solution of the BYBE (3.8) which is given by

\[
K_{j+1}(u) = I
\] (3.10)

which is the aforementioned vacuum solution labelled by \( (r, s) = (1, 1) \).

Together, the face weights from the Temperley-Lieb algebra and the fusion projectors allow the construction of further solutions \( K_j^{(r,s)} \) of the BYBE

\[
K_{j+1}^{(r,s)}(u,\xi) = P_{j+2}^r P_{j+r+1}^s \prod_{k=1}^{r-1} X_{j+k}(u-\xi_{r-k}) \prod_{\ell=1}^{s-1} X_{j+\ell+r-1}(-i\infty) \\
\times \prod_{\ell=s-1}^{1} X_{j+\ell+r-1}(i\infty) \prod_{k=r-1}^{1} X_{j+k}(u+\xi_{r-k}) P_{j+2}^r P_{j+r+1}^s
\] (3.11)

which is depicted graphically as
3.2 Proof of BYBE for \((r, s)\) type boundary condition

Using (3.4) and (3.2), the full expression for the BYBE for the right boundary becomes

\[
K_{j+1}^{(r,s)}(u, \xi) = \begin{pmatrix}
1 & \cdots & \cdots & 1 \\
\lambda & u & v & \cdots & u \\
& u & u & \cdots & u & v \\
\end{pmatrix}
\]

Where the red lines indicate connectivities between face weights and boundary triangles. Using the fact that the face weights satisfy the YBE (2.13), it is possible to use the local crossing relation (2.5) to express the YBE as

\[
\begin{pmatrix}
\lambda - u - v \\
u - v \\
u - v
\end{pmatrix} = \begin{pmatrix}
\lambda - u - v \\
u - v \\
u - v
\end{pmatrix}
\]

Where the fusion projectors are explicitly shown. The \((1, s)\) boundary triangle \(K_{j+1}^{(1,s)}(\xi = i\infty)\) occurring on the right side has the form (3.5). Care must be taken for \(s > m\), since the projectors may not exist in this regime. Instead, the projectors are omitted and the action is restricted to the vector space of link states \(V^{(s)}\) [59].

\[ (3.13) \]

3.2 Proof of BYBE for \((r, s)\) type boundary condition

Using (3.4) and (3.2), the full expression for the BYBE for the right boundary becomes

\[
K_{j+1}^{(r,s)}(u, \xi) = \begin{pmatrix}
1 & \cdots & \cdots & 1 \\
\lambda & u & v & \cdots & u \\
& u & u & \cdots & u & v \\
\end{pmatrix}
\]

Where the red lines indicate connectivities between face weights and boundary triangles. Using the fact that the face weights satisfy the YBE (2.13), it is possible to use the local crossing relation (2.5) to express the YBE as

\[
\begin{pmatrix}
\lambda - u - v \\
u - v \\
u - v
\end{pmatrix} = \begin{pmatrix}
\lambda - u - v \\
u - v \\
u - v
\end{pmatrix}
\]

Where the fusion projectors are explicitly shown. The \((1, s)\) boundary triangle \(K_{j+1}^{(1,s)}(\xi = i\infty)\) occurring on the right side has the form (3.5). Care must be taken for \(s > m\), since the projectors may not exist in this regime. Instead, the projectors are omitted and the action is restricted to the vector space of link states \(V^{(s)}\) [59].

\[ (3.14) \]
Looking at the right hand side of (3.13), the face with weight $\lambda - u - v$ and the faces it is connected to (indicated by the red arcs) are weighted such that (3.14) can be repeatedly employed to yield

\[(3.15)\]

\[
\begin{array}{cccc}
\text{(r, 1)} & \text{(1, s)} & \text{(1, 1)} \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{r - 1 columns} & \text{s - 1 columns} \\
\end{array}
\]

and again, applying (3.14) to the remaining face with weight $u - v$

\[(3.16)\]

\[
\begin{array}{cccc}
\text{(r, 1)} & \text{(1, s)} & \text{(1, 1)} \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{r - 1 columns} & \text{s - 1 columns} \\
\end{array}
\]

