Epistemic Fidelity of Didactical Models
for the Teaching of Negative Numbers.

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Abstract

This thesis will examine the different didactical models used to teach negative numbers. Each model will be evaluated with regards to its treatment of the mathematical concepts required for a full understanding of directed number, the ways in which it represents the underlying number system, and its transparency and accessibility to students. Isomorphic models will be grouped, and the strengths and weaknesses of each model will be summarised.
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1 Introduction

This thesis is aimed generally at two classes of readers: mathematicians who may find it difficult to imagine being unable to understand the directed number system, and educationalists who understand how to operate with the directed number system but may know little of its formal construction.

This thesis will describe a range of didactical models which are used to introduce negative numbers to students. The term model will refer both to concrete models with physical manipulatives, and also to non-physical ‘real-life’ scenarios which are described verbally or represented pictorially.

The applicability of each model for illustrating the various properties of the directed number system will be discussed, as will the degree to which it accurately represents the underlying formal mathematical system.

A model is a map between a physical situation and a formal number system. This thesis will examine how close this map is for each of the models examined. Regardless of the strength of the map between any model and its target concept, the model must allow for this mapping to be made clear. Research studies have shown that students are not always able to make links between isomorphic problems[13] so they may not be able to link operations using a model with the equivalent mathematical concept.

Some educationalists believe that, due to the failure of any didactical model to completely capture every aspect of the directed number system, negative numbers should be taught as formal constructions without any recourse to incomplete didactical models[8, 11].

Despite this, the use of physical manipulatives or descriptions of ‘real-life’ situations is widespread in the teaching of mathematics, including the teaching of negative numbers. Some educationalists, including Baratta-Lorton[2], believe that abstract symbols cannot be used to help a child visualise a concept, and that introducing a concept in a purely symbolic fashion can actually interfere with understanding.
In a meta-analysis of 60 studies on the effects of manipulatives in mathematics instruction, Sowell[29] concludes that concrete manipulatives can increase mathematical achievement and improve student attitudes towards a subject.

Very little of this research refers specifically to directed numbers, and Freudenthal[11] suggests that negative numbers are the first concept encountered by students which are not well described by models which rely on intuition or real-life experience. It is therefore educationally important that more quantitative research is performed on the application of different models to the teaching of negative numbers.

Despite the lack of a homogeneous opinion, it is likely that teachers will continue to use didactical models in their treatment of negative numbers. It is then useful to examine the available didactical models so that their suitability to the purpose can be assessed.

It is clear that any model that is used must be introduced with an eye to its degree of representational accuracy, its applicability to the aspect of the directed number system in focus, its epistemic fidelity and its accessibility to students.

This thesis will address these issues for a range of didactical models. The strengths and weaknesses of each model will be discussed, and summarised in a table after the discussion.
1.1 Educational Context

Directed numbers are usually introduced at either late primary or early secondary school levels and by this stage, students should have a good grasp of the natural numbers and be familiar with the operations of addition, subtraction, multiplication and division.

For example, in the Victorian Essential Learning Standards, the Victorian Curriculum Assessment Authority recommends that students at Grade 6 level be introduced to negative numbers and able to place them on a number line, and that they be taught about operations on negative numbers at Grade 8 level[1].

Negative numbers are usually introduced after rational numbers in both fractional and decimal form. For example, in Victoria fractions are introduced in early primary school, and different operations, comparisons and manipulations with fractions are taught throughout primary school, with students usually expected to be proficient in fractions by the end of Grade 7[1].

It is sometimes implicitly assumed that students will be able to combine what they know of rational numbers and what they know of negative numbers to be able to work correctly with negative rationals. There is often little attention paid to negative rationals specifically.

Negative numbers are often taught as a single unit of work at a particular grade level, which in Victoria is usually Grade 8, and usually deal only or predominantly with integer calculation. Later topics will then assume knowledge of negative numbers in different contexts such as algebra or rational numbers, without further teaching of the subject.

For example in two commonly used textbooks in Victoria, ‘Maths Dimensions’[4] and ‘Heinemann Maths Zone’[5], the Grade 8 textbooks contain a section on directed numbers, which includes very few questions with non-integer values, and directed numbers are not covered explicitly in the textbook for any other grade level.
1.2 Conceptual Difficulties: Current and Throughout History

Directed or signed numbers are a concept that many students find difficult or confusing, which can be surprising to people with a higher degree of mathematics education, to whom they present no problem. For students, a difficulty with negative numbers can cause problems with other topics, and lead to an inability to interpret numbers and situations correctly[23].

One source of difficulty that students have with negative numbers relates to the conception of numbers as measures. This mindset leads to the conviction that it is impossible to have less than nothing - something even the great mathematician Blaise Pascal believed[14].

Historically, the restriction of number as a measure of magnitude or quantity continued even after the use of negative numbers was acceptable in calculations - they were considered as allowable in intermediate stages in the process of calculating an ‘acceptable’ or positive answer, but were not considered to have real meaning on their own[8].

Another source of difficulty is the dual purpose of the ‘minus’ sign signifying both an object and a process[27]. Negative numbers have historically been considered the result of a process of subtraction, rather than objects in themselves[16]. Part of the goal of any teaching sequence on directed numbers is for students to see negative numbers as entities in their own right.

No physical manipulative set or ‘real-life’ scenario used to model negative numbers completely captures every facet of the number system, despite centuries of work by mathematicians, including some of the great names in mathematics, such as MacLaurin, Euler, Laplace and Cauchy[8].

Hermann Hankel sidestepped this dilemma by separating the directed number system from reliance on physical representation, and defining it using formal number theoretical constructions, such that restricted numerical systems used to model physical domains could be embedded in the complete number system. This formal construction then
had the requirement that any calculation laws which hold in the restricted domain must not meet with any contradiction in the formal number system, a property which is known as the ‘algebraic permanence principle’[12, 8].

Hankel proved in order for the extended number system to respect the distributive law of multiplication over addition, the negative numbers must conform to the rule of signs which mathematicians take for granted.
1.3 Epistemic Fidelity

Epistemic fidelity refers to the strength and nature of the relationship between features of the model and features of the target mathematical system[31].

The relevant features of the target mathematical system, the directed numbers, are the skills needed to operate with, manipulate, and compare both integer and non-integer values. The density of the number line is also an important concept for students to understand.

Each model illustrates these various skills and ideas in different ways, with different degrees of adherence to the underlying mathematical concepts. Each model will be discussed in terms of how well it illustrates:

- Addition and Subtraction
- Multiplication and Division
- Size Comparison
- Number Density and Placement of Non-Integer Numbers

Subtraction has been grouped with addition because of the nature of formal construction of subtraction as adding the additive inverse number, and similarly division is formally constructed as multiplying by the multiplicative inverse number.

Working with negative numbers also requires the ability to compare two different numbers or place a group of numbers in order. It is also important to be able to do this with non-integer values. Discussion of the placement of non-integer values has been grouped with an understanding of the density of the rational numbers, as both require an appreciation of the existence of numbers other than integers, something that not all models illustrate well.

Any model that is used to introduce a mathematical concept must first be mathematically accurate, in the sense that it must not introduce misconceptions about the mathematical topic. A model need not capture every feature of the target domain to be useful,
since several models could be used in conjunction, or a model could be used as an
introduction to a topic, with further lessons involving symbolic or formal representations.
However it is important that the limitations in such cases are made apparent.

