Pricing Defaultable Bonds
under a Marked Point Process Model
for Interest Rates

Gary A. Katselas

Supervised by Prof. Kostya Borovkov

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Abstract

In this thesis we aim to review the framework used in the pricing of defaultable securities. We pay particular attention to reduced form treatments of default risk. Our original contribution is the extension of an existing interest rate model to include default risk. We also investigate some closed-form pricing formulae that can be derived. We propose a model where the risk-free interest rate is modeled as a marked Poisson point process. We model the default time through the use of a stochastic intensity which we allow to depend on the risk-free interest rate. We find that by imposing restrictions on the relationship between the default intensity and the risk-free interest rate, we can derive a simple expression for the no-arbitrage price of a defaultable bond. Under the same assumptions, we are also able to derive expressions for the prices of European call and put options on an underlying defaultable bond. We conclude by considering a specific case of our model and use simulation techniques to verify the expression we obtained for the defaultable bond price.
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List of notation

a.s. almost surely
1ₐ indicator function for the set A
\( \mathbb{R}_+ \) the non-negative real numbers
\( \overline{\mathbb{R}}_+ \) the non-negative real numbers, extended to include \( \infty \)
\( \mathbb{Z}_+ \) the non-negative integers
\( \overline{\mathbb{Z}}_+ \) the non-negative integers, extended to include \( \infty \)
\( X \sim F \) the random variable \( X \) has distribution \( F \)
\( \mathcal{B}(A) \) Borel \( \sigma \)-algebra of subset of \( A \)
\( \mathbb{P} \prec \mathbb{Q} \) the probability measure \( \mathbb{P} \) is absolutely continuous with respect to the probability measure \( \mathbb{Q} \)
\( \mathbb{P} \sim \mathbb{Q} \) the probability measures \( \mathbb{P} \) and \( \mathbb{Q} \) are equivalent
EMM equivalent martingale measure
\( \mathbb{P}^* \) notation for the EMM
\( \mathbb{E}_{\mathbb{P}^*} \) expectation under the EMM
SDE stochastic differential equation
\( p(t, T) \) time \( t \) price of a unit face value zero-coupon default free bond maturing at time \( T \)
\( p_d(t, T) \) time \( t \) price of a unit face value zero-coupon defaultable bond maturing at time \( T \)
\( S(\cdot) \) price process of a risky asset
\( \tilde{S}(\cdot) \) discounted price process of a risky asset
\( x^+ \) \( \max(0, x) \)
\( x^- \) \( -\min(0, x) \)
\( \Lambda \) mean measure of a Poisson process
\( \mathcal{P}_{\lambda, L} \) compound Poisson probability with parameter \( \lambda \) and compounding law \( L \)
PPP Poisson point process
MPPP marked Poisson point process
$L_N(f)$ Laplace functional for the PPP
$L_M(f)$ Laplace functional for the MPPP
$B(\cdot)$ bank account process
$\phi$ general trading strategy
$V_\phi(\cdot)$ value process
$\hat{V}_\phi(\cdot)$ discounted value process
$G_\phi(\cdot)$ gains process
$\hat{G}_\phi(\cdot)$ discounted gains process
$H(\cdot)$ default process
$D(\cdot)$ dividend process
$X^d(t,T)$ time $t$ price of a defaultable claim maturing at time $T$
$K$ probability kernel governing interest rate jumps
$\gamma$ intensity of default
$Call(t, s, K, T)$ time $t$ price of a call with strike $K$ maturing at time $s$. The underlying asset is a defaultable bond maturing at time $T$
$Put(t, s, K, T)$ time $t$ price of a put with strike $K$ maturing at time $s$. The underlying asset is a defaultable bond maturing at time $T$
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Chapter 1
Introduction

Financial markets are inherently risky. Asset prices fluctuate continuously and bankruptcy announcements are a relatively frequent occurrence. The pool of funds invested in financial markets has grown rapidly in recent decades and funds have flowed into the derivative and corporate bond markets as institutional investors seek to maximise profits. Given the volume of funds at risk, research has focused on developing mathematical models that allow for the associated risks to be appropriately managed.

In managing the risks of corporate bonds and over-the-counter financial contracts (such as derivatives), there is a need to consider the usual risks of value changes arising from shifting market conditions as well as the default risk stemming from the possibility that the counter-party to a contract may fail to meet their obligations.

The treatment of default risk in the literature has fallen into two broad categories: structural and reduced form models.

Merton’s famous paper [21] is considered to be the originator of what has come to be known as the structural approach. Merton modeled the default event as the time when the firm’s total value falls below some threshold. This assumption was then used to develop a pricing approach for corporate bonds. Early developments in the modeling of default risk centered around structural models as they naturally relate bond values to economic fundamentals, a fact that was considered particularly appealing at the time. Black and Cox [4] extended Merton’s approach to include some of the realities of corporate bonds, such as debt covenants, subordination arrangements and the financing of interest and dividend payments. These initial models assumed that the firm’s value can be modeled by a geometric Brownian motion process and that the risk-free interest rate is non-random and constant. Longstaff and Schwartz [19] produced a structural model where the firm’s value is driven by a geometric Brownian motion process and the risk-free interest rate varies randomly according to an arithmetic Brownian motion process that is independent of the firm’s value process. Bryis and de Varenne [7] continued with this line of extension by modeling the firm value and risk-free interest rate with more general diffusion processes. In his 1997 paper, Zhou [24] proposed a jump-diffusion model for the firm value and found that his model was more successful than the previous
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diffusion approaches at capturing basic features of default risk (such as the shapes of yield spread curves). Despite the considerable progress shown in the development of structural models, their practical applications have been limited by the fact that, under fairly weak assumptions, the default event is predictable. Thus there is an inherent unrealistic assumption in these models that default is signaled prior to its occurrence. This has led to considerable effort being directed toward another class of default risk models, ones that aren’t hindered by this restriction.

The second category of default risk models are known as reduced form approaches. In the reduced form setting, the relationship between firm value and default is not explicitly modeled. It is instead assumed that the default time is specified exogenously with this random time being modeled directly. A feature of reduced form models is that the default event is unpredictable, it comes as a complete shock to the market, a feature that is more realistic than the predictable nature of default in structural models. Jarrow et al. [14] modeled the default event as the time at which a markov process specifying a firm’s credit rating enters the “default” state. Cathcart and El-Jahel [8] take an entirely different approach, they propose that default is related to certain signaling variables. These signaling variable were modeled by geometric Brownian motion processes and the default event was taken to be a function of their trajectories. By far the most prevalent reduced form models are those that use hazard rates and stochastic intensities to model the default time. The papers produced by Duffie and Singleton [13] and Madan and Unal [20] provide examples of the stochastic intensity approach. The appeal of stochastic intensities is that they are flexible and easy to implement in practice.

In this thesis we will review the framework used to price defaultable securities. We will focus on the reduced form approach, paying particular attention to the use of the stochastic intensity as a tool for modeling the default time. We will develop a reduced form model for pricing zero-coupon defaultable bonds in a setting where interest rates follow a particular pure jump process. We specify the risk-free interest rate according to the marked Poisson point process model introduced by Borovkov et al. [6]. We allow our default intensity to be related to the risk-free interest rate and place restrictions on the functional form of this relationship to facilitate calculations. We use the compound Poisson nature of our modeled processes, as well as some properties of the Laplace functional to derive the price of a zero-coupon defaultable bond as well as the prices of standard European call and put options on these bonds.

The structure of our thesis will be as follows: Chapter 2 will provide a brief overview of probability, stochastic processes, stochastic calculus, Poisson point processes, Laplace functionals and no-arbitrage pricing. Chapter 3 will present the theory required to establish the pricing framework for a reduced form model. Chapter 4 will contain our model formulation, pricing results and their proofs. Chapter 5 will contain some computational results, including a verification of our bond pricing result and a brief investigation of the behaviour of option prices under our model. Finally, Chapter 6 will contain some concluding remarks.
Chapter 2

Preliminaries

In this chapter we will give a brief survey of those areas of probability and financial mathematics that provide important background for the theory involved in the pricing of defaultable claims. We will present some basic notions relating to general probability theory and provide definitions and results that are necessary precursors to the introduction of valuation techniques for defaultable claims.

2.1 Probability and Stochastic Processes

2.1.1 Probability as a measure

Definition 2.1.1 (Measurable spaces.) Given an underlying space $\Omega$, a $\sigma$-algebra $\mathcal{F}$ on $\Omega$ is a collection of subsets of $\Omega$ satisfying the following three conditions:

i) $\Omega \in \mathcal{F}$;

ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$, where $A^c = \Omega \setminus A$;

iii) $A_1, A_2, A_3, \ldots \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a measurable space. We often refer to the elements of $\mathcal{F}$ as events or measurable sets.