Now it is possible to apply the standard BYBE (2.17), since the face weights now act on the trivial $(r, s) = (1, 1)$ boundary condition. This results in

\[(3.17)\]

\[
\begin{array}{cccc}
\text{(r, 1)} & \text{(1, s)} & \text{(1, 1)} \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{r - 1 columns} & \text{s - 1 columns} \\
\end{array}
\]

Now the face weights are in a configuration such that it is possible to apply the YBE (3.14) and ‘push’ both face weights back through to the left hand side of the $(r, 1)$ and $(1, s)$ type boundary
3.3 Background on CFT/LCFT

Conformal Field Theory was conceived in 1984 by Belavin, Polyakov and Zamoldichikov [1] to describe scale invariant statistical systems at criticality in two dimensions. Since then it has found a number of applications in both physics and pure mathematics;

- CFT plays a fundamental role in the description of nature offered by string theory [60] - one of the few consistent theories which unifies the fundamental forces.

- CFTs describe fixed points of the renormalization group in the theory of two dimensional critical systems [61].

- In modern mathematics, CFT is used in the theory of vertex operator algebras and Borcherds algebras [62] as well as low-dimensional topology.

- CFTs on surfaces with boundaries in condensed matter physics [63].

- Chiral CFTs in quantum field theory.

This list is by no means exhaustive and CFT still enjoys a wealth of treatment from a wide range of mathematical disciplines. From the aforementioned paper [1], emerged the first proposal of so called minimal models - in which the Hilbert space of the CFT decomposes into finitely many irreducible representations of the Virasoro algebra.

Efforts attempting to classify CFTs have been primarily through the study of representations of the Virasoro algebra of conformal transformations. In 1993 Gurarie [64] found that in order to have a consistent field theory when the central charge vanishes \((c = 0)\), logarithmic operators had to be introduced. He described the emergence of logarithmic singularities in the correlation functions of certain CFTs - termed logarithmic CFTs. This sparked a considerable amount of effort in analyzing and attempting to classify LCFTs by using the tools of rational CFTs such as fusion rules, modular invariance, etc and extending them to the logarithmic case.

Ludwig and Gurarie [65] introduced an indecomposability parameter \(b\), which was to play the role of the central charge when \(c = 0\). They were able to produce limited but successful predictions...
for $b$. However, it would appear though that this is not the case; rather than a single parameter $b$, it is accepted that a LCFT is underpinned by indecomposable Virasoro modules - with an infinite number of indecomposability parameters. Thus it is expected that the Virasoro generators act on some representations through Jordan cells in a LCFT.

In attempts to classify LCFTs, two general approaches have emerged — to study the indecomposable Virasoro modules directly or to study lattice models which can be thought of as lattice regularizations of LCFTs. The first, more algebraic approach has led to new predictions for indecomposability parameters and certain classifications of some of the modules ([66], [67], [68]). The second, and somewhat less abstract approach has found that for certain values of the parameters, the associated lattice Hamiltonians have a Jordan cell structure which replicates that of the continuum theory. The second approach is underpinned by the integrability of the lattice model and relies on the association between the indecomposable modules of the Temperley-Lieb algebra and the Virasoro algebra [69]. For example, rational minimal CFTs have been realized from RSOS lattice models [70] and the previous chapter outlines how to realize LCFTs from lattice models built from the Temperley-Lieb algebra. The next section provides a summary of results and motivations for both rational and logarithmic minimal models.