Another criterion against which to assess a didactical model is transparency. Transparency
is the extent to which it is clear to the student how features of the model correspond
to features of the target mathematical concept.

Transparency also requires the absence of extraneous features in the model which do
not correspond to features of the underlying number system, or that any such features
do not interfere with interpretation of the target domain.

Lave and Wenger[19] hold the view that transparency depends not just on the model
itself, but on the ways in which it is employed, and hence cannot be evaluated in
isolation, however it is possible to make an assessment about how the features of the
model are likely to interact with its modes of employment.

A model should also be accessible to students. This has several different aspects,
including familiarity with the metaphorical domain, the degree to which the use of
the model is intuitive, and the degree to which abstraction from the model to the
target concept is intuitive. These will all depend on the students involved and the
circumstances surrounding the use of the model in the classroom.

Familiarity is vital to any didactical model: models are metaphors for an underlying
number system, a means of describing one concept in terms of another concept. If
students are not familiar or comfortable with the concepts used in the model, then
the metaphor is useless.
2 Number Theory Constructions

This section outlines the formal number-theory construction of the integers and the rational numbers. These systems are extensions of the natural number system[3].

These formal constructs are not generally taught in secondary school, and are part of a tertiary course in number theory or pure maths. They form a framework against which more everyday uses of numbers can be understood.

The algebraic permanence principle[12, 8] which guides the creation of numerical extensions ensures that laws which govern calculations in one system, in this case the natural numbers, are not contradicted by laws governing the extended system.

By the stage that integers or rationals are introduced, students are expected to be familiar with the natural numbers. In this context, the set of natural numbers is assumed to include zero. Both integers and rationals are taught as an extension to the natural numbers, and can also be constructed formally as numerical extensions.

When teaching directed number in primary or secondary school, the goal is for students to be able to perform calculations that conform to the rules outlined in the formal system. Teaching of negative numbers should extend students’ existing calculation skills in the domain of natural numbers, in the sense that algebraic permanence ensures that these formal number systems will not contradict anything they have already been taught.

The ways in which negative numbers, fractions and irrational numbers are introduced in the classroom forms a parallel to the methods used to construct integers, rationals numbers and real numbers from the set of the natural numbers[3].

Bruno and Martinon[3] note that there are several paths that the formal construction of number systems can take, and also several paths for teaching of these number systems. Leaving aside consideration of irrational numbers, one path is for positive rational numbers to be defined as an extension of the natural numbers, and then for rational directed numbers to be defined as an extension of the positive rationals.
Another method is for integers to be defined first, as an extension of the natural numbers, and then for rational directed numbers to be defined as an extension of the integers.

Similarly, students can first be taught about rational numbers, in both fractional and decimal form, and can then be taught about negative numbers, including negative rationals. Or they can first be taught about negative integers, and then be taught about rational numbers, including negative rationals.

In the next section, rational numbers are constructed as an extension of the natural numbers, and then directed rationals as an extension of the positive rationals. This mirrors the order in which rational numbers and negative numbers are usually taught in schools.

Unfortunately, the ways in which these number systems interact is seldom taught explicitly: it is common for a curriculum to include only positive rational numbers and only integer negative numbers, or to be taught completely separately[3], as discussed in a previous section on educational context.
2.1 Equivalence Relations

Rational numbers and directed numbers are defined via equivalence relations. Each rational or directed number is represented by a number pair, which is equivalent to another number pair under certain conditions defined by the relation. Equivalence relations hold between members of a set, in this case, the set of pairs of numbers, and is of the form $x \sim y$. $\sim$ is an equivalence relation if:

- $\sim$ is reflexive: $\forall x, x \sim x$
- $\sim$ is symmetric: $x \sim y \implies y \sim x$
- $\sim$ is transitive: $x \sim y$ and $y \sim z \implies x \sim z$

This means, in the case where the set is number pairs, that:

- Every element in the set is equivalent to itself, so $(a, b) \sim (a, b)$.
- If $(a, b)$ is equivalent to $(c, d)$, then symmetrically, $(c, d)$ is equivalent to $(a, b)$.
- If $(a, b)$ is equivalent to $(c, d)$ which is equivalent to $(e, f)$, then $(a, b)$ and $(e, f)$ are equivalent.

This will be familiar in the context of the rationals as equivalent fractions:

\[
\frac{1}{2} \sim \frac{2}{4} \sim \frac{3}{6} \sim \frac{4}{8} \ldots
\]

Thus the equivalence relation used to define the positive rationals as pairs of natural numbers will be familiar. The equivalence relation used to define the directed numbers as pairs of positive rationals will be defined in a similar way.
2.2 Construction of the Rationals

Once the natural numbers are defined, they can be used to create the positive rational numbers\[18\]:
\[
\mathbb{Q}^+ := \mathbb{N} \times \mathbb{N} / \sim,
\]
where \(\mathbb{Q}^+\) denotes the positive rationals, and \(\sim\) is an equivalence relation defined by:
\[(a, b) \sim (c, d) \text{ if } ad = bc.\]

A rational number given by \((a, b)\) is commonly called a fraction and denoted \(\frac{a}{b}\), and addition and multiplication are performed as a fraction in a way students are familiar with, using addition and multiplication from the natural numbers:
\[
\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}
\]
\[
\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}
\]
2.3 Construction of the Directed Numbers

Once the positive rational numbers are defined as an extension of the natural numbers, directed numbers can be constructed as follows[17]:

\[ \mathbb{Q} := \mathbb{Q}^+ \times \mathbb{Q}^+/\sim, \]

where \( \sim \) is an equivalence relation defined by:

\[ (a, b) \sim (c, d) \text{ if } a + d = b + c. \]

A representative can be chosen from each equivalence class in the following manner, and then a single number can be used to identify this representative in a familiar way using negative and positive notation:

For \( x = (a, b) \):

- \( x = (a', 0) \) \quad s.t. \( a = a' + b \) \quad \( (a > b) \)
- \( x \mapsto +a' \)
- \( x = (0, 0) \) \quad \( (a = b) \)
- \( x \mapsto 0 \)
- \( x = (0, b') \quad s.t. \ b = a + b' \) \quad \( (b > a) \)
- \( x \mapsto -b' \)

Then we can define addition and multiplication to create \( \mathbb{Q} \) as a field, using addition and multiplication from the positive rational numbers:

\[ (a, b) + (c, d) = (a + c, b + d) \]
\[ (a, b) \times (c, d) = (ac + bd, ad + bc) \]

Since this construction in this form is dependent on algebraic theory, it is clearly inappropriate for use in the initial teaching of negative numbers. However, considering a negative number as the difference between two positive numbers can be used as an introduction to negative numbers, as in the ‘Passengers on a Bus’ model [34] which will be discussed later.

Considering this construction in this manner, if a number \( x \) is given by two numbers \( a \) and \( b \), then \( x = a - b \). In this way, negative numbers are defined as a subtraction.
of two positive numbers in which the second number, the subtrahend, is larger than the first number, the minuend.

Using this perspective, the equivalence class \((a, b) \sim (c, d)\) if \(a + d = b + c\) can be thought of as \((a, b) \sim (c, d)\) if \(a - b = c - d\).

This is the standard method for constructing the set of directed numbers, but an alternative construction could be proposed, and parallels the way in which negative numbers are sometimes considered:

\[
\mathbb{Q} = \mathbb{Q}^+ \cup (\mathbb{Q}^+)^{-1}
\]

Where \((\mathbb{Q}^+)^{-1}\) is defined as the set of numbers such that for each number \(q \in \mathbb{Q}\), there exists a number \(-q \in (\mathbb{Q}^+)^{-1}\) such that \(q + (-q) = 0\).