Definition 2.1.2 (Generated $\sigma$-algebras.) Given a measurable space $(\Omega, \mathcal{F})$ and $\mathcal{A}$, a class of subsets of $\Omega$, we define the $\sigma$-algebra generated by $\mathcal{A}$ (denoted by $\sigma(\mathcal{A})$) by

$$\sigma(\mathcal{A}) := \bigcap \left\{ \mathcal{H} : \mathcal{H} \text{ is a } \sigma\text{-algebra and } \mathcal{A} \subseteq \mathcal{H} \right\},$$

i.e. $\sigma(\mathcal{A})$ is the (unique) smallest $\sigma$-algebra containing $\mathcal{A}$.

An important example of a generated $\sigma$-algebra is $\mathcal{B}(\mathbb{R})$ which is generated by the open subsets of $\mathbb{R}$ and referred to as the $\sigma$-algebra of Borel subsets of $\mathbb{R}$. Elements of $\mathcal{B}(\mathbb{R})$ are referred to as Borel subsets of $\mathbb{R}$.

For two $\sigma$-algebras $\mathcal{F}$ and $\mathcal{G}$, we use the notation $\mathcal{F} \vee \mathcal{G}$ to denote $\sigma(\mathcal{F}, \mathcal{G})$, the smallest $\sigma$-algebra containing both $\mathcal{F}$ and $\mathcal{G}$.
Definition 2.1.3 (Standard product spaces.) Given measurable spaces \((S_1, \mathcal{S}_1)\) and \((S_2, \mathcal{S}_2)\), we define the direct product of \(S_1\) and \(S_2\), denoted by \(S_1 \otimes S_2\), to be the \(\sigma\)-algebra generated by sets of the form \(B_1 \times B_2\), where \(B_1 \in \mathcal{S}_1\) and \(B_2 \in \mathcal{S}_2\). Then \((S_1 \times S_2, \mathcal{S}_1 \otimes \mathcal{S}_2)\) forms a measurable space which we call the standard product space. As a convention, when no \(\sigma\)-algebra on \(S_1 \times S_2\) is mentioned, the implicit assumption is that \(\mathcal{S}_1 \otimes \mathcal{S}_2\) is used.

Definition 2.1.4 (Probability spaces.) Given a measurable space \((\Omega, \mathcal{F})\), a probability measure \(P\) on it, is a mapping \(P: \mathcal{F} \to \mathbb{R}\) satisfying the following conditions:

i) \(P(A) \geq 0, \forall A \in \mathcal{F}\);

ii) \(P(\Omega) = 1\);

iii) If \(A_1, A_2, A_3, \ldots \in \mathcal{F}\) with \(A_i \cap A_j = \emptyset, \forall i \neq j\), then \(P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)\).

The triplet \((\Omega, \mathcal{F}, P)\) is called a probability space.

Note: a mapping \(\mu: \mathcal{F} \to \mathbb{R}\) that satisfies properties i) and iii) is called a measure, and the triplet \((\Omega, \mathcal{F}, \mu)\) is referred to as a measure space.

Definition 2.1.5 (Absolute continuity and equivalence.) If \(P\) and \(Q\) are probability measures on the same measurable space \((\Omega, \mathcal{F})\) then we say that \(P\) is absolutely continuous with respect to \(Q\), denoted by \(P \prec Q\), if \(Q(A) = 0 \Rightarrow P(A) = 0, \forall A \in \mathcal{F}\).

We say that the measures are equivalent, denoted by \(P \sim Q\), if \(P \prec Q\) and \(Q \prec P\), i.e. \(P \sim Q\) if for all \(A \in \mathcal{F}\), \(P(A) = 0 \iff Q(A) = 0\).

Definition 2.1.6 (Filtrations.) Given a measurable space \((\Omega, \mathcal{F})\), a filtration \(\mathcal{F}\) is a set of \(\sigma\)-algebras \(\{\mathcal{F}_t\}_{t \in I}\), indexed by the set \(I \subset \mathbb{R}\), with \(\mathcal{F}_t \subset \mathcal{F}\) for each \(t \in I\) and

\[ t_1 \leq t_2 \Rightarrow \mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2} \text{ for any } t_1, t_2 \in I. \]

The collection \((\Omega, \mathcal{F}, \mathcal{F}, P)\) is called a filtered probability space.

Definition 2.1.7 (Completeness and continuity of filtered probability spaces.) A probability space \((\Omega, \mathcal{F}, P)\) is said to be complete if \(B \subset A, P(A) = 0 \Rightarrow B \in \mathcal{F}\). A filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}, P)\) is complete if \((\Omega, \mathcal{F}, P)\) is complete in the previous sense and \(\mathcal{F}_0\) contains all sets \(A \in \mathcal{F}\) such that \(P(A) = 0\).

We define the right limit of a filtration \(\mathcal{F}\) by

\[ \mathcal{F}_{t+} := \bigcap_{s > t} \mathcal{F}_s \]

and we say that \(\mathcal{F}\) is right-continuous if

\[ \mathcal{F}_t = \mathcal{F}_{t+} \forall t \]

We define the left limit of a filtration \(\mathcal{F}\) by

\[ \mathcal{F}_{t-} := \sigma\left(\bigcup_{s < t} \mathcal{F}_s\right) \]
and we say that $\mathcal{F}$ is left-continuous if

$$\mathcal{F}_t = \mathcal{F}_{t-}, \forall t$$

**Definition 2.1.8** ($\sigma$-finiteness.) For a measure space $(\Omega, \mathcal{F}, \mu)$, $\mu$ is said to be $\sigma$-finite if there exists a partition $A_1, A_2, \ldots \in \mathcal{F}$ of $\Omega$ (i.e. $A_i \cap A_j = \emptyset \forall i \neq j$ and $\bigcup_{n=1}^{\infty} A_n = \Omega$) such that $\mu(A_n) < \infty$ for all $n$.

**Definition 2.1.9** Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we say that $A \in \mathcal{F}$ holds almost surely (a.s.) if $\mathbb{P}(A) = 1$. For a measure space $(S, \mathcal{S}, \mu)$ we say that a condition holds almost everywhere (a.e.) if there exists a $B \in \mathcal{S}$ such that $\mu(B^c) = 0$ and the condition holds for all $s \in B$.

Note that the notions of a.e. and a.s. are equivalent in a probability space.

### 2.1.2 Random variables and stochastic processes

**Definition 2.1.10** (Random variables.) Given a measurable space $(\Omega, \mathcal{F})$, we say that a function $X : \Omega \to \mathbb{R}$ is a random variable (r.v.) if, for any open set $B \subset \mathbb{R}$, we have

$$X^{-1}(B) := \{ \omega \in \Omega : X(\omega) \in B \} \in \mathcal{F}.$$  

Note: we alternatively refer to $X$ as $\mathcal{F}$-measurable or, when the $\sigma$-algebra is clear from the context, as simply measurable.

**Definition 2.1.11** (Convergence in probability.) Given a sequence $\{X_n\}_{n \geq 0}$ of r.v.'s on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we say that the sequence $\{X_n\}_{n \geq 1}$ converges in probability to the r.v. $X_0$, if for all $\varepsilon > 0$

$$\lim_{n \to \infty} \mathbb{P}(|X_n - X_0| \geq \varepsilon) = 0.$$  

**Definition 2.1.12** ($\sigma$-algebras generated by r.v.'s.) For a r.v. $X$, on a measurable space $(\Omega, \mathcal{F})$, we define $\sigma(X)$, the $\sigma$-algebra generated by the r.v. $X$, by

$$\sigma(X) := \{ X^{-1}(B) : B \in \mathcal{B}(\mathbb{R}) \} .$$  

Equivalently, we could define $\sigma(X)$ to be the smallest $\sigma$-algebra on $\Omega$ such that $X$ is $\sigma(X)$-measurable.

**Definition 2.1.13** (Stochastic process.) Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a stochastic process is a collection of random variables indexed by a set $I$ (which is often thought of as “time”), i.e. a stochastic process $X$ is a collection

$$\{ X(t) \}_{t \in I},$$

where $X(t)$ is a random variable for each $t \in I$.

The index set $I$ is often taken to be a subset of $\mathbb{Z}_+$ in the case of discrete time, or an interval in $\mathbb{R}_+$ in the case of continuous time.
The trajectories of a process are the functions we obtain by fixing an \( \omega \) and allowing \( t \) to vary, i.e.

\[
X(t, \cdot) \text{ is a random variable for each } t, \\
X(\cdot, \omega) \text{ is a trajectory of the process for each } \omega.
\]

We can consider a stochastic process to be a mapping \( X: I \times \Omega \rightarrow \mathbb{R} \) which is a random variable when the first argument is held fixed.

**Definition 2.1.14 (Càdlàg processes.)** A stochastic process \( X \) is càdlàg (abbreviation of the French, “continu à droite, limite à gauche”) if its trajectories are right continuous with finite left limits (a.s.) at any point \( t \in I \).

**Definition 2.1.15 (Càglàd processes.)** A stochastic process \( X \) is càglàd if its trajectories are left continuous with finite right limits (a.s.) at any point \( t \in I \).