### 3.3.1 Rational minimal models $\mathcal{M}(m, m')$

As mentioned in Section (1.5), rational minimal models have only a finite number of conformal weights and hence are identified with finite irreducible highest weight representations of the Virasoro algebra. These CFTs are characterized by central charge

$$c = c(m, m') := \frac{6(m - m')^2}{mm'}$$

and conformal weights

$$h_{r,s} = \frac{(m'r - ms)^2 - (m - m')^2}{4mm'}$$

for $1 \leq r \leq m - 1$ and $1 \leq s \leq m' - 1$. The symmetry of a conformal weight $h_{r,s} = h_{m'-r,m-s}$ associated with a primary field $\phi_{r,s}$ gives $\frac{m-1}{2} \text{ distinct primary fields}$. Also, as seen in the introduction, these models are unitary for $m' = m + 1$.

### 3.3.2 Logarithmic minimal models $\mathcal{LM}(p, p')$

In some circumstances, it is possible to realize $\mathcal{LM}(p, p')$ from $\mathcal{M}(m, m')$ as a limit of a sequence of minimal models [71]. This is achieved by employing the following logarithmic limit

$$\lim_{m,m' \rightarrow \infty, \frac{m}{m'} \rightarrow \frac{p}{p'}} \mathcal{M}(m, m') = \mathcal{LM}(p, p')$$

Where the limit is taken after the thermodynamic limit and such that the ratio of coprime pairs $(m, m')$ is the same as the ratio of $(p, p')$. In this limit, the associated central charge becomes

$$c(m, m') = 1 - \frac{6(m - m')}{mm'} \rightarrow 1 - \frac{6(p - p')^2}{pp'} = c(p, p')$$

with conformal weights

$$h_{r,s}(m, m') = \frac{(rm' - sm)^2 - (m - m')^2}{4mm'} \rightarrow \frac{(rp' - sp)^2 - (p - p')^2}{4pp'}$$
With $1 \leq r \leq p - 1$ and $1 \leq s \leq p' - 1$ and $r, s = 1, 2, 3, \ldots$. Unlike the case of rational minimal models, these conformal weights lie in an infinitely extended Kac table. It must be noted that this is most definitely not a complete classification of LCFTs and there are many subtleties involved in taking this sequence of rational minimal models [71,72]. One can fix the values $(p, p') = (m, m + 1)$ called the principal series, of which the the first few members $(m = 1, m = 2, m = 3$ and $m = 4)$ correspond to critical dense polymers, critical percolation, the logarithmic Ising model and logarithmic tricritical Ising model respectively. See the extended Kac table below.
### 3.3 Background on CFT/LCFT

$m = 1, c = -2$

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<td>...</td>
</tr>
</tbody>
</table>

$m = 3, c = 1/2$

<table>
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<th>153</th>
<th>899</th>
<th>549</th>
<th>909</th>
<th>39</th>
<th>390</th>
<th>14</th>
<th>99</th>
<th>59</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>135</td>
<td>11</td>
<td>51</td>
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<td>3</td>
<td>2</td>
<td>7</td>
<td>16</td>
<td>...</td>
</tr>
<tr>
<td>91</td>
<td>193</td>
<td>18</td>
<td>143</td>
<td>5</td>
<td>50</td>
<td>7</td>
<td>3</td>
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<td></td>
</tr>
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<td>33</td>
<td>323</td>
<td>21</td>
<td>10</td>
<td>63</td>
<td>50</td>
<td>10</td>
<td>3</td>
<td>80</td>
<td>...</td>
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<td>29</td>
<td>1</td>
<td>1</td>
<td>16</td>
<td>0</td>
<td>1</td>
<td>7</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>99</td>
<td>3</td>
<td>10</td>
<td>-1</td>
<td>3</td>
<td>99</td>
<td>80</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>38</td>
<td>5</td>
<td>16</td>
<td>21</td>
<td>43</td>
<td>38</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>14</td>
<td>...</td>
<td></td>
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<tr>
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<td>1</td>
<td>5</td>
<td>3</td>
<td>33</td>
<td>65</td>
<td>32</td>
<td>48</td>
<td>...</td>
<td></td>
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<tr>
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<td>1</td>
<td>2</td>
<td>5</td>
<td>7</td>
<td>2</td>
<td>6</td>
<td>55</td>
<td>6</td>
<td>...</td>
</tr>
</tbody>
</table>