In this way, the negative numbers are seen as a ‘mirror’ of the positive numbers. This construction is simpler at first glance, but does not give a clear picture of the operations. \(\mathbb{Q}^{-1}\) is not a group under multiplication, so multiplication of two negative numbers is undefined, and multiplication between elements in different subsets is also not given.

In this construction, the negative numbers are seen as a ‘mirror’ of the known positive numbers, and using a mirror as a metaphor for inverses can cause confusion between negative numbers, fractions and decimals[30], since explanations of fractions and decimals also often utilise the mirror metaphor.
2.4 Irrational Real Numbers

The construction of the real numbers is beyond the scope of this discussion. At the stage of education at which negative numbers are usually introduced, students generally have very little concept of the difference between rational and irrational numbers\cite{10}, and conflate an irrational number with the decimal representation given by their calculators\cite{38}.

Since this suggests that students do not generally have a different way of thinking of irrational numbers as opposed to rationals, in the following discussion, any reference to real numbers will identify a real number with its decimal representation to an appropriate degree of precision.
2.5 Inverses

The above constructions only give definitions for addition and multiplication, yet methods for performing subtraction and division are also required. In order to define these, the concept of inverses is introduced.

Once negative integers and rationals have been constructed, negative numbers can be considered as additive inverses of natural numbers and of positive rationals. If \( n \) is a positive number, then its inverse, denoted \(-n\), is uniquely defined as a number such that \( n + (-n) = 0 \).

Subtraction of a number is then formally defined as adding the inverse of that number: \( a - b = a + (-b) \). Subtraction can only be defined in this way for a group that contains additive inverses, hence this definition works for integers and signed rationals, but does not work for natural numbers, as the for any natural number \( a \), the solution \( b \) to the equation \( a + b = 0 \) is not an element of the natural numbers.

The use of negative numbers as the inverse of a positive number, and subtraction as addition of the inverse number, has been used in the classroom, where subtraction is described as ‘adding the opposite number’[22].

Multiplicative inverses are defined similarly: for some number \( n \), its multiplicative inverse, denoted \( n^{-1} \), or \( \frac{1}{n} \), is a number such that \( n \times n^{-1} = n \times \frac{1}{n} = 1 \).

Division by a number is then formally defined as multiplying by its inverse: \( a \div b = a \times \frac{1}{b} \). Similarly to subtraction, division can only be defined in this way in a group that contains multiplicative inverses, hence the definition works for rational numbers but not for integers.
A negative number \((-a)\) can also be considered as \((-1) \times a\), and a positive number \(b\) as \((+1) \times b\), with \(a\) and \(b\) positive. If multiplication and division of positive numbers is understood, then multiplication and division including negative numbers can be reduced to the cases involving \((+1)\) and \((-1)\):

\[
(+1) \times (+1) = (+1) \\
(-1) \times (+1) = (-1) \\
(+1) \times (-1) = (-1) \\
(-1) \times (-1) = (+1)
\]

The first three identities can be shown using the fact that \((+1)\) is the multiplicative identity. Since this holds for the natural numbers, the algebraic permanence principle requires that it also hold for the extended number system of integers.

The last one can be shown using the distributive law, in a way similar to Hankel’s proof of the rule of signs\([33]\):

\[
0 = (-1).0 \\
= (-1).((+1) + (-1)) \\
= (-1).(+1) + (-1).(-1)) \\
= (-1) + (-1).(-1)) \\
(+1) = (-1).(-1)
\]

Once the rule of signs for multiplication has been established, each equation can be rearranged to give the rule of signs for division:

\[
(+1) \div (+1) = (+1) \\
(-1) \div (+1) = (-1) \\
(+1) \div (-1) = (-1) \\
(-1) \div (-1) = (+1)
\]
3 Didactical Models

Janvier[15] separates the different models used to teach directed numbers into two groups. One group is based on a number line, with numbers represented by positions on the line and operations represented by movement along the line or a change in position. The other group is based on combining two types of objects of ‘opposing nature’, such that one represents negative and one represents positive.

In this thesis, the models within each group will be further separated into subgroups, in order to facilitate more detailed evaluation of the different strengths and weaknesses of each model.
3.1 Number Line Type Models

Number lines are often used to teach directed numbers, and are frequently included in textbooks[5, 4]. They are sometimes drawn free from any context, but there are also several different ‘real-life’ situations which are modelled by a number line, and with which students may already be familiar. These situations are often used to teach directed numbers and may be used with a number line drawn, or the number line may be assumed.

Common models are temperature, sea-level, and building floor levels. There are some slight differences in the models, for example, building floor levels are limited to integers, whereas temperature and sea-level models are not.

In each case, the number line is arranged around an essentially arbitrary zero, with the negative part mirroring the positive part.

In these cases, an arbitrary point is chosen to be labelled as the origin or zero, a zero which exists in the mathematical formulation, but need not represent that arbitrary point: for example, the Celsius and Fahrenheit scales each place the zero at different arbitrary points, neither of which actually corresponds to ‘nothing’ of some quantity.

The fact that the zero is arbitrary can lead to misconceptions. Students are used to numbers representing quantities or magnitudes, with zero representing ‘nothing’. It is natural to say that ‘30° is twice as hot as 15°’ when in fact, because 0° is not the absence of heat, this statement is not true.

Number lines can run from the most negative numbers at the left to the most positive numbers at the right, or from the most negative numbers at the bottom to the most positive numbers at the top. The bottom-to-top orientation matches with students’ perception of smaller numbers being lower than bigger numbers, and the left-to-right orientation matches students’ knowledge of a non-negative number line, with 0 being at the left or start when reading from left to right.

Additionally, both of these correspond to axes in the Cartesian plane, so that when
students are introduced to the Cartesian plane, they will already be familiar with the orientations of both axes.

The word ‘lower’ can mean closer to the bottom of a bottom-to-top number line, or smaller in value. In practice, these two meanings correspond, but for clarity in this discussion, number lines will be described as running from left-to-right.

A slight variation on the number line model explicitly involves movement along the number line. This can sometimes involve students physically walking backwards and forwards on a number line on the floor, having them move manipulatives on a number line, such as in the ‘Small Trains’ model[34] model, which involves positive trains facing in one direction and negative trains facing in the opposite direction, or simply moving a finger or pointer along a number line.

Another variation involves treating the numbers as vectors or arrows. In one version, the first number, the minuend or first addend, is treated as a point on the number line, and the second number, the subtrahend or second addend, is treated as an arrow or vector originating from the point represented by the first and having length given by the second number, pointing in the direction of the polarity of the number.

In a second version, the first number is also treated as an arrow or vector, originating from the point zero on the number line, with length and direction described by the second number[12].

In either case, this provides a link between a structural and an operational conception of the negative numbers. An arrow from zero to the number it represents can be seen both as the number itself, and as the act of moving from zero to that number.

Freudenthal[12] also treats such arrows or vectors as members of equivalence classes, in the same way that a two-number representation treats number pairs. This treatment will be discussed further in the section on two-number representations.
3.1.1 Addition and Subtraction

The number line model can be very good at modeling subtraction as a difference: ‘How much colder is it in Moscow than Melbourne?’, ‘How much higher is the bottom of the Atlantic Ocean than the bottom of the Pacific Ocean?’, ‘How much higher is the penthouse than the basement?’. This can then be counted or measured on the number line.