**Definition 2.1.16 (Adapted processes.)** Given a filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, P) \), a stochastic process \( X = \{X(t)\}_{t \in I} \) is said to be \( \mathcal{F} \)-adapted if \( X(t) \) is \( \mathcal{F}_t \)-measurable for all \( t \) in \( I \).

**Definition 2.1.17 (Progressively measurable processes.)** Given a filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, P) \), a stochastic process \( X: [0, \infty) \times \Omega \rightarrow \mathbb{R} \) is said to be \( \mathcal{F} \)-progressively measurable if, for every time \( t \), the map \([0, t] \times \Omega \rightarrow \mathbb{R}\) given by \((s, \omega) \mapsto X(s, \omega)\) is \( \mathcal{B}([0, t]) \otimes \mathcal{F}_t \)-measurable (where \( \mathcal{B}([0, t]) \) is the \( \sigma \)-algebra of Borel subsets of \([0, t])\).

**Definition 2.1.18 (Predictable processes.)** Given a filtration \( \{\mathcal{F}_t\}_{t \in \mathbb{Z}^+} \) on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), a stochastic process \( X = \{X(t)\}_{t \in \mathbb{Z}^+} \) is predictable if \( X(t) \) is \( \mathcal{F}_{t-1} \)-measurable for every \( t \geq 1 \) and \( X(0) \) is \( \mathcal{F}_0 \)-measurable. This amounts to the requirement that the value of \( X(t) \) is known at the previous time step. The analogue of this concept in continuous time is less obvious. A stochastic process \( X: \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R} \) is predictable if it is measurable with respect to the predictable \( \sigma \)-algebra on \( \mathbb{R}_+ \times \Omega \) that makes all left-continuous adapted processes measurable.

**Definition 2.1.19 (Stopping times.)** A random variable \( \tau: \Omega \rightarrow [0, \infty] \) is a stopping time if it is finite (a.s.) and \( \{\tau \leq t\} \in \mathcal{F}_t \) for all \( t \geq 0 \).

**Definition 2.1.20 (Stopped processes.)** For \( X = \{X(t)\} \) a stochastic process and \( \tau \) a stopping time, we define the process \( X \) stopped at \( \tau \) by

\[
X^\tau(t) := X(t \wedge \tau),
\]

where \( a \wedge b = \min(a, b) \) for \( a, b \in \mathbb{R} \).

**Definition 2.1.21 (Brownian motion.)** The Brownian motion (or Wiener process) \( W = \{W(t)\}_{t \geq 0} \) is a stochastic process satisfying the following three properties:
i) $W(0) = 0$;

ii) the trajectories $W(t)$ are continuous (a.s.);

iii) $W$ has independent increments with

$$W(t) - W(s) \sim \mathcal{N}(0, t - s) \quad \text{for } 0 \leq s \leq t$$

where $\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with expected value $\mu$ and variance $\sigma^2$.

**Definition 2.1.22** (Convolution.) Let $L$ be a probability distribution for a random variable $X$, i.e.

$L(A) = \mathbb{P}(X \in A).$

Then the $k$-fold convolution $L^k$ of $L$ with itself is the distribution of the sum of $k$ independent copies of the random variable $X$.

### 2.1.3 Expectations

**Definition 2.1.23** (Indicator functions.) Given a measurable space $(\Omega, \mathcal{F})$, we define the indicator function $1_A$ of a set $A \in \mathcal{F}$ by

$$1_A(\omega) = \begin{cases} 
1 & \text{if } \omega \in A; \\
0 & \text{if } \omega \notin A.
\end{cases}$$

**Definition 2.1.24** (Simple random variables.) Given a measurable space $(\Omega, \mathcal{F})$, $X$ is a simple random variable on it if it can be written in the form

$$X(\omega) = \sum_{k=1}^{m} a_k 1_{A_k}(\omega), \quad \text{where } a_k \in \mathbb{R} \text{ and } A_k \in \mathcal{F} \text{ for all } k.$$ 

**Definition 2.1.25** (Expectations of simple random variables.) For a simple random variable $X = \sum_{k=1}^{m} a_k 1_{A_k}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we define its expectation by

$$\mathbb{E}(X) = \int_{\Omega} X \, d\mathbb{P} := \sum_{k=1}^{m} a_k \mathbb{P}(A_k).$$

**Definition 2.1.26** (General expectations.) For a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we define the expectation of a non-negative random variable $X$ on it by

$$\mathbb{E}(X) := \sup \left\{ \int_{\Omega} Y \, d\mathbb{P} : Y \geq 0 \text{ is a simple r.v. and } Y \leq X (a.s.) \right\}.$$ 

We extend this to a general random variable $X$ by introducing the random variables $X^+$ and $X^-$:

- $X^+(\omega) := \max(0, X(\omega)),$
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- $X^-(\omega) := - \min(0, X(\omega))$.

If $\mathbb{E}(X^+)$ and $\mathbb{E}(X^-)$ are both finite, we say that $X$ is integrable and we define the expectation of $X$ by:

$$\mathbb{E}(X) := \mathbb{E}(X^+) - \mathbb{E}(X^-).$$

**Lemma 2.1.1** Suppose we have a measurable space $(S, \mathcal{S})$ and a measurable function $f : S \to \mathbb{R}$. If $f \geq 0$ (a.e.) ($f \leq 0$ (a.e.)) then there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of simple functions such that

$$0 \leq f_n \leq f \quad \forall n \quad \text{and} \quad f_n \uparrow f \quad (a.e.)$$

$$(f \leq f_n \leq 0 \quad \forall n \quad \text{and} \quad f_n \downarrow f \quad (a.e.))$$

**Theorem 2.1.1** (Monotone Convergence Theorem) Let $(f_n)_{n \in \mathbb{N}}$ be a monotone sequence of measurable functions on a measure space $(S, \mathcal{S}, \mu)$ such that the pointwise limit of the sequence $(f_n)_{n \in \mathbb{N}}$, exists:

$$f(s) := \lim_{n \to \infty} f_n(s), \quad \forall s \in S.$$ 

Then $f$ is measurable and

$$\lim_{n \to \infty} \int_S f_n d\mu = \int_S f d\mu.$$

**Theorem 2.1.2** (Lebesgue’s Dominated Convergence Theorem) Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of real-valued measurable functions on a measure space $(S, \mathcal{S}, \mu)$. Assume that the sequence converges pointwise to a function $f$ (a.e.) and is dominated by some integrable function $g \geq 0$, i.e.

$$f(s) = \lim_{n \to \infty} f_n(s) \text{ is defined a.e.,}$$

$$|f_n| \leq g \quad (a.e.) \quad , \forall n, \quad \text{and} \int_S g d\mu \text{ exists and is finite.}$$

Then $f$ is integrable and

$$\lim_{n \to \infty} \int_S f_n d\mu = \int_S f d\mu.$$ 

**Definition 2.1.27** (Conditional expectation.) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $X$ an integrable random variable on it and $\mathcal{A}$ a sub-$\sigma$-algebra (i.e. $\mathcal{A}$ is a $\sigma$-algebra and $\mathcal{A} \subset \mathcal{F}$). Then the conditional expectation of $X$ given $\mathcal{A}$ is an $\mathcal{A}$-measurable function $\mathbb{E}(X|\mathcal{A}) : \Omega \to \mathbb{R}$ which satisfies

$$\int_A \mathbb{E}(X|\mathcal{A}) d\mathbb{P} = \int_A X d\mathbb{P}, \quad \forall A \in \mathcal{A}.$$
We can also define conditional expectations given random variables by

\[ E(X|Y) := E(X|\sigma(Y)), \]

where \(Y\) is another random variable defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

**Theorem 2.1.3** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with random variables \(X\) and \(Y\) defined on it and sub-\(\sigma\)-algebras \(\mathcal{A}\) and \(\mathcal{G}\). We have the following results regarding conditional expectations:

i) \(E(X|\mathcal{A}) = E(X)\) if \(X\) is independent of \(\mathcal{A}\);

ii) \(E(E(X|\mathcal{A})) = E(X)\);

iii) \(E(E(X|\mathcal{A})|\mathcal{G}) = E(X|\mathcal{G})\) if \(\mathcal{G} \subset \mathcal{A}\);

iv) \(E(XY|\mathcal{A}) = XE(Y|\mathcal{A})\) if \(X\) is \(\mathcal{A}\)-measurable.

**Definition 2.1.28** (Laplace transform.) For a random vector \(X = (x_1, \ldots, x_d)' \in \mathbb{R}^d\) and vector \(w = (w_1, \ldots, w_d)' \in \mathbb{R}^d\) (where \(a'\) denotes the transpose of \(a\)), we define it’s Laplace transform \(\mu\) by

\[ \mu(w) = Ee^{-w'X} \]

as long as this expectation is defined.

There is a one-to-one correspondence between Laplace transforms and distributions. So knowing the Laplace transform of a variable fully specifies its distribution.