Table 5: Lower left corner of the extended Kac table of conformal weights $h_{r,s}$ for $m = 1, 2, 3, 4$ corresponding, respectively, to critical dense polymers ($c = -2$), critical percolation ($c = 0$), the logarithmic Ising model ($c = 1/2$), and the logarithmic tricritical Ising model ($c = 7/10$).
3.3.3 Characters of $\mathcal{LM}(p, p')$

Every pair of coprime integers $(p, p')$ gives rise to a highest weight Verma module $V_{hr,s}$, where $r, s$ are positive integers. This Verma module is reducible since it has a submodule $V_{h_{r,s}}$ of highest weight $h_{r,s} = h_{r,s} + rs$. The quotient module $Q_{r,s} := V_{hr,s}/V_{h_{r,s}}$ is not necessarily irreducible but reducible, even though it arises from the limit of minimal models in which the associated Verma modules are all irreducible [71]. The character associated with $Q_{r,s}$ is

$$\chi_{r,s}(q) = q^{-c(m,m') \frac{h^{hr,s}-q^{hr,s}}{PP'}} \prod_{n=1}^{\infty} (1-q^n)$$

(3.24)

It is strongly suggested ([59, 71]) that the characters $\chi_{r,s}(q)$ describe the conformal spectra of $\mathcal{LM}(p, p')$. It is also known that the characters of the representations $Q_{r,s}$ decompose into a finite number of characters of irreducible representations of the Virasoro algebra (section 2.2 of [59]).

It is important to note that specifying the central charge and conformal weights does not necessarily uniquely determine a LCFT and thus it is not possible to give an exhaustive classification of LCFTs. Instead, one defines minimal LCFTs as the continuum scaling limit of logarithmic minimal models - and then study their conformal properties.

4 Logarithmic Superconformal Minimal Models

Since the logarithmic superconformal minimal models are formed from the logarithmic minimal models, they are also non-unitary.

4.1 Characters of $\mathcal{LSM}(p, p')$

For general $n \times n$ fusion, the rational superconformal weights are given by

$$h_{r,s}^{n} = \frac{(rP' - sP)^2 - (P' - P)^2}{4nP'P}$$

(4.1)

where $(P, P') = (|2p - p'|, p')$. In the logarithmic limit, the conformal weights are

$$h_{r,s}^{n} = h_{r,s}^{P,P'n} + h_{m}^{l,n}$$

(4.2)

with

$$h_{m}^{l,n} = \text{Max}[h(m, l, n), h(4 - m, l, n), h(2 - m, 2 - l, n)]$$

(4.3)

and

$$h(m, l, 2) = \frac{\ell(\ell + 2)}{16} - \frac{m^2}{8}$$

(4.4)

with $0 \leq m \leq \ell \leq n$ and $\ell - m \in 2\mathbb{Z}$. These values have been taken from [73] and can be specialized to the $2 \times 2$ fused case. The associated superconformal characters for $2 \times 2$ fusion in their respective sectors are

$$\text{NS: } \ell = 0, 2; \ r \pm s \text{ even} : \chi_{r,s}^{P,P'2}(q) = \begin{cases} q^{-\frac{c_{\ell}}{8}+h_{r,s}^{P,P'}2}(1-q^{\frac{c_{\ell}}{2}})c_{m-}^{l}, & m_- = m_+ \\ q^{-\frac{c_{\ell}}{8}+h_{r,s}^{P,P'}2}c_{m-}^{l}(q)-q^{\frac{c_{\ell}}{2}}c_{m+}^{l}(q), & m_+ = 2 - m_- \end{cases}$$