One consideration with this kind of scenario is that students can complete these kinds of problems by circumventing negative number operations[25]. If one place is 6 degrees below zero, and another place is 2 degrees above zero, students will work out the difference is 6 + 2 = 8, rather than 2 − (−6) = 8, which is what the didactical model requires. While these calculations are perfectly valid, they do not advance the goal of increasing a student’s knowledge of operations with signs.

Another problem with treating subtraction in this manner is that the difference is often expressed as an absolute value: Students may think that, since 8 and 5 are 3 apart, 5 - 8 = 3. In order to clarify this without combining a difference model with another model, a teacher can impose a seemingly arbitrary condition that if the first number or minuend is less than the second number or subtrahend, then the difference must be expressed as a negative.

Modeling subtraction as take-away is not particularly well facilitated by this model, as it does not transparently represent objects to remove. A better alternative to a difference idea is to treat the subtrahend as a movement along the number line.

Addition and subtraction can both be modelled as movements along the number line. Using this model, start at the position of the minuend on the number line and then move the number of steps corresponding to the subtrahend. However, the direction in which to move is not necessarily immediately clear, so students may have to resort to direct use of the rule of signs to complete the calculation.

This problem can be partially overcome with some of the movement models that
involve either a student or a manipulative facing a particular direction. In these cases, a negative number is represented by facing backwards, and a subtraction is represented by walking backwards. Thus subtracting a negative involves walking backwards while facing backwards, which is equivalent to walking forwards.

Both a difference model and a movement model allow for the addition and subtraction of non-integer numbers, as the number line can include numbers between to the integers. A number line can be drawn to any scale: each unit can be divided into intervals of any size, however students may have difficulty in placing decimals in the correct positions[36], as will be discussed later.

### 3.1.2 Multiplication and Division

Multiplication of negative numbers by positive numbers is well modelled by a number-line. Multiplication can be performed easily as repeated addition: starting at the origin zero, the negative number is added a number of times corresponding to the positive number.

Multiplication of negative numbers by negative numbers needs to be thought of as repeated subtraction: the first negative number is subtracted a number of times corresponding to the magnitude of the second number[9].

Dividing a negative number by a positive number is best thought of as partition or sharing division[9]: breaking the negative dividend into the number of groups represented by the positive divisor, whereas dividing a negative number by a negative makes less sense in this context: students cannot think of a negative number of groups.

Dividing a negative number by a negative number is better thought of as quotation or measurement division[9]: breaking the negative dividend into groups the size of the negative divisor[28].

Since students are unable to envisage a negative number of groups, neither partition or quotation serve well to model a positive number divided by a negative number, so again, students may resort to the rule of signs directly to complete the calculation.
3.1.3 Size Comparison

This is perhaps the most transparent model for size comparisons. A student merely locates the two numbers on the number line, and whichever is closest to the left (or bottom, depending on the orientation of the number line) is the smallest, as these orientations have a strong degree of familiarity based on left-to-right reading and vertical positive number lines.

However, some students will have difficulty drawing or visualising the negative part of the number line, for instance, visualising the negative numbers on a separate positively-oriented number line translated downwards from the positive number line some distance such that it encompasses the number to be placed, for example, when placing \(-2\), students might label a number line: \((-0, -1, -2, 0, 1, 2...)[36]\).

3.1.4 Number Density

Using a number line allows students to place and visualise non-integer values, as a number line can be drawn to any desired scale. Since the negative half of the number line is treated as a mirror of the positive half, the degree to which students comprehend the density of the number line in the negative polarity will depend on their understanding of the density of the positive rationals.

While the number line allows rational numbers to be drawn, non-integer negative numbers are rarely taught explicitly, and students are not always able to place such numbers correctly[36].

Some students will correctly label the integers, but will have the decimal numbers in the intervals between integers either oriented incorrectly or placed between the incorrect integer values. This is in addition to students who cannot correctly place the integers.
3.1.5 Representational Mapping

This model is more closely analogous to the construction of the directed numbers as the union of the positive rationals and their inverses, than it is to the two-number equivalence class construction, as it represents the negative part of the number line as a mirror-image of the familiar positive number line, each natural number being reflected to its additive inverse.

Treating subtraction as walking or moving backwards is analogous to the formal construction of subtraction as addition of an inverse.

Since the formal construction:

\[ \mathbb{Q} = \mathbb{Q}^+ \cup (\mathbb{Q}^+)^{-1} \]

Does not describe coherently the rules for multiplication or division of elements from each of the subsets, there can be no analogue of multiplication or division to the formal construction.

3.1.6 Transparency

Using a bare number line does not introduce any extraneous details which could cause confusion, however the linking of the number line to ‘real-life’ concepts such as temperature, sea- or floor-level, or trains could introduce irrelevant details, depending on a student’s previous experience with these topics.

3.1.7 Accessibility

Students are generally familiar with the idea of negative temperatures and the thermometer as a number line. They are also familiar with the positive number line, which is often used in earlier mathematics teaching, and the idea of using a number line to represent relative position.

A number line is also convenient, in that even without any other concrete manipulatives, students are able to draw a number line next to their work and use it for calculations.
3.2 Debts and Assets

It is possible to model positive numbers as money and negative numbers as debts. Students are generally familiar with the idea of owing somebody money, so they have some concrete frame of reference for this model. It is also possible to provide physical manipulatives, in the form of model currency and debt-notes.

This is the first of the models which Janvier[15] would classify as being based on two objects of opposing nature, in this case assets and debts, but because this model does not rely on concrete manipulatives, allows calculations with non-integer values, can be used in a number of different ways and was historically the first application of negative numbers to real situations[24], it is treated separately.

When used with physical manipulatives, this model can be used in a similar way to the Algebra Tiles model (discussed in the next section), in that there are positive pieces and negative pieces which cancel each other out in, and can be made into ‘zero pairs’. The differences are in the way computations are performed with Algebra Tiles in a strict rules-based manner without reference to a familiar scenario such as money.

A debts and assets model can also be considered as a two-number model, as will be discussed later.

Pelled and Carraher[25] note that students can find this model ambiguous: If a person has no money, and borrows $20, do they now have a) $20 because they have $20 in their hand? b) $0 because they have none of their own money? or c) −$20 because they owe $20? Students can give coherent arguments for all of these positions, so while the didactical model requires that the answer be $0, this is not always clear to students.

3.2.1 Addition and Subtraction

Addition and subtraction are modelled similarly to Algebra Tiles discussed in the next section. For addition, money notes or debt notes representing the two addends are placed together, and any debt-notes that can be repayed are removed along with
the repayment money, essentially removing zero pairs as in the Algebra Tiles model.

Since this model can also be used without physical notes, first calculate if any debts can be paid off, and if so, how much is left over, and if debts cannot be fully paid off, how much would still be owing if all your money has gone towards partially paying off the debt.

Subtraction can also be modelled similarly to Algebra Tiles, by first placing the notes representing the minuend or first number into the working space, then placing sufficient zero pairs of debt notes and money notes into the working area so that the notes representing the subtrahend, the second number, can be removed. After some experimentation, students will find that instead of removing a debt, they can put in the equivalent magnitude in money. Similarly for subtracting a positive number, if the minuend is not sufficiently large to remove the positive number, students can ‘borrow money’ which gives them money notes and the equivalent in debt notes, ie ‘zero pairs’, and then remove the correct number of money notes.