### 2.1.4 Martingales

**Definition 2.1.29** (Martingales.) Given a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\), a stochastic process \(X = \{X(t)\}_{0 \leq t < \infty}\) is a martingale relative to the filtration \(\mathbb{F}\) (or an \(\mathbb{F}\)-martingale) if

i) \(X\) is adapted to \(\mathbb{F}\),

ii) \(E|X(t)| < \infty\) for all \(0 \leq t < \infty\),

iii) \(E(X(t)|\mathcal{F}_s) = X(s)\) (a.s.) for all \(0 \leq s \leq t\).

The process \(X\) is a supermartingale if in place of iii) we have

\[ E(X(t)|\mathcal{F}_s) \leq X(s). \]

The process \(X\) is a submartingale if in place of iii) we have

\[ E(X(t)|\mathcal{F}_s) \geq X(s). \]
Definition 2.1.30 (Local martingales.) An adapted stochastic process $X$ defined on $(\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})$ is a local martingale if there is an increasing (to infinity) sequence of stopping times $\{\tau_n\}$ such that the stopped processes

$$X^{\tau_n} = \{X(\tau_n \wedge t)\}$$

are $\mathcal{F}$-martingales for each $n$.

Note that every martingale is necessarily a local martingale.

Definition 2.1.31 (Local boundedness.) An adapted stochastic process $X$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})$ is locally bounded if there is an increasing (to infinity) sequence of stopping times $\{\tau_n\}$ such that, for each $n$, the stopped process

$$X^{\tau_n} = \{X(\tau_n \wedge t)\}$$

is bounded.

Definition 2.1.32 (Partition.) By a partition $\pi$ of an interval $[a, b]$ we mean

$$\pi = \{t_i\}_{i=0,1,...,m}$$

where $a = t_0 < t_1 < \ldots < t_m = b$ for some $m \in \mathbb{N}$

We define the mesh width $||\pi||$ of a partition by

$$||\pi|| = \max_{j=1,...,m} \{|t_j - t_{j-1}|\}$$

Definition 2.1.33 (Local finite variation.) An adapted stochastic process $X$ defined on $(\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})$ is locally of finite variation if there is an increasing (to infinity) sequence of stopping times $\{\tau_n\}$ such that, for each $n$, the stopped process

$$X^{\tau_n} = \{X(\tau_n \wedge t)\}$$

is of finite variation.

We say that a process is of finite variation if its trajectories, which we will denote by $f$ for simplicity, satisfy the property that the supremum of their variation

$$\sum_{\{x_i\} \in \pi} |f(x_{i+1}) - f(x_i)|$$

over partitions $\pi$ is finite. This is equivalent to requiring that $f$ can be represented as the difference of two increasing functions.

Definition 2.1.34 (Semi-martingale.) A process $X$ defined on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})$ is called a semi-martingale if it can be decomposed as

$$X(t) = M(t) + A(t),$$

where $M$ is a local martingale and $A$ is a càdlàg adapted process locally of finite variation.
2.1.5 Stochastic calculus

In this section we will provide a very brief overview of stochastic integration and stochastic differential notation. The interested reader is referred to [22] for a rigorous treatment of the material.

Definition 2.1.35 (Stochastic integral.) Let $X$ be a semi-martingale and $H$ a locally bounded, adapted càglàd process given on a common filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Let $\pi_n$ be a sequence of partitions of $[0,t]$ with $\lim_{n \to \infty} \|\pi_n\| \to 0$. Then we define the stochastic integral (or Itô integral) of $H$ with respect to $X$ by

$$\int_0^t H(u) dX(u) := \lim_{n \to \infty} \sum_{t_{i-1}, t_i \in \pi_n} H(t_i) \left( (X(t_i) - X(t_{i-1})) \right),$$

where this limit is defined in terms of convergence in probability. The stochastic integral defined in this manner is itself a semi-martingale.

This definition can be extended to allow for predictable and locally bounded integrands. For conditions detailing the existence and construction of the stochastic integral in this situation the reader is referred to Ch 4 of [22]. We will spend the remainder of this section recounting useful results relating to the stochastic integral and specifying notation for stochastic differentials.

Definition 2.1.36 (Quadratic variation/covariation.) Let $X$ and $Y$ be stochastic processes defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\pi_n$ be a sequence of partitions of $[0,t]$ with $\lim_{n \to \infty} \|\pi_n\| \to 0$. Then we define the quadratic covariation of $X$ and $Y$ by

$$[X,Y](t) := \lim_{n \to \infty} \sum_{t_{k-1}, t_k \in \pi_n} (X(t_k) - X(t_{k-1})) (Y(t_k) - Y(t_{k-1})), $$

where this limit is defined in terms of convergence in probability. We define the quadratic variation of a single process $X$ by

$$[X](t) = [X,X](t).$$

Definition 2.1.37 (Stochastic differentials.) Given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, a semi-martingale $S$ and a stochastic process $X$ on it expressible in the form

$$X(t) = X(0) + \int_0^t A(u) du + \int_0^t B(u) dS(u)$$

for some $\mathbb{F}$-progressively measurable process $A$ and $B$, we say that the process $X$ has stochastic differential $dX(t)$ given by

$$dX(t) = A(t) dt + B(t) dS(t).$$

Lemma 2.1.2 Let $X$ be a semi-martingale and $H$ a process of finite variation. Then, if either $X$ or $H$ is continuous, we have $[X,H](t) = 0$ for all $t$ (and thus also $d[X,H](t) = 0$).
The following theorem gives some conditions under which the process of stochastic integration relative to a local martingale preserves the local martingale property. The proof is omitted and the reader is referred to Section 2 of Ch 4 of [22] for further details.

**Theorem 2.1.4** Let $M$ be a local martingale and let $H$ be a predictable, locally bounded stochastic process. Then the stochastic integral $\int_0^t H(u)dM(u)$ is a local martingale.

We now quote without proof two results proved by K. Itô for stochastic integrals relative to the Brownian motion process and extended to general stochastic integrals with respect to semi-martingales.

**Theorem 2.1.5** (Itô’s Product Rule) For semi-martingales $X$ and $Y$, the stochastic differential of $XY$ is given by

$$d(X(t)Y(t)) = X(t-)dY(t) + Y(t-)dX(t) + dX(t)dY(t),$$

where $dX(t)dY(t) = d[X,Y](t)$, the differential of the quadratic covariation process.

**Theorem 2.1.6** (Itô’s Formula) Given a stochastic process $X$ and a function $f(t, x)$, continuously differentiable in $t$ and twice differentiable in $x$, the stochastic differential of the process $Y(t) := f(t, X(t))$ is given by

$$dY(t) = \frac{\partial f}{\partial t}(t, X(t))dt + \frac{\partial f}{\partial x}(t, X(t))dX(t) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X(t))(dX(t))^2.$$ 

### 2.2 Poisson point processes

Point processes provide a useful tool for modeling interest rates with jumps. We will give a brief introduction to Poisson point processes (PPPs) on general measurable spaces. For a more detailed exposition of the underlying theory the reader is referred to Ch 9 of the text [10] and Ch 12 of [16] (the latter of which was taken as the basis for the notation used here).

We start with the following definitions:

**Definition 2.2.1** (Poisson distribution.) The **Poisson distribution** with parameter $\lambda$ is the distribution of a r.v. $X$ which has probabilities

$$\mathbb{P}(X = x) = \begin{cases} \frac{e^{-\lambda} \lambda^x}{x!} & \text{if } x = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the mean and variance of $X$ are both equal to $\lambda$. We write $X \sim \text{Poisson}(\lambda)$ to indicate that $X$ is a Poisson distributed random variable with parameter $\lambda$. 

Definition 2.2.2 (Kernel.) Given two measurable spaces \((S, \mathcal{S})\) and \((T, \mathcal{T})\), a mapping \(\mu : S \times T \to [0, \infty)\) is called a (probability) kernel from \(S\) to \(T\) if the function \(\mu(s, B)\) is \(\mathcal{S}\)-measurable in \(s \in S\) for any fixed \(B \in \mathcal{T}\) and a (probability) measure in \(B \in \mathcal{T}\) for any fixed \(s \in S\).

Definition 2.2.3 (Random measures.) Given an arbitrary measurable space \((S, \mathcal{S})\) and a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) we define a random measure on \(S\) to be a \(\sigma\)-finite kernel \(\xi\) from \(\Omega\) into \(S\), i.e. the mapping \(\xi : \Omega \times S \to [0, \infty]\) satisfies the properties:

i) \(\xi(\omega, B)\) is \(\mathcal{A}\)-measurable in \(\omega \in \Omega\) for any fixed \(B \in \mathcal{S}\),

ii) \(\xi(\omega, B)\) is a measure in \(B \in \mathcal{S}\) for any fixed \(\omega \in \Omega\).

It is convenient to use the notation \(\xi_B := \xi(\cdot, B)\). Note that \(\xi_B\) is an r.v. in \([0, \infty]\) for every \(B \in \mathcal{S}\). The \(\sigma\)-finiteness condition is taken here to mean the existence of a partition \(B_1, B_2, \ldots \in \mathcal{S}\) of \(S\) such that \(\xi_{B_k} < \infty\) a.s. for all \(k\).