(4.5)

$$\text{R: } \ell = 1; \ r \pm s \text{ odd} : \chi_{r,s}^{P,P'2}(q) = q^{-\frac{c_{\ell}}{8}+h_{r,s}^{P,P'}2}(1-q^{\frac{c_{\ell}}{2}})c_{1}^{l}(q)$$
where \( m_- = r - s \) mod 4 and \( m_+ = r + s \) mod 4. The sectors \( \ell \) and \( n - \ell \) can be combined into symmetric and anti-symmetric superconformal characters. In the Neveu-Schwarz sector with \( n = 2 \), this gives

\[
NS: \quad \chi_{r,s,0}^{p,p',2}(q) \pm \chi_{r,s,2}^{p,p',2}(q) = \begin{cases} 
q^{-\frac{\ell}{48} + h_{r,s}^{p,p'}(1) \frac{c_0(q) \pm c_2(q)}{c_0(q) \pm c_2(q)}} & \text{if } m_+ = m_- \\
q^{-\frac{\ell}{48} + h_{r,s}^{p,p'}(1)} \frac{c_0(q) \pm c_2(q)}{c_0(q) \pm c_2(q)} & \text{if } m_+ = 2 - m_- 
\end{cases}
\]

with

\[
c_0(q) \pm c_2(q) = \frac{\chi_0(q) \pm \chi_4(q)}{(q)_{\infty}} \prod_{k=1}^{\infty} (1 \pm q^{k-\frac{1}{2}}), \quad c_1(q) = \frac{\chi_4(q)}{(q)_{\infty}} = \frac{q^{\frac{1}{2}}}{(q)_{\infty}} \prod_{k=1}^{\infty} (1 + q^n)
\]

and

\[
(q)_{\infty} = \prod_{k=1}^{\infty} (1 - q^k)
\]

Which are simple expressions, written in terms of the Ising characters (1.45) – (1.47). These formulas are obtained through the aforementioned logarithmic limit. See [73] for explicit details.

### 4.2 2 \times 2 fused boundary conditions and the elementary BYBE

This section shows the explicit construction of the \( 2 \times 2 \) vacuum boundary condition and the proof of the elementary BYBE for \( 2 \times 2 \) fusion.

#### 4.2.1 Notation

Associated edges or connected faces are denoted by the following equivalent notation

\[
\begin{array}{c}
\text{u} \\
\text{v}
\end{array} = \begin{array}{c}
\text{u} \\
\text{v}
\end{array}
\]

The action of projectors on fused face weights are denoted by either a black dot, or an ellipse between two connectivities

\[
\begin{array}{c}
\text{u} \\
\text{u} + \lambda \\
\text{u} - \lambda \\
\text{u}
\end{array} = \begin{array}{c}
\text{u} \\
\text{u} + \lambda \\
\text{u} - \lambda \\
\text{u}
\end{array}
\]

An equivalent expression would have resulted if the projectors were placed at both ends of the connectivities, as indicated below.
This equivalence comes from using the YBE with the projectors and face weights, along with crossing symmetry and the projector property $P^2 = P$.

### 4.2.2 $2 \times 2$ fused vacuum boundary condition

At level $2 \times 2$ fusion, a new vacuum solution for the boundary condition is constructed from two elementary 1-triangles and an elementary 2-box.

Where the above equality is due to the action of the projectors, which do not allow any half arcs. It is now possible to state and prove the BYBE for this new boundary condition and the $2 \times 2$ fused face weights.

### 4.2.3 $2 \times 2$ fused BYBE

In order to prove the $2 \times 2$ BYBE, it is necessary to use (2.6) of [3] and specialize $\mu = 2\lambda$ for this model, which results in

This statement has a great amount of detail which is suppressed in this pictorial representation of the fused BYBE. The full extent of the statement is only realized once it is written in terms of the elementary face weights of which the fused face weights and vacuum boundary conditions are
composed of. This is detailed below

\[
\begin{align*}
\lambda - 2v & \quad \lambda - u - v \\
-u - v & \quad v \\
u - v & \quad 2\lambda - u - v \\
u - v - \lambda & \quad \lambda + u - v \\
u - v & \quad u - v \\
u - v & = u + \lambda
\end{align*}
\]