This can also be done without manipulatives: after some experience with this model, students can learn to consider removing a debt to be increasing the money, and removing money to be increasing the debt.

### 3.2.2 Multiplication and Division

Multiplication can be seen as repeated addition or repeated subtraction: multiplication of a negative by a positive is either repeated removal of money, or a repeated increase in debt. Multiplication of a negative by a negative can be seen as repeated removal of debt. Since removing a debt is equivalent to increasing money, represented removal of debt can also be seen as repeated increase in money.

Similarly to the number-line models, dividing a negative number by a position number is best thought of as partition: splitting the debt between the number of people represented by the positive divisor, and working out how large each debt is, whereas dividing a negative number by a negative makes less sense in this context: students cannot think of a negative number of people.
Dividing a negative number by a negative number is better thought of as quotition: breaking the debt into parts the size of the negative divisor[28], and working out how many parts there are.

Since students are unable to invisage a negative number of people, neither partition or quotition serve well to model a positive number divided by a negative number, so students may resort to a direct use of the rule of signs.

3.2.3 Size Comparison

The introduction of a familiar framework makes size comparison somewhat intuitive. Students will see that any amount of debt is less than having any amount of money, and that if you owe more, you are poorer than if you don’t owe as much, so it is more intuitive to see more negative numbers as smaller than less negative numbers.

3.2.4 Number Density

This framework allows some consideration of non-integer values: if the units are dollars, then students are familiar with parts of dollars, and so will be able to consider decimals up to two decimal places. Students may be able to extrapolate to smaller intervals, but it may also have the effect that students can only consider numbers up to two decimal places, and will not see any further decimal places as sensible in any context, even once they have moved away from considering the model directly, since they are not sensible in the model with which they originally learned negative numbers.

3.2.5 Representational Mapping

This model is analogous to the formal construction of integers using pairs of numbers, in this case the two numbers representing assets and debts. The mapping of addition from this model to this construction is strong: if one is adding the net values of two people, one adds their assets and adds their debts to get a new pair of numbers giving a new net value, in an identical manner to the description of addition in the formal
construction: \((a, b) + (c, d) = (a + c, b + d)\).

Subtraction however does not make quite as neat an analogy to a two number construction, especially in question worded as 'find the difference between the net values of two people', since either the difference in assets or the difference in debts may itself be a negative number, whereas in the formal construction, both these numbers are positive by definition.

In this case, it would make more sense to treat the net value as a single number, and then find the difference either by the rule of signs or using a number-line representation.

While multiplication and division are closed in the formal construction, they are not closed if numbers are tied to assets and debts: it does not make sense to multiply one net value by another net value, so the operation cannot be completed without using numbers represented in a different form, such as a number of people each with some net value.

For division, a net value can be divided by a net value, if a question is given a wording such as 'If you share $x$ between a number of people so that each gets $y$, how many people can have a share?', but in this case, the answer is not a net value, so again this shows that division cannot be considered to be closed when a 'real-life' scenario is imposed.

### 3.2.6 Transparency

Creating a link to a debts and assets model can cause confusion for some students - many students only consider assets when thinking about their net value, and are unable to consider debts. If they have $20 dollars in hand, they consider that they have $20, even if they owe $30 to somebody[25].

Students may also have other preconceptions about the idea of money or debt that may cause confusion.
3.2.7 Accessibility

At the stage at which directed numbers are usually introduced, all students should be familiar with the concept of assets and money, and most will be familiar with the idea of owing somebody money.

However, they will also have familiar ways of working with money, which will probably not utilise negative numbers, and may interfere with the imposition of a negative number system onto a known framework.
3.3 Algebra Tiles

Algebra Tiles or Algebra Blocks are used for the teaching of algebraic equations, but are also often used for teaching directed numbers[20]. They consist of integer tiles, which are small squares, and $x$ and $y$ tiles which are the same width as the integer tiles but longer, different lengths to each other. They all come in two colours, one colour (denoted white) representing positive numbers, and another colour (denoted black) representing negative numbers. Sometimes the colours are separate pieces, or sometimes as opposite sides of the same piece.

Since this discussion is only of directed real numbers, this discussion will only consider integer tiles, not $x$ or $y$ tiles.

Examples:

The number $(+5)$

\[
\begin{array}{cccc}
\square & \square & \square & \square \\
\end{array}
\]

The number $-4$

\[
\begin{array}{cccc}
\text{■} & \text{■} & \text{■} & \text{■} \\
\end{array}
\]

This model falls into Janvier’s[15] group of models that use two objects of opposing natures, but has a complete system of working that allows for object manipulation rules to model all the operations.

There are also several other didactical models that share similarities with the Algebra Tiles model. One is ‘The Witch’[25] [34], which asks students to imagine a witch’s cauldron which can contain hot cubes and cold cubes which cancel each other out.

Another involves positive weights and negative balloons (as in hot air or helium balloons, which rise and reduce the weight). Janvier[15] reports that students who were introduced to signed numbers using a balloon and weight model had better success at addition only than students who were taught using a coloured tile model. There was no difference noted in performance at subtraction, and the research did not cover multiplication or division.

Scenarios such as weights and balloons, hot and cold, or other opposing properties may make the cancellation of hot with cold or up with down seem more intuitive than
cancelling black and white, however they do not allow for the complete modelling of multiplication and division by negative numbers as Algebra Tiles do.

3.3.1 Addition and Subtraction

Algebra blocks can only be used to model integer addition and subtraction, as the tiles to not split up into smaller parts.

To model addition using Algebra Tiles, the tiles representing the each addend are placed into the working area. Then any ‘zero pairs’ are removed. A ‘zero pair’ is created when a white or positive tile can be matched with a black or negative tile. Since adding or subtracting zero makes no difference to the sum, ‘zero pairs’ can be removed. When all ‘zero pairs’ have been removed, the remaining tiles represent the answer to the sum.

Examples for addition:

$$(+4) + (-3) \quad \text{(remove zero pairs)} = (+1)$$

$$\begin{array}{cccc}
\square & \square & \square & \square \\
\quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad \\
\end{array}$$

$$\begin{array}{ccc}
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\end{array}$$

$$(−5) + (+2) \quad \text{(remove zero pairs)} = (−3)$$

$$\begin{array}{cccccc}
\quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\end{array}$$

$$(−2) + (−3) \quad \text{(no zero pairs to remove)} = (−5)$$

$$\begin{array}{ccc}
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\quad & \quad & \quad \\
\end{array}$$

The use of zero pairs models equivalence classes in the formal construction of integers. If a number $x = (a, b)$, then the same number with an extraneous zero pair is given by $(a+1, b+1)$. We can show that the two numbers are equivalent using the definition of equivalence classes in the construction of integers:

$$(a + 1, b + 1) = (a, b) \text{ since } a + (b + 1) = (a + 1) + b$$

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as the natural numbers are associative and commutative.

To model subtraction using Algebra Tiles, the tiles representing the minuend are placed into the working area. Then from this, tiles representing the subtrahend are removed. If there are not enough tiles of the required colour in the working area to represent the subtrahend, zero pairs are placed into the working area until there are sufficient tiles, since adding zero makes no difference to the equation. Once the subtrahend tiles have been removed from the working area, the tiles remaining represent the answer to the problem.