The intensity of \(\xi\) is given by the measure \(\Lambda\) on \((S, \mathcal{S})\) defined by \(\Lambda(B) = \mathbb{E}(\xi_B), B \in \mathcal{S}\).

Definition 2.2.4 (Point process.) A point process on \(S\) is an integer-valued random measure \(\xi\), i.e. \(\xi_B\) is a \(\mathbb{Z}_+\)-valued r.v. for every \(B \in \mathcal{S}\).

Definition 2.2.5 (Independent increments.) We say that a random measure \(\xi\) on a measurable space \(S\) has independent increments if the random variables \(\xi_{B_1}, \xi_{B_2}, \ldots, \xi_{B_n}\) are independent for any disjoint sets \(B_1, B_2, \ldots, B_n \in \mathcal{S}\).

Definition 2.2.6 (Poisson point process.) We define a Poisson point process (PPP) on \(S\) with intensity measure \(\Lambda\) (where \(\Lambda\) is a \(\sigma\)-finite measure on \(S\)) to be a point process \(\xi\) on \(S\) with independent increments such that \(\xi_B\) is a Poisson r.v. with mean \(\Lambda(B)\) whenever \(\Lambda(B) < \infty\).

Definition 2.2.7 (Compound Poisson distribution.) We say that a random variable \(X\) follows a compound Poisson distribution with parameter \(\lambda\) and compounding law \(L\) if it has the following distribution:

\[
P_{\lambda, L}(B) := \mathbb{P}(X \in B) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} L^*(B),
\]

where \(L\) is a probability distribution and \(L^*(B)\) is the \(k\)-fold convolution of \(L\) with itself. The compound Poisson distribution arises naturally in the situation where we sum a Poisson distributed number \(N\) of conditionally independent (given \(N\)) random variables.

If we denote by \(\mu\) the Laplace transform of distribution \(L\), then the Laplace transform of the compound Poisson distributed \(X\) is given by

\[
\mathbb{E} e^{-uX} = e^{\lambda(\mu(u) - 1)}.
\]
2.2.1 The Laplace functional

**Definition 2.2.8** (Laplace functional.) Given a Poisson point process $N$ with intensity (or mean) measure $\Lambda$, we have the following definition for the Laplace functional $L_N(f)$:

$$L_N(f) := \mathbb{E}\left(e^{-N(f)}\right),$$

where $f : S \to \mathbb{R}$ is a measurable function and $N(f)$ is the integral of the function $f$ with respect to the process $N$, i.e. $N(f) = \int f \, dN = \sum f(S_j)$, where the sum is taken over all points $S_j$ in the process.

With very general restrictions on the function $f$ we can give simplified integral expressions for the Laplace functional.

**Lemma 2.2.1** Let $f : S \to \mathbb{R}$ be a measurable, bounded function. Let $N$ be a PPP on $S$ with mean measure $\Lambda$. Then

$$L_N(f) = \exp\left\{ \int (e^{-f(v)} - 1) \, \Lambda(\,dv\,) \right\}.$$  

**Proof** First consider a random variable $X \sim \text{Poisson}(\lambda)$. Then for any $c \in \mathbb{R}$ we have

$$\mathbb{E}(e^{-cX}) = \sum_{k=0}^{\infty} e^{-ck} \frac{e^{-\lambda} \lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^{-c})^k}{k!} = \exp\{\lambda(e^{-c} - 1)\}. \quad (2.2)$$

Now let $h$ be a simple r.v., i.e. $h = \sum_{k \leq m} c_k 1_{B_k}$, where $c_k \in \mathbb{R}$ and the sets $B_k \in \mathcal{B}(\mathbb{R})$ are disjoint with $\Lambda(B_k) < \infty$. Noting that $N(B_k) := N(\cdot, B_k)$ is a Poisson r.v. with mean $\Lambda(B_k)$ we have

$$\mathbb{E}\left(e^{-N(h)}\right) = \mathbb{E}\left(e^{-N(\sum_{k \leq m} c_k 1_{B_k})}\right)$$

$$= \mathbb{E}\left(e^{-\sum_{k \leq m} c_k N(B_k)}\right)$$

$$= \prod_{k \leq m} \mathbb{E}\left(e^{-c_k N(B_k)}\right) \quad \text{by independent increments}$$

$$= \prod_{k \leq m} \exp\left\{ \Lambda(B_k)(e^{-c_k} - 1) \right\} \quad \text{using (2.2)}$$

$$= \exp\left\{ \sum_{k \leq m} \Lambda(B_k)(e^{-c_k} - 1) \right\}$$

$$= \exp\left\{ \int (e^{-h(v)} - 1) \, \Lambda(\,dv\,) \right\}. \quad (2.3)$$
For a general bounded $g \geq 0$ ($g \leq 0$) the result of Lemma 2.1.1 tells us that we can find a sequence $(g_n)_{n \in \mathbb{N}}$ of non-negative (non-positive) simple functions with $g_n \uparrow g$ (a.e.) ($g_n \downarrow g$ (a.e.)) and therefore we can apply monotone convergence (Theorem 2.1.1) to conclude that

$$N(g_n) \to N(g) \text{ monotonically as } n \to \infty. \quad (2.4)$$

Note also that we have

$$e^{-g_n} - 1 \to e^{-g} - 1 \text{ monotonically as } n \to \infty. \quad (2.5)$$

Since $g$ is bounded, there exists a positive constant $C \in \mathbb{R}$ such that $|g| \leq C$ and hence

$$e^{-g} \leq \max\{e^C, 1\}, \quad (2.6)$$

$$e^{-N(g)} \leq \max\{e^{N(C)}, 1\}, \quad (2.7)$$

and so again by monotone convergence (Theorem 2.1.1) and the result for the Laplace functional of simple functions (given by (2.3)) we have, as $n \to \infty$,

$$\mathbb{E}(e^{-N(g_n)}) = \exp \left\{ \int (e^{-g_n(v)} - 1)\Lambda(dv) \right\} \to \exp \left\{ \int (e^{-g(v)} - 1)\Lambda(dv) \right\}, \quad (2.8)$$

where (2.6) ensures the finiteness of the integrals in question. Using the bound from (2.7) and given (2.4) we can apply dominated convergence (Theorem 2.1.2) to conclude that

$$\mathbb{E}(e^{-N(g_n)}) \to \mathbb{E}(e^{-N(g)}) \text{ as } n \to \infty. \quad (2.9)$$

Combining (2.8) and (2.9) we have, as $n \to \infty$,

$$\mathbb{E}(e^{-N(g_n)}) \to \exp \left\{ \int (e^{-g(v)} - 1)\Lambda(dv) \right\}$$

and

$$\mathbb{E}(e^{-N(g_n)}) \to \mathbb{E}(e^{-N(g)})$$

and the uniqueness of the limit gives us

$$\mathbb{E}(e^{-N(g)}) = \exp \left\{ \int (e^{-g(v)} - 1)\Lambda(dv) \right\}. \quad (2.10)$$

For a general bounded function $f$, we decompose it as $f = f^+ - f^-$ where $f^+ = 0 \lor f$ and $f^- = -(0 \land f)$. It is obvious that $f^\pm \geq 0$ and they have disjoint supports, i.e. if we define the sets $B_{\pm} = \{v: f_{\pm} > 0\}$, then $B_+ \cap B_- = \emptyset$. With this decomposition in mind, we have

$$\mathbb{E}(e^{-N(f)}) = \mathbb{E}(e^{-N(f^+-f^-)})$$

$$= \mathbb{E}(e^{-N(f_+)}e^{N(f_-)})$$

$$= \mathbb{E}(e^{-N(f_+)}) \mathbb{E}(e^{-N(-f_-)}),$$
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where the last line follows from the independent increments of the PPP and the disjoint supports of \( f_+ \) and \( f_- \).

We can now use (2.10) to prove the assertion of Lemma 2.2.1. Noting that \( f_+ = 0 \) on \( B_- \) and \( f_- = 0 \) on \( B_+ \), we have

\[
\mathbb{E}(e^{-N(f)}) = \mathbb{E}(e^{-N(f_+)} \mathbb{E}(e^{-N(-f_-)})
\]

\[
= \exp \left\{ \int_{B_+} (e^{-f_+(v)} - 1) \Lambda(dv) + \int_{B_-} (e^{-f_-(v)} - 1) \Lambda(dv) \right\}
\]

\[
= \exp \left\{ \int_{B_+} (e^{-f_+(v)} - 1) \Lambda(dv) + \int_{B_-} (e^{f_-(v)} - 1) \Lambda(dv) \right\}
\]

\[
= \exp \left\{ \int_{B_+} (e^{-f_+(v)} - f_-(v)) - 1) \Lambda(dv) + \int_{B_-} (e^{-f_+(v)} - f_-(v)) - 1) \Lambda(dv) \right\}
\]

\[
= \exp \left\{ \int_{B_+} (e^{-f_+(v)} - f_-(v)) - 1) \Lambda(dv) \right\}
\]

\[
= \exp \left\{ \int_{B_-} (e^{-f_+(v)} - f_-(v)) - 1) \Lambda(dv) \right\}.
\]

\[\square\]

**Definition 2.2.9** (Marked point process.) Given measurable spaces \((S, \mathcal{S})\) and \((T, \mathcal{T})\), we define a \(T\)-marked point process on \( S \) as a point process \( \xi \) on \( S \times T \) in the usual sense satisfying \( \xi(\{s\} \times T) \leq 1 \) for all \( s \in S \) and such that the projections \( \xi(\cdot \times T_j) \) are \( \sigma \)-finite point processes on \( S \) for some measurable partition \( T_1, T_2, \ldots \) of \( T \).