Where there are many equivalent configurations of the placement of projectors \( P \) due to the fact that they satisfy the YBE and \( P^2 = P \). Starting with the left hand side of (4.13) and leaving the projectors on the peripheral, it is possible to apply the elementary YBE (3.14) to the indicated face weights as follows

\[
\begin{align*}
\lambda + u - v & \quad \lambda - u - v \\
u - v & \quad -u - v \\
u - v & = u + \lambda \\
u - v - \lambda & \quad \lambda + u - v \\
u - v & = u - v \\
u - v & = u - v \\
u - v - \lambda & \quad 2\lambda - u - v \\
u - v & = \lambda - 2u \\
\end{align*}
\]

Where the YBE has once again been used in order to ‘push’ the face weight all the way through to the other side. The dotted lines indicate coincident edges of the face weights. Now, applying (4.12) to the boundary and face weights on the right hand side of the above equation (as indicated
by the bold edges), one obtains

\[
\begin{align*}
\lambda - 2u & \quad \lambda - 2v \\
\lambda - u - v & \quad \lambda - u - v \\
u - v & \quad 2\lambda - u - v \\
u - v & \quad u - v \\
u - v & \quad u - v \\
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\end{align*}
\]
The YBE is again employed on the indicated face weights on the left hand side, and then on the right hand side of the above equation to obtain

\[
\begin{align*}
&\lambda - 2u & \lambda - u & v + \lambda \\
&u & -u & -v \\
&2\lambda - u & -v & -u - v - \lambda \\
&\lambda - 2v & v & & \lambda + u - v
\end{align*}
\]

\[
= \begin{align*}
&\lambda - 2u & \lambda - u & v + \lambda \\
&u & -u & -v \\
&2\lambda - u & -v & -u - v - \lambda \\
&\lambda - 2v & v & & \lambda + u - v
\end{align*}
\]

(4.18)

Applying (4.12) to the indicated area results in

\[
\begin{align*}
&\lambda - 2u & \lambda - u & v + \lambda \\
&u & -u & -v \\
&2\lambda - u & -v & -u - v - \lambda \\
&\lambda - 2v & v & & \lambda + u - v
\end{align*}
\]

\[
= \begin{align*}
&\lambda - 2u & \lambda - u & v + \lambda \\
&u & -u & -v \\
&2\lambda - u & -v & -u - v - \lambda \\
&\lambda - 2v & v & & \lambda + u - v
\end{align*}
\]

(4.19)

Where the appropriate face weights have been by 45° clockwise so the connectivities are implicit.

\[
\begin{align*}
&\lambda - 2u & \lambda - u & v + \lambda \\
&u & -u & -v \\
&2\lambda - u & -v & -u - v - \lambda \\
&\lambda - 2v & v & & \lambda + u - v
\end{align*}
\]

(4.20)

Again, using (4.12) to the indicated boundary and face weights, one finds
Bringing together the associated edges, the expression becomes

Applying the YBE to the indicated faces a final time yields

Which is almost what is required, except that two of the elementary face weights need to be swapped — the elementary face weights with weights $\lambda + u - v$ and $u - v - \lambda$ in the upper $2 \times 2$ block. This is easily fixed by the action of the projectors, which are still sitting on the peripheral of the calculation. Explicitly drawing the position of the projectors results in the following picture.
It is possible to apply the BYBE to the bottom most projector (indicated by the white filling), then the YBE to push it through to the position shown below.

Note that neither the face weights nor boundary weights have been switched when employing the YBE and BYBE, respectively. This is because the fact that $P^2 = P$ enables one to send a copy of the projector back to its original position, thus undoing any of the switching of face weights. The same projector is now pushed through to the left, upper most projector and then is annihilated using $P^2 = P$. The focus now turns to the indicated projector in the diagram below.
A copy of this projector is created using $P^2 = P$ and the YBE is applied to obtain

\[
\lambda - u - v - \lambda - u - v
\]

Once the identification (4.9) is made, then this is readily seen as the right hand side of (4.13).