Examples for subtraction:

\[
(+4) - (+3) \quad \text{(remove subtrahend)} \quad = (+1)
\]

\[
\begin{array}{c}
\Box \quad \Box \quad \Box \quad \Box
\end{array}
\]

removed: \[
\begin{array}{c}
\Box \quad \Box
\end{array}
\]

\[
(-6) - (-2) \quad \text{(remove subtrahend)} \quad = (+1)
\]

\[
\begin{array}{c}
\blacksquare \quad \blacksquare \quad \blacksquare \quad \blacksquare \quad \blacksquare \quad \blacksquare \quad \blacksquare
\end{array}
\]

removed: \[
\begin{array}{c}
\blacksquare \quad \blacksquare
\end{array}
\]

\[
(-5) - (+2) \quad \text{(insert zero pairs)} \quad = (-3)
\]

\[
\begin{array}{c}
\blacksquare \quad \blacksquare \quad \blacksquare \quad \blacksquare \quad \blacksquare \quad \blacksquare
\end{array}
\]

\[
\begin{array}{c}
\Box \quad \Box
\end{array}
\]

removed: \[
\begin{array}{c}
\Box \quad \Box
\end{array}
\]

\[
(-2) - (-3) \quad \text{(insert zero pairs)} \quad = (+1)
\]

\[
\begin{array}{c}
\blacksquare \quad \blacksquare
\end{array}
\]

\[
\Box
\]

removed: \[
\begin{array}{c}
\blacksquare \quad \blacksquare \quad \blacksquare
\end{array}
\]
Note that the operations of addition and subtraction are kept distinct. The minus sign is being used both structurally and operationally in separate ways. Sfard[27] suggests that the way in which negative numbers are learned is a process of progressing from an operatational to a structural approach. This may then interfere with one of the goals for the teaching of negative numbers, which is for students to progress from an operational to a structural viewpoint.

3.3.2 Multiplication and Division

Algebra blocks can only be used to model integer multiplication, and division where the quotient is also an integer.

To model multiplication using Algebra Tiles, the working area is set up as a rectangle. The tiles representing the multiplicand, or second number, are layed out in a horizontal line at the top, and the tiles representing the multiplier, or first number, are layed out in a vertical line down the side. Next to each tile in the multiplier, a copy of the multiplicand is layed out in a horizontal line. Then, if the multiplier is black or negative, the tiles in each copy of the multiplicand are swapped to the opposite colour.

To model division using Algebra Tiles, the working area is again set up as a rectangle. The tiles representing the divisor, the number to divide by, are layed out in a horizontal line, and the tiles representing the dividend, or the number to be divided, are layed out in a rectangle underneath, so that each row of the rectangle contains the same number of tiles as the divisor. If the divisor is black or negative, then each tile in the rectangle is swapped. Then, for each row in the rectangle, a tile is placed in a line along the side, which represents the quotient for the equation.
Examples for multiplication:

\[
(+4) \times (+3) = (+12)
\]

\[
\begin{array}{c|cccc}
\times & \square & \square & \square & \square \\
\hline
\square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square \\
\end{array}
\]

\[
(+4) \times (-3) = (-12)
\]

\[
\begin{array}{c|cccc}
\times & \black & \black & \black \\
\hline
\square & \black & \black & \black & \black \\
\square & \black & \black & \black & \black \\
\square & \black & \black & \black & \black \\
\square & \black & \black & \black & \black \\
\square & \black & \black & \black & \black \\
\end{array}
\]

\[
(-4) \times (+3) \quad \text{(multiplier negative)} = (-12)
\]

\[
\begin{array}{c|cccc}
\times & \square & \square & \square & \square \\
\hline
\black & \square & \square & \square & \square \\
\black & \square & \square & \square & \square \\
\black & \square & \square & \square & \square \\
\black & \square & \square & \square & \square \\
\black & \square & \square & \square & \square \\
\end{array}
\]

\[
(-4) \times (-3) \quad \text{(multiplier negative)} = (+12)
\]

\[
\begin{array}{c|cccc}
\times & \black & \black & \black & \black \\
\hline
\black & \black & \black & \black & \black \\
\black & \black & \black & \black & \black \\
\black & \black & \black & \black & \black \\
\black & \black & \black & \black & \black \\
\black & \black & \black & \black & \black \\
\end{array}
\]

35
Examples for division:

<table>
<thead>
<tr>
<th></th>
<th>(place side tiles)</th>
<th>= (+4)</th>
</tr>
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<tbody>
<tr>
<td>(+12) ÷ (+3)</td>
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<td>□ □ □</td>
</tr>
<tr>
<td>(−12) ÷ (−3)</td>
<td></td>
<td>= (+4)</td>
</tr>
<tr>
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</tr>
</tbody>
</table>

36
Algebra tiles allow for calculation following a set of rules, but the rules are imposed and can seem arbitrary or unjustified\cite{22}, leading to students becoming disconnected from the learning of the rules.

### 3.3.3 Size Comparison

Algebra Tiles do not lend themselves to size comparison of anything but pairs of positive numbers. For two numbers $a$ and $b$, if both are negative, and $a < b$, $a$ will be represented by a greater number of black tiles, giving the impression that $a > b$. If $a$ is negative and $b$ is positive, then the number of tiles representing each is immaterial to the relative size of the numbers.

### 3.3.4 Number Density

Since Algebra Tiles only deal with integers, they do not give students an idea of any fractional or decimal numbers.

### 3.3.5 Representational Mapping

Algebra tiles allow for mappings to both the two-number formal construction and the union of positive numbers and their inverses. The concept of equivalence in two-number representation is illustrated by the fact that a representation of a number remains in the same equivalence class with the insertion or removal of zero pairs.

The concept of flipping a collection of tiles (or replacing them with the opposite colour tiles) is a direct analog of the concept of an additive inverse used in the construction of directed numbers.

### 3.3.6 Transparency

Since the rules for the manipulation of the objects may seem imposed, students may find it easy to perform the manipulations to arrive at a correct solution, but may have difficulty interpreting that solution in symbolic terms or describing why the solution is correct, and may have difficulty replicating the correct working in the absence of
Some teachers encourage students to use the manipulatives for as long as they need to in a form of scaffolding, or encourage students to represent the tiles pictorially in the absence of the physical objects.

### 3.3.7 Accessibility

Students will be comfortable with a one-to-one correspondence of number to object utilised by this model, and will be familiar with the use of counters to represent numbers. The rules of manipulation are simple, and students will likely find the model itself accessible.

However, because of the possible disconnect between manipulations of objects and manipulations of numbers, students may fail to find the model to be an accessible method of learning operations on directed numbers.
3.4 Two-number difference models

The theoretical construction of integers which gives rise to negative numbers defines a directed number as a combination of two positive rationals. There are several scenarios which can be used to introduce this model of directed numbers. One is the ‘passengers on a bus’ model, which is introduced by Van Den Brink[35], and extended to represent negative numbers by Streefland[34], or similarly, ‘patrons entering a venue’[22]. Another is an extension to the Debts and Assets model.

In the ‘passengers on a bus’ model, a bus is thought of as having some unknown number of passengers on it. At each stop a number of passengers get off and a number of passengers get on, and these two numbers form the net change in the number of passengers on the bus, creating a negative number if more people get off than get on. Students will find that if one extra or fewer person gets on, and one extra or fewer person gets off, the net change remains the same.

The ‘patrons in a venue’ model is similar, with each two-number set representing the number of people entering or leaving.