**Definition 2.2.10** (Independent increments.) Given measurable spaces \((S, \mathcal{S})\) and \((T, \mathcal{T})\), we say that a \( T \)-marked point process \( \xi \) on \( S \) has independent increments if the point processes \( \xi(B_1 \times \cdot), \ldots, \xi(B_n \times \cdot) \) on \( T \) are independent for any disjoint sets \( B_1, \ldots, B_n \in S \).

### 2.3 No-arbitrage pricing

In financial markets, asset prices shift constantly, and mathematical methods are employed to try to accurately determine the “true” price of an asset. A key technique in determining asset prices is the notion of no-arbitrage pricing. The central idea is that market participants should not be able to earn a riskless profit through a trading strategy that involves zero net-investment. If such an outcome is possible then we say that arbitrage is present and the asset is mispriced.

There has been extensive development of the mathematical framework of no-arbitrage pricing. We will present the basics here, and the reader is referred to Ch 6 of [3] for a concise summary of the material. The principles developed here will provide the foundations for the theory used in the pricing of defaultable claims.

We will assume that we are operating in a frictionless market, i.e. there are no transaction costs involved with trading and trades occur instantaneously. We will
also assume that all securities are perfectly divisible, i.e. we can purchase and sell portions of a single unit of an asset.

In constructing our market model, we first assume an underlying probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and a filtration \( \mathcal{F} = \{ \mathcal{F}_t \}_{0 \leq t \leq T} \) satisfying the conditions of completeness and right-continuity (see Definition 2.1.7). We assume that \( \mathcal{F}_T = \mathcal{F} \) and that \( \mathcal{F}_0 \) is trivial in the sense that, for every \( A \in \mathcal{F}_0 \), either \( \mathbb{P}(A) = 0 \) or \( \mathbb{P}(A) = 1 \). We think of the filtration as modeling the flow of information available to traders in the market. The probability measure \( \mathbb{P} \) is called the real world probability measure.

We assume that there are \( d + 1 \) primary traded assets (stocks, bonds or options), whose prices are given by stochastic processes \( S_0, \ldots, S_d \). It is assumed that \( S = (S_0, \ldots, S_d) \) is an adapted, continuous and strictly positive semi-martingale (recall Definition 2.1.34). We make this technical assumption so that we can interpret stochastic integrals relative to the stock prices in accordance with the general theory set out e.g. in [22].

The price processes are implicitly measured in units relative to some common measure of value, known as a numéraire. The formal definition is given below.

**Definition 2.3.1 (Numéraire.)** A numéraire is a price process \( X(t) \) that is strictly positive (a.s.) for each \( t \in [0, T] \).

We now assume \( S_0(t) \) to be the price process of a non-dividend paying asset, which is strictly positive (a.s.) and so can be used as our numéraire. Traditionally the “bank account” \( B(t) \) is used as a numéraire.

**Definition 2.3.2 (Bank account.)** The bank account \( B(t) \) specifies the value at time \( t \in [0, T] \) of 1 unit invested at time 0. It is usually specified by

\[
B(t) = e^{R(t)},
\]

where \( R(t) \) is a positive process and \( R(0) = 0 \).

We will, however, refrain from explicitly equating \( S_0 \) to \( B \) so as not to limit the flexibility of our results.

We want to use our model to value contingent claims, which we interpret in the economic sense as being financial contracts whose value is determined exactly by the price of an underlying financial asset. We also have the following formal definition of a contingent claim.

**Definition 2.3.3 (Contingent claim.)** A contingent claim \( X \) with maturity date \( T \) is an arbitrary \( \mathcal{F}_T \)-measurable random variable.

A key tool in the set-up of no-arbitrage type arguments is the concept of a trading strategy.
Definition 2.3.4 (Trading strategy.) A trading strategy is an $\mathbb{R}^{d+1}$-valued predictable locally bounded process

$$\phi(t) = (\phi_0(t), \ldots, \phi_d(t)), \quad t \in [0, T],$$

satisfying $\int_0^T \mathbb{E}(\phi_0(t))dt < \infty$, $\sum_{i=0}^d \int_0^T \mathbb{E}(\phi_i^2(t))dt < \infty$, so that the stochastic integral $\int_0^t \phi(u)dS(u)$ exists.

Here $\phi_i(t)$ is the number of shares of asset $i$ held in the portfolio at time $t$. The predictability of $\phi$ means that the composition of the portfolio at time $t$ is entirely determined by information available before time $t$, i.e. the investor determines how many units of each stock to hold at time $t$ based on the stock prices $S(t-)$.

A negative value of a component of $\phi$ indicates that the particular stock has been short sold. The ability for the components of $\phi$ to take non-integer values represents the assumption that the traded stocks are perfectly divisible, i.e. we can purchase/sell a portion of 1 stock unit.

Definition 2.3.5 (Value process.) The value process of the trading strategy $\phi$, denoted by $V_\phi(t)$, is given by

$$V_\phi(t) := \sum_{i=0}^d \phi_i(t)S_i(t), \quad t \in [0, T],$$

which, as its name suggests, is simply the value of the portfolio at time $t$.

Definition 2.3.6 (Gains process.) We define the gains process $G_\phi(t)$ by

$$G_\phi(t) := \sum_{i=0}^d \int_0^t \phi_i(u)dS_i(u),$$

which represents the capital gains generated by the portfolio.

Definition 2.3.7 (Self-financing trading strategies.) A trading strategy is called self-financing if $V_\phi(t)$ satisfies

$$V_\phi(t) = V_\phi(0) + G_\phi(t) \quad \text{for all } t \in [0, T],$$

which says that changes in the value of our portfolio come only from capital gains and not from injections or withdrawals of funds.

We now express our processes in terms of our designated numéraire $S_0(t)$ (which we can think of as discounting by the bank account $B(t)$).

Definition 2.3.8 (Discounted price process.) We define the discounted price process by

$$\tilde{S}(t) := \frac{S(t)}{S_0(t)} = (1, \tilde{S}_1(t), \ldots, \tilde{S}_d(t)),$$

where $\tilde{S}_i(t) = S_i(t)/S_0(t), \ i = 1, 2, \ldots, d.$
Definition 2.3.9 (Discounted wealth process.) The discounted wealth process is defined by
\[ \hat{V}_\phi(t) := \frac{V_\phi(t)}{S_0(t)} = \phi_0(t) + \sum_{i=1}^{d} \phi_i(t) \tilde{S}_i(t). \]

Definition 2.3.10 (Discounted gains process.) The discounted gains process is defined by
\[ \hat{G}_\phi(t) := \sum_{i=1}^{d} \int_{0}^{t} \phi_i(u) d\tilde{S}_i(u). \]

As was mentioned earlier, the presence of arbitrage gives investors the ability to generate riskless profits from no initial outlay and thus represents a market failure. The aim of no-arbitrage pricing is to establish conditions on our market model that eliminate potential arbitrage opportunities. To do this we require a formal definition of what arbitrage means in our market model:

Definition 2.3.11 (Arbitrage opportunity.) A self-financing trading strategy \( \phi \) is an arbitrage opportunity if \( V_\phi \) satisfies the conditions:

i) \( V_\phi(0) = 0 \) (zero initial net-investment),

ii) \( \mathbb{P}(V_\phi(T) \geq 0) = 1 \) (no chance of a loss),

iii) \( \mathbb{P}(V_\phi(T) > 0) > 0 \) (positive probability of a gain).

A key tool in no-arbitrage pricing is the concept of equivalent martingale measures. We give the definition here:

Definition 2.3.12 (Equivalent martingale measure, EMM.) Given a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), we say that \( \mathbb{Q} \) is an equivalent martingale measure (EMM) if:

i) \( \mathbb{Q} \sim \mathbb{P} \),

ii) the discounted price process \( \tilde{S} \) is a \( \mathbb{Q} \)-martingale.

The following theorem (proved in Section 6.1.2 of [3]) gives the link between EMM’s and arbitrage:

Theorem 2.3.1 If the market model admits an EMM, then it contains no arbitrage opportunities.