### 4.3 BYBE solutions for $\mathcal{L}_S \mathcal{M}(p, p')$

#### 4.3.1 $1 \times 2$ fused face weights and associated $(r, s)$ boundary conditions

Following the construction in (3.33) and (3.53) of [3], the definition of the $1 \times 2$ fused face weights and associated boundary conditions is as follows.

The $1 \times 2$ fused face weights are composed of elementary face weights (2.4) and take the form

\[
X^{(1,2)}(u) = \begin{array}{c}
\begin{array}{c}
\text{u} \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{u + \lambda} \\
\text{u}
\end{array}
\end{array}
\] (4.28)

The $(r, s) = (1, 1)$ boundary triangles in the $1 \times 2$ case take the form

\[
\begin{array}{c}
\begin{array}{c}
\text{u} \\
\text{\lambda - 2u}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{u + \lambda} \\
\text{\lambda - 2u} \\
\text{u}
\end{array}
\end{array}
\] (4.29)

In this section, the notion of $(r, s)$ type boundary conditions (3.13) with elementary face weights is extended to constructing $(r, s)$ type boundary conditions using the $1 \times 2$ face weights (4.28). The full boundary condition is graphically depicted below in terms of elementary face weights.

45
4.3 BYBE solutions for $\mathcal{LSM}(p, p')$

\[ \rho - 1 \text{ columns of elementary face weights} \]

\[ s - 1 \text{ columns of } 1 \times 1 \text{ face weights} \]

The structure of the $1\times2$ face weights are indicated by the bold outlines of the elementary face weights. $\rho$ is defined by $\lfloor \frac{r p}{p'} \rfloor$ - where $\lfloor x \rfloor$ is the floor function of $x$, defined as the largest integer not greater than $x$. These boundaries in fact satisfy a BYBE analogous to (4.13). The proof of such claim is omitted, since it shares the same logic and steps in proving it for the $2 \times 2$ case - which is done explicitly in the next section.

4.3.2 $2 \times 2$ fused face weights and associated $(r, s)$ boundary conditions

In a way exactly analogous to the previous section, the $2 \times 2$ fused face weights are defined (using (3.33) of [3]) as

\[ X^{(2,2)}(u) = \begin{array}{c}
\begin{array}{ccc}
& u & \\
\hline
u & & \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{ccc}
& u & u + \lambda \\
\hline
u - \lambda & & \\
\end{array}
\end{array} \quad (4.30) \]

With the same $(r, s) = (1, 1)$ boundary triangles defined in (4.29). The full $(r, s)$ boundary conditions constructed out of the $2 \times 2$ elementary face weights are detailed below
4.3 BYBE solutions for $\mathcal{LSM}(p, p')$

Where the $(1, s)$ boundary is defined similarly to (3.4) as

$$
\lambda - u - v = \lim_{\xi \to -i\infty} u, \xi
$$

2 $(s - 1)$ columns of $1 \times 1$ face weights

For $r + s$ even, these boundary conditions fall into the Neveu-Schwarz sector, while for $r + s$ odd they are in the Ramond sector. They also satisfy the BYBE, which will be shown in the next section.

4.3.3 Proof of $2 \times$ BYBE for $(r, s)$ type boundary conditions

The BYBE for the $2 \times 2$ fused face weights and $(r, s)$ boundary conditions defined in (4.31) is

$$
\begin{array}{c}
(r, s) \\
\lambda - u - v \\
\downarrow u - v \\
\downarrow v, \xi \\
\end{array} = 
\begin{array}{c}
(r, s) \\
\lambda - u - v \\
\downarrow u - v \\
\downarrow v, \xi \\
\end{array}
$$