In the debts and assets model, rather than paying off the largest amount possible towards any debts as in the previous scenario, a person is assumed to keep a certain amount of money, and owe a certain amount of money, which is somewhat more in line with a person’s usual finances, and these two numbers form the person’s net worth. If person pays some of their money towards a debt, or borrows some more money, their net worth remains the same.

In all these cases, if there is a change to the number representing the positive part that is balanced by an equal change to the number representing the negative part, the net value remains the same. This allows the set of two numbers to form an equivalence class, as in the algebraic construction of the integers. Students who have previously been introduced to fractions will be familiar with the concept of different representations of the same number: In the same way that, in the domain of fractions, \( \frac{1}{2} = \frac{2}{4} = \frac{3}{6} \), in the domain of integers, \((0 - 3) = (1 - 4) = (2 - 5)\)[26].
The bus and venue models have the extra complication that there is an extra unseen value - the number of people already inside the bus or venue, so the number modelled by the people entering or leaving represents the net change to this unseen value. This could be confusing for students. However, it does also highlight the creating of arbitrary zero values, such as are used in temperature scales.

The two numbers used in this model can also be represented on a double abacus[6]. Linchevski and Williams[22] discuss a teaching sequence in which a double abacus is used to model the ‘patrons in a venue’ scenario, one wire representing people entering and the other people leaving, and note that once the number on either wire exceeds the available beads, students are required to consider the equivalence classes of number pairs.

Note that the equivalence classes in two-number models make more explicit the idea of zero pairs used in the Algebra Tiles manipulatives model. It is also possible to combine this model with a number line model which uses arrows or vectors to represent numbers. In this combined model, each end-point of the arrow represents one of the numbers in a pair. If a number is represented by the number pair \((a - b)\), then the origin of the arrow represents \(b\), and the head of the arrow represents \(a\). The fact that a number can be represented in several different ways is modelled by moving the arrow along the number line - as long as the orientation of the arrow is kept consistent, the end-points of the arrow give a number pair for different elements of the equivalence class for the number represented by the arrow[12].

### 3.4.1 Addition and Subtraction

To model addition and subtraction using these models, one can consider either two buses, two gates at a venue or two people. For subtraction, one can calculate the difference between the changes on one bus or gate and the changes on the other, or the difference in net worth of two people.

For addition, one can calculate the total changes in the number of people on both buses, two gates at the venue, or the total net worth of two people.
In either case, students can perform calculations with the net change or net worth, or they can calculate the total value of the positive aspect (people entering or assets) and the total value of the negative aspect (people leaving or debts) to create two new numbers which represent the new net value.

3.4.2 Multiplication and Division

A two-number model with the following scenarios as context do not represent multiplication or division well, as it does not make sense to multiply or divide a two-number net amount by a two-number net amount.

However, one can model multiplication by using values that do not have a two-number representation, for example, multiplying the change in each bus by a number of buses, or the net worth of each person by a number of people. Similarly, one can divide the net change in passengers on a bus among a number of buses, or the net worth of a person among a number of people.

In these cases, each number in a pair is either multiplied or divided by the multiplier or divisor which is represented by a single number, to give a new number pair.

While it is possible to use the formal construction of multiplication using a two-number form, the formal definition, \((a, b) \times (c, d) = (ac + bd, ad + bc)\) does not correspond to anything students are likely to be able to make sense of.

3.4.3 Size Comparison

This model does not allow for the intuitive comparison of two numbers, unless it is used in conjunction with arrows or vectors in a number-line model, since it is the difference between the two natural numbers used to represent each integer, rather than the magnitude of either of them, which must be considered.

If used with arrows or vectors in conjunction with a number-line model, it suffers from the same problem as the Algebra Tiles model, in that it is only the magnitude of the number which is obvious intuitively, so comparison is only intuitively obvious.
for two positive numbers.

### 3.4.4 Number Density

While a two-number model as constructed in number theory is excellent at modelling non-integer values, the scenarios which are usually used to introduce them impose restrictions. A bus model does not allow for non-integer values for the number of people, and the assets and debts model allows only decimals to two decimal points, the cents, if the unit is considered as dollars. However, the use of models that use only money to illustrate decimal values may encourage the misconception known as ‘money thinking’[32].

### 3.4.5 Representational Mapping

As with the debts and assets model, this model is analogous to the formal construction of integers using pairs of numbers. The mapping of addition from this model to this construction is strong: if one is adding a pair of numbers such as net changes on a bus, one adds the people entering each bus and adds the people leaving each bus to get a new pair of numbers giving a new change in the number of passengers, in an identical manner to the description of addition in the formal construction: 

\[(a, b) + (c, d) = (a + c, b + d).\]

When using a two-number model with a debts and assets framework, subtraction is similar, except that the formal construction does not contain subtraction as an operation, but rather defines it as adding the inverse. Since the magnitudes of debts and assets are positive numbers, there can be no inverse numbers, so subtraction cannot be defined in this way.

While one can validly define subtraction as: 

\[(a, b) - (c, d) = (a - c, b - d),\]

this is not clearly in line with the formal construction, as either \(a - c\) or \(b - d\) may themselves be negative numbers.
3.4.6 Transparency

Using a bus model or venue model introduces an extra unseen number, the number of people already in the bus or venue, which is treated as the base level or zero, and which can cause unnecessary confusion for students, depending on how it is introduced. With a debts and assets framework of a two-number model, the base level is itself zero, and so no unseen number is required.

3.4.7 Accessibility

All of the ‘real-life’ concepts will be familiar to students, however they may have some difficulty with the idea that different number pairs can represent the same number, despite previous exposure to equivalent fractions.
3.5 Mathematical Patterns and the Rule of Signs

While more a method than a model, numerical patterns are often used to introduce negative numbers, and to allow students to convince themselves about the rule of signs, for example in the table below[12]:

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>$3 + 2 = 5$</td>
<td>$3 - 2 = 1$</td>
<td>$3 \times 2 = 6$</td>
<td>$(-3) \times 2 = -6$</td>
</tr>
<tr>
<td>$3 + 1 = 4$</td>
<td>$3 - 1 = 2$</td>
<td>$3 \times 1 = 3$</td>
<td>$(-3) \times 1 = -3$</td>
</tr>
<tr>
<td>$3 + 0 = 3$</td>
<td>$3 - 0 = 3$</td>
<td>$3 \times 0 = 0$</td>
<td>$(-3) \times 0 = 0$</td>
</tr>
<tr>
<td>$3 + (-1) = ...$</td>
<td>$3 - (-1) = ...$</td>
<td>$3 \times (-1) = ...$</td>
<td>$(-3) \times (-1) = ...$</td>
</tr>
<tr>
<td>$3 + (-2) = ...$</td>
<td>$3 - (-2) = ...$</td>
<td>$3 \times (-2) = ...$</td>
<td>$(-3) \times (-2) = ...$</td>
</tr>
</tbody>
</table>

Once these patterns are explored, students use the rule of signs, frequently given in textbooks[5, 4], to perform calculations, with the rules given below, with a pronumeral without a prefix being a positive number:

<table>
<thead>
<tr>
<th>Addition and Subtraction</th>
<th>Multiplication and division</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pm a + (+b) = \pm a + b$</td>
<td>$a \times b = a \times b$</td>
</tr>
<tr>
<td>$\pm a + (-b) = \pm a - b$</td>
<td>$a \times (-b) = -(a \times b)$</td>
</tr>
<tr>
<td>$(a) \times b = -(a \times b)$</td>
<td>$(a) \times (-b) = a \times b$</td>
</tr>
<tr>
<td>$\pm a - (+b) = \pm a - b$</td>
<td>$a \div b = a \div b$</td>
</tr>
<tr>
<td>$\pm a - (-b) = \pm a + b$</td>
<td>$a \div (-b) = -(a \div b)$</td>
</tr>
<tr>
<td>$(a) \div b = -(a \div b)$</td>
<td>$(a) \div (-b) = a \div b$</td>
</tr>
</tbody>
</table>

Note that with the addition and subtraction rules, no distinction is generally made between a positive and negative minuend or first addend, therefore there is generally no explicit instruction for rearranging addition sums to an easier form, for example \((-2) + 7\) is more easily calculated as \(7 - 2\), but this is not always made explicit.

Students can often confuse these rules: they know that ‘a negative and a negative is a positive’, so when faced with an addition problem of the form \((-a) + (-b)\), or
its equivalent formulation as a subtraction, they will over-apply the rule and assume that the answer has to be positive.

These rules allow most calculations to be reduced to an equivalent calculation in the positive realm, with the exception of addition and subtraction in which the first addend or the minuend is of a different polarity to the answer, such that, modelled on a number line the calculation crosses zero, a case which will be discussed later.

If the rules are presented as a *fait accompli*, perhaps without any student experimentation with number patterns or other models, and without understanding, students are required to rote-memorise the rules, which in general does not promote deep learning. However, used well, using number patterns allows students to work the rules out for themselves. This is in accord with a constructivist approach[37], and studies have shown that rules or facts developed oneself or in discourse with peers have a more salient effect on students[7].

### 3.5.1 Addition and Subtraction

By following the rule of signs, most cases are easily reduced to known calculations with positive numbers. The exceptions are:

\[ a - b, b > a \]

and

\[ -a + b, b > a \]

Problems of second form could be converted to \( b - a \), but students are often unaware of this, despite having some understanding of commutativity with positive numbers, or they may incorrectly rearrange such a problem as \(-b + a\).

### 3.5.2 Multiplication and Division

By following the rule of signs, performing multiplication and division is reduced to known calculations for positive numbers.
3.5.3 Size Comparison

This method is more a method for calculation than for making sense of the numbers involved, and following the rule of signs does not give students an intuitive sense of the relative size of two numbers, except for the basic idea that any negative number is smaller than any positive number.

3.5.4 Number Density

This method essentially reduces students’ knowledge of negative numbers to the corresponding positive values, and hence depends on students’ existing knowledge of the density of the positive numbers.

3.5.5 Representational Mapping

Being a method rather than a model, there is not a representational mapping to be made to a formal construction of the number system.

Using number patterns and experimentation can allow students to discover for themselves the rule of signs, or allow them to convince themselves that it is correct. However this does not allow for a direct mapping to either the two-number construction or the union of the positive numbers with their inverses.

3.5.6 Transparency

Since this is a method for working directly with numbers rather than assigning external meaning, there is no issue of extraneous details obscuring the underlying concept for students. However the transparency and accessibility will depend on how the students are encouraged to look for number patterns.

3.5.7 Accessibility

By the stage that negative numbers are usually taught, students are generally familiar with number patterns, so if students are encouraged to look for patterns and guided
to make sense of them, this method will allow students to make rules for themselves. However, if the rule of signs is simply given to students, some students may find this quite inaccessible, and may lead to confusion and mis-application of rules.
## Summary of Strengths and Weaknesses of Models

<table>
<thead>
<tr>
<th>Model</th>
<th>Strengths</th>
<th>Weaknesses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number Line Type Models</td>
<td>• Excellent size comparison</td>
<td>• Rule of signs not readily apparent</td>
</tr>
<tr>
<td></td>
<td>• Good subtraction-as-difference visualisation</td>
<td>• Difficulty modelling positive divided by negative</td>
</tr>
<tr>
<td></td>
<td>• Can be drawn to any scale for decimal placement</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Good modelling of addition and multiplication</td>
<td></td>
</tr>
<tr>
<td>Debts and Assets</td>
<td>• Excellent real-life example of two-number construction</td>
<td>• Students’ tendency to ignore debts</td>
</tr>
<tr>
<td></td>
<td>• Size comparison can be intuitive</td>
<td>• Difficulty modelling positive divided by negative</td>
</tr>
<tr>
<td>Algebra Tiles</td>
<td>• Models all operations</td>
<td>• Manipulation rules may seem arbitrary</td>
</tr>
<tr>
<td></td>
<td>• Analogous to both formal constructions</td>
<td>• Does not intuitively allow for size comparison</td>
</tr>
<tr>
<td></td>
<td>• Object manipulation rules relatively simple</td>
<td>• Links between model and number system may not be apparent</td>
</tr>
<tr>
<td>Two-number difference</td>
<td>• Excellent correspondence with formal number system construction</td>
<td>• Rule of signs not readily apparent</td>
</tr>
<tr>
<td>models</td>
<td>• Allows student to develop ideas of equivalence classes</td>
<td>• Size comparison not intuitive</td>
</tr>
<tr>
<td>Mathematical Patterns</td>
<td>• No chance of extraneous details confusing students</td>
<td>• Some implementations may contain confusing unseen number</td>
</tr>
<tr>
<td>/Rule of Signs</td>
<td>• Student exploration allows students to develop rules themselves</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Rule of signs can be imposed rather than explored</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• May not give understanding of reasons behind rules</td>
</tr>
</tbody>
</table>
4 Conclusion

There are a wide variety of didactical models used by teachers in a variety of ways to introduce and illustrate various concepts and skills involved in working with negative numbers. This thesis has examined the ways in which each model corresponds to a formal number theory construction of negative numbers, and the ways in which each model illustrates different features of the negative number system.

No one model can will completely encapsulate the underlying number system in a way that is transparent and accessible to students. Since no one model is complete, various models can be used in combination. Number lines are frequently used as one component in a program for the teaching of negative numbers, due to the fact that they give excellent intuitive size comparison, have several ‘real-world’ situations which can be applied to them, and allow for the placement of decimal numbers to any scale.

Number lines are good for modelling addition and subtraction in terms of the scenarios applied to them such as sea-level or temperature, and may be used to complement other models for the calculation of division and multiplication.

Debts and assets as a model is also frequently used, possibly because this was the first application of negative numbers to a practical problem, because it allows for an excellent representation of a two-number construction of the negative numbers, including demonstrating the concept that two pairs of numbers can represent the same net value, or in formal terms, equivalence class.

Since some questions cannot be meaningfully formulated entirely as a change in debts or assets, this model can be used in conjunction with another model, perhaps number lines or the rule of signs, to enable students to perform all required calculations.

Having students experiment with number patterns, while not technically a model but rather a method, can allow them to come to their own conclusions, which can provide more meaningful learning. This can also be used in conjunction with other models.
which provide a meaningful metaphor for the calculations.

Using Algebra Tiles provides a complete method for performing all calculations, but should be used in conjunction with other models which provide students with a more intuitive link between operations and object manipulations.

Overall, the success of any model depends on the ways in which it is used in the classroom. Educators selecting models to use with their students should consider the students’ familiarity with the metaphors used in the various models, their exposure to the types of calculations or object manipulations, and the ways in which models can be combined to provide the most meaning for their students, in the context of the entire mathematics curriculum.
References