The market model we have formulated treats time as a continuous variable. In the case of discrete time it can also be shown that an arbitrage free market model must admit an EMM (see Ch 4 of [3]) yielding what is known as the ‘fundamental theorem of asset pricing’ which states that for a market model, the No Arbitrage (NA) condition is equivalent to the existence of an EMM. In continuous time models we need a stronger condition than NA. A suite of potential conditions have been
proposed in the literature. A brief overview of these conditions can be found in [15] where the relationship between the ‘No Free Lunch’ (NFL), ‘No Free Lunch with Bounded Risk’ (NFLBR) and ‘No Free Lunch with Vanishing Risk’ (NFLVR) conditions is discussed. This will be the extent of our discussion of these conditions. The interested reader is referred to the papers [17], [11] and [12] in which NFL, NFLBR and NFLVR, respectively, were first introduced.

At this point we assume that there exists an EMM \( \mathbb{P}^\ast \) for our market model (and thus no arbitrage opportunities) and consider a subclass of trading strategies.

**Definition 2.3.13 (Admissible trading strategies.)** A self-financing trading strategy \( \phi \) is called \( (\mathbb{P}^\ast)\)-admissible if the discounted gains process \( \tilde{G}_\phi(t) \) is a \( \mathbb{P}^\ast \)-martingale.

The link between our model and the valuation of contingent claims is the concept of a replicating strategy, which we define as follows.

**Definition 2.3.14 (Replicating strategy.)** A replicating strategy for a contingent claim \( X \) is an admissible trading strategy \( \phi \) such that

\[
V_\phi(T) = X.
\]

**Definition 2.3.15 (Attainable claims.)** We say that a contingent claim \( X \) is attainable if a replicating strategy for \( X \) exists.

Thus, for an attainable contingent claim, we can construct a portfolio which produces the same cashflow at maturity and is thus equivalent to holding the claim itself. So the price of the contingent claim \( X \) at time \( t \), denoted by \( P_X(t) \), should satisfy

\[
P_X(t) = V_\phi(t) \quad \text{for all } t \in [0, T]
\]

for there to be no arbitrage opportunities (otherwise we could buy the cheaper of the claim or portfolio and sell the dearer asset to produce a positive profit at present with no net future obligation, i.e. an arbitrage opportunity would exist).

The natural question arising here is what happens if we have two replicating strategies \( \phi \) and \( \psi \) for \( X \)? We have the following key result (proved in Section 6.1.3 of [3]) which addresses this question:

**Theorem 2.3.2 (Risk-neutral Valuation Formula.)** The arbitrage price process \( P_X(t) \) of any attainable claim \( X \) is given by

\[
P_X(t) = S_0(t)\mathbb{E}_{\mathbb{P}^\ast} \left[ \frac{X}{S_0(T)} \bigg| \mathcal{F}_t \right].
\]

**Proof** Since \( X \) is attainable, there is a replicating strategy \( \phi \), so that \( V_\phi(T) = X \) and \( P_X(t) = V_\phi(t) \). Since \( \phi \) is an admissible and self-financing trading strategy,
\( \tilde{V}_\phi(t) \) is a \( \mathbb{P}^* \)-martingale, so

\[
P_X(t) = V_\phi(t) = S_0(t) \tilde{V}_\phi(t) \\
= S_0(t) \mathbb{E}_{\mathbb{P}^*} \left[ \tilde{V}_\phi(T) \bigg| \mathcal{F}_t \right] \\
= S_0(t) \mathbb{E}_{\mathbb{P}^*} \left[ \frac{V_\phi(T)}{S_0(T)} \bigg| \mathcal{F}_t \right] \\
= S_0(t) \mathbb{E}_{\mathbb{P}^*} \left[ \frac{X}{S_0(T)} \bigg| \mathcal{F}_t \right]
\]

\( \Box \)

We can see from (2.11) that the price process of \( X \) is independent of the replicating strategy, and thus if \( \phi \) and \( \psi \) are two replicating strategies for \( X \) then

\[
V_\phi(t) = V_\psi(t).
\]

**Definition 2.3.16** (Market completeness) We describe our market model as being complete if any contingent claim is attainable.

We have the following result regarding completeness (appearing as Theorem 6.1.5 in [3]), which we quote without proof for conciseness.

**Theorem 2.3.3** If there exists a unique EMM for our market model then our market is complete.

It should be noted that in the discrete time case the converse is also true, and thus the existence of a unique EMM is equivalent to market completeness.

## 2.4 Interest rates

In Section 2.3 we gave a passing mention to the bank account \( B(t) \). We will examine the process \( B(t) \) in more detail in this section, defining the concepts of forward, spot and short interest rates.

We again assume the presence of a filtered probability space \( (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \), where \( \mathbb{F} = \{ \mathcal{F}_t \}_{0 \leq t \leq T^*} \) satisfies the conditions of right-continuity and completeness over the fixed time horizon \( T^* \). We assume that we can observe the price of default-free, zero-coupon bonds with face value 1.

**Definition 2.4.1** (Zero-coupon bond.) A zero-coupon bond with maturity \( T \leq T^* \) is a financial contract that guarantees the holder a cash payment of 1 unit at the maturity date \( T \). For convenience we will refer to a zero-coupon bond maturing at time \( T \) as a \( T \)-bond and denote its price at time \( t \) by \( p(t, T) \).
It is clear from the definition that \( p(T, T) = 1 \) for all \( T \). In addition we shall assume that the price process \( \{p(t, T)\}_{0 \leq t \leq T} \) is adapted, strictly positive and, for every fixed \( t \), is continuously differentiable in the \( T \) variable.

We assume that interest is compounded continuously, i.e. if we are quoted an interest rate of \( c\% \) per annum, then an investment of 1 unit will accumulate to \( e^{\frac{c}{100} \times t} \) after \( t \) years.

We will now define a selection of risk-free interest rates by considering a simple arbitrage argument. Given three times \( t < T_1 < T_2 \), a simple arbitrage argument can be used to find the risk-free rate of return over the interval \([T_1, T_2]\), determined at time \( t \) of an investment of 1 at time \( T_1 \). Table 2.1 shows the set-up for the arbitrage argument.

<table>
<thead>
<tr>
<th>Time</th>
<th>( t )</th>
<th>( T_1 )</th>
<th>( T_2 )</th>
</tr>
</thead>
</table>
| Sell \( T_1 \)-bond (receive \( p(t, T_1) \)) | Buy \( \frac{p(t, T_1)}{p(t, T_2)} \) \( T_2 \)-bonds (costing \( p(t, T_1) \)) | Pay out 1 | Receive \( \frac{p(t, T_1)}{p(t, T_2)} \)
| Net payoff | 0 | -1 | \( \frac{p(t, T_1)}{p(t, T_2)} \)

We can see that at time \( t \) a planned investment of 1 unit at time \( T_1 \) will yield a payoff of \( \frac{p(t, T_1)}{p(t, T_2)} \) at time \( T_2 \). To avoid arbitrage in this situation, the constant risk-free rate of interest \( R \) that should apply over the period \( T_1 \) to \( T_2 \) needs to satisfy:

\[
e^{R(T_2 - T_1)} = \frac{p(t, T_1)}{p(t, T_2)} \quad \text{i.e.} \quad R = \frac{1}{T_2 - T_1} \ln \left( \frac{p(t, T_1)}{p(t, T_2)} \right) = R^*
\]

If \( R \) is smaller than this value we can generate a risk-free profit by agreeing at time \( t \) to borrow 1 unit at time \( T_1 \) to be repaid at time \( T_2 \) with interest charged at the continuously compounded rate of \( R \). This situation is illustrated in Table 2.2.

<table>
<thead>
<tr>
<th>Time</th>
<th>( t )</th>
<th>( T_1 )</th>
<th>( T_2 )</th>
</tr>
</thead>
</table>
| Sell \( T_1 \)-bond (receive \( p(t, T_1) \)) | Buy \( \frac{p(t, T_1)}{p(t, T_2)} \) \( T_2 \)-bonds (costing \( p(t, T_1) \)) | Pay out 1 | Receive \( \frac{p(t, T_1)}{p(t, T_2)} \)
| Borrow 1 at \( T_1 \) and repay at \( T_2 \) | Receive 1 | Repay \( \frac{p(t, T_1)}{p(t, T_2)} \) | > 0
| Net payoff | 0 | 0 | > 0

If \( R \) is larger than \( R^* \) we can generate a risk-free profit by reversing our bond positions and agreeing at time \( t \) to lend 1 unit at time \( T_1 \) to be returned at time \( T_2 \) with interest charged at the continuously compounded rate of \( R \). This situation is illustrated in Table 2.3.

In light of this arbitrage argument we formally define the following interest rates:
Table 2.3: $R > R^*$

<table>
<thead>
<tr>
<th>Time</th>
<th>$t$</th>
<th>$T_1$</th>
<th>$T_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell</td>
<td>$\frac{p(t,T_1)}{p(t,T_2)}$ $T_2$-bonds (receive $p(t,T_1)$)</td>
<td>Receive 1</td>
<td>Pay out $\frac{p(t,T_1)}{p(t,T_2)}$</td>
</tr>
<tr>
<td>Buy</td>
<td>$T_1$-bond (costing $p(t,T_1)$)</td>
<td>Lend 1</td>
<td>Receive $&gt; \frac{p(t,T_1)}{p(t,T_2)}$</td>
</tr>
<tr>
<td>Lend 1</td>
<td>at $T_1$ to be repaid at $T_2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Net payoff</td>
<td>0</td>
<td>0</td>
<td>$&gt; 0$</td>
</tr>
</tbody>
</table>

**Definition 2.4.2** Given the times $t < T_1 < T_2$, we specify the following interest rates:

i) The *forward rate* at time $t$ for the period $[T_1, T_2]$ is defined by

$$R(t; T_1, T_2) = -\frac{\ln p(t, T_2) - \ln p(t, T_1)}{T_2 - T_1},$$

which gives the risk-free rate of return we would get over the period $[T_1, T_2]$ if we contracted at time $t$ to invest 1 unit at time $T_1$.

ii) The *spot rate* $R(T_1, T_2)$ for the period $[T_1, T_2]$ is defined by

$$R(T_1, T_2) = R(T_1; T_1, T_2),$$

which gives the risk-free rate of return we would get over the period $[T_1, T_2]$ if we contracted to invest 1 unit at time $T_1$.

If we fix $T_1$ to be the present time and allow $T_2$ to vary across all future times, the resulting function of $T_2$ is known as the *yield curve*.

iii) The *instantaneous forward rate* with maturity $T$, at time $t$, denoted by $f(t, T)$ is defined as

$$f(t, T) = -\frac{\partial \ln p(t, T)}{\partial T},$$

which gives the risk-free rate of return we would get over the period $[T, T + dT]$ if we contracted at time $t$ to invest 1 unit at time $T$, where $dT$ is an infinitesimally small increment.

iv) The *short rate* at time $t$, $r(t)$, is defined by

$$r(t) = f(t, t),$$

which represents the interest we would receive at time $t + dt$ from an investment of 1 unit at time $t$, where $dt$ is an infinitesimally small increment.

We can now return to the bank account $B(t)$ and specify it in further detail relative to the short rate.
Definition 2.4.3 (Bank account and short rates.) In a market model supporting the definition of the short rate \( r \), we define the bank account process by

\[
B(t) = \exp \left\{ \int_0^t r(u) du \right\}.
\]

So far we have assumed that the prices of zero-coupon bonds are observable, and we derive from these prices the values of selected rates. In modeling scenarios we often start with a model for the short rate \( r(t) \) specifying the short rate, defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P}^*)\), where \( \mathbb{P}^* \) is the EMM for our market model, then we have the following result regarding bond prices.

Theorem 2.4.1 The time \( t \) price of a zero-coupon bond maturing at time \( T \) is given by

\[
p(t, T) = \mathbb{E}_{\mathbb{P}^*} \left( \exp \left\{ - \int_t^T r(u) du \right\} \Big| \mathcal{F}_t \right).
\] (2.12)

Proof In our no-arbitrage set-up from Section 2.3 we take the bank account \( B \) to be our numéraire. A zero-coupon bond represents a contingent claim paying 1 unit (measured relative to the numéraire) at maturity, so we can apply the result of Theorem 2.3.2 to obtain

\[
p(t, T) = B(t) \mathbb{E}_{\mathbb{P}^*} \left( \frac{1}{B(T)} \Big| \mathcal{F}_t \right)
\]

\[
= \mathbb{E}_{\mathbb{P}^*} \left( \frac{B(t)}{B(T)} \Big| \mathcal{F}_t \right) \quad \text{since } r \text{ is } \mathbb{F}-\text{adapted } \Rightarrow B \text{ is } \mathbb{F}-\text{adapted}
\]

\[
= \mathbb{E}_{\mathbb{P}^*} \left( \frac{\exp \left\{ \int_0^t r(u) du \right\}}{\exp \left\{ \int_0^T r(u) du \right\}} \Big| \mathcal{F}_t \right) \quad \text{using the definition of } B
\]

\[
= \mathbb{E}_{\mathbb{P}^*} \left( \exp \left\{ - \int_t^T r(u) du \right\} \Big| \mathcal{F}_t \right),
\]

and so (2.12) holds. \( \square \)
Chapter 3

Defaultable Claims

In reality, it is often the case that a contingent claim will be subject to default risk, i.e. there is a chance that the counterparty to the claim will fail to make their contractually obligated payments. In this chapter we will follow the text [2] and give a review of the main techniques used to model default risk and price defaultable claims. These techniques build on the framework of no-arbitrage pricing which was presented in Section 2.3.

3.1 Structural vs reduced form approaches

The modeling approaches for defaultable claims fall into two broad categories. The first category features what are known as structural models. Structural approaches model the value of the assets of a party to a defaultable claim and specify that default occurs when the value of these assets falls below some threshold. Under certain broad assumptions (see Ch 2 of [2] for more detail), structural models produce a default time that is predictable in the sense that market participants will be able to forecast the occurrence of default based on observable signals. The second category of defaultable claim pricing approaches contains what are known as reduced form models. Reduced form models do not require the modeling of a contractual party’s assets. Instead, the default time is modeled by a stopping time which precludes investors from forecasting default. Thus in reduced form models the default event comes as a complete shock to the market.

The remainder of this chapter will be dedicated to pricing defaultable claims in the context of reduced form models.
CHAPTER 3. DEFAULTABLE CLAIMS

3.2 Defaultable claims in the reduced form setting

3.2.1 Definitions and framework

We will recount here the basic set-up for general reduced form models as is presented in [2].

We assume an underlying probability space \((\Omega, \mathcal{G}, \mathbb{P})\) with the market filtration \(\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}\) (which is assumed to satisfy the usual conditions of right-continuity and completeness). As was mentioned in Section 2.3, the EMM is a key concept in the pricing of contingent claims. We will assume that \(\mathbb{P}^*\) is an EMM relative to our filtration \(\mathbb{F}\) for our market model, meaning that the price process of any tradeable security paying no coupons or dividends follows a \(\mathcal{G}\)-martingale under \(\mathbb{P}^*\) when discounted by the numéraire. We will define our random objects on the probability space \((\Omega, \mathcal{G}, \mathbb{P}^*)\) unless otherwise specified. We have the following formal definition for a defaultable claim:

**Definition 3.2.1 (Defaultable claim.)** A defaultable claim is a quintuple \((X, A, \hat{X}, Z, \tau)\), where:

- \(X\) is the promised contingent claim to be received at time \(T\); \(X\) is taken to be a \(\mathcal{F}_T\)-measurable random variable.

- \(A\) is the process representing the promised dividends. The process \(A\) is assumed to be progressively measurable with respect to the filtration \(\mathbb{F}\). We assume that \(A_0 = 0\) and that \(A\) is a predictable process of finite variation so that we can define integrals with respect to it.

- \(Z\) is the recovery process – the payoff to be received at the time of default. The process \(Z\) is assumed to be a progressively measurable predictable process relative to \(\mathbb{F}\).

- \(\hat{X}\) is the recovery claim – the payoff to be received at time \(T\) if default occurs prior to maturity. The random variable \(\hat{X}\) is assumed to be \(\mathcal{F}_T\)-measurable.

- \(\tau\) is an arbitrary non-negative random variable representing the time of default. We assume that \(\tau\) is finite (a.s.), i.e. \(\mathbb{P}^*(\tau < \infty) = 1\), and that default can not occur at the instant when a defaultable claim is formulated, i.e. \(\mathbb{P}^*(\tau = 0) = 0\). For convenience we also assume that \(\tau\) is unbounded, i.e. \(\mathbb{P}^*(\tau > t) > 0\) for every \(t \in \mathbb{R}_+\).

The way we choose to model \(\hat{X}\) and \(Z\) depends on the rules we believe will apply in the event of default, e.g. if we are likely to receive a payout at the time of default, then we would model our recovery using the process \(Z\) rather than the r.v. \(\hat{X}\).

We use the default time \(\tau\) to specify a new process:
Definition 3.2.2 (Default process.) For a given default time $\tau$ we define the right-continuous jump process $H$ by

$$H(t) = 1_{\tau \leq t}, \quad t \in \mathbb{R}_+.\)$$

The process $H$ is referred to as the default process, and we let $\mathbb{H}$ be the filtration generated by $H$, i.e.

$$\mathcal{H}_t = \sigma(H(u): u \leq t) = \sigma(\{\tau \leq u\}: u \leq t).$$

We now enlarge the filtration on our probability space by defining a new filtration $\mathcal{G} := \mathcal{H} \vee \mathcal{F}$, i.e. for every $t$ we set $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t = \sigma(\mathcal{H}_t, \mathcal{F}_t)$ (the smallest $\sigma$-algebra containing both $\mathcal{H}_t$ and $\mathcal{F}_t$). It is clear that $\tau$ is a stopping time relative to the filtration $\mathcal{G}$, but this is not necessarily the case when considered relative to the market filtration $\mathcal{F}$.

We will assume that we have access to the short rates for the market and we will use the bank account $B$ (given by Definition 2.4.3) as the numéraire for our market model, i.e. $\{r(t)\}_{t \geq 0}$ is an $\mathcal{F}$-progressively measurable