Where each boundary triangle is of the form (4.31) and each face carries the weights from (4.13). The left hand side of (4.31) is expanded in an analogous way to (3.13)
Since the boundary weights are such that the constituent faces satisfy (3.14), it is possible to ‘push’ through the elementary face weights one at a time. Pushing through the face with weight $\lambda - u - v$ first, then $-u - v$ and $2\lambda - u - v$ and finally the second face with weight $\lambda - u - v$, the expression becomes

$$2(\rho - 1) \text{ columns of elementary face weights}$$

Where the elementary face weights have been pushed through through the $2(\rho - 1)$ seams and now it is possible to apply the following, in order to push the face weights through the $s$-type boundary

$$2(s - 1) \text{ columns of } 1 \times 1 \text{ face weights}$$

Thus, after applying this to (4.33), this $2 \times 2$ block sits wedged between the two vacuum boundary solutions of the form (4.11). Now, the remaining $2 \times 2$ face weight is pushed through in an identical fashion - it passes through the $r$-type boundary seam using the elementary YBE on each of the
constituent faces of the block. Then it is pushed through the $s$-type seam using the previous step in reverse. At this point one can apply (4.13) and then push both weights back through the $s$-type boundary seams and the resultant picture is

$$
(r, 1) \quad \quad \quad \quad \quad \quad (1, s)
$$

![Diagram showing the process]

Carrying out the same procedure as before and pushing the individual face weights through, one at a time using (3.14) results in

$$
(r, 1) \quad \quad \quad \quad \quad \quad (1, s)
$$

![Diagram showing the process]

$$2(p-1) \text{ columns of } 1 \times 1 \text{ face weights}
$$

$$2(s-1) \text{ columns}
$$
Thus is exactly the expanded form of the right hand side of (4.31). This holds for arbitrary \( \xi \) but in this specific case, it is specialized to \( \xi = \frac{\lambda}{2} \).

<table>
<thead>
<tr>
<th>( u-v )</th>
<th>( \lambda + u-v )</th>
<th>( u-v-\lambda )</th>
<th>( \lambda - u-v )</th>
<th>( 2\lambda - u-v )</th>
</tr>
</thead>
<tbody>
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<td>( \lambda + u-v )</td>
</tr>
</tbody>
</table>

2\( (r-1) \) columns of elementary face weights

Thus is exactly the expanded form of the right hand side of (4.31). This holds for arbitrary \( \xi \) but in this specific case, it is specialized to \( \xi = \frac{\lambda}{2} \).
5 Conclusion

Starting from elementary principles in statistical mechanics, the first half of this thesis has given a whirlwind tour of the motivations and interplay between statistical mechanics and conformal field theory. By exploring results associated with the seminal paper by Belavin, Polyakov and Zamolodchikov, the concept of unitary CFTs and relevance to integral lattice models was outlined by outlining key results for the 2-d square lattice Ising model. A proof that satisfaction of the YBE together with the BYBE and the inversion identity implies commutativity of double row transfer matrices is given.

The second half of this thesis gives a brief historical overview of the development of CFT and the emergence of LCFTs. This motivates the construction of lattice models out of elements of the Temperley-Lieb algebra which yield $\mathcal{LM}(p, p')$ in the conformal limit. An alternative derivation of this logarithmic CFT by taking the logarithmic limit of $\mathcal{M}(p, p')$ is also recalled.

Starting with $\mathcal{LM}(p, p')$, fusion is employed, in which elementary face weights are fused together and the unphysical states are projected out. It is proposed that such a procedure results in $\mathcal{LSM}(p, p')$ in the conformal limit. Boundary conditions of $(r, s)$ type are then constructed using this method and their integrability is shown explicitly in a proof of the $2 \times 2$ fused BYBE. For $(r + s)$ even, it is proposed that the boundary condition falls into the Neveu-Schwarz sector, while for $(r + s)$ odd, then it is in the Ramond sector of the theory.

The $2 \times 2$ fused face weights constructed out of elementary Temperley-Lieb face weights form a special case of the Birman-Wenzl-Murakami (BMW) algebra [74]. It would be interesting to extend the above construction of integrable fused face weights to the construction of integrable models built from the full BMW algebra. Another interesting problem would be to investigate the conditions for the projectors to exist in general and what their explicit form is for the aforementioned BMW lattice models.

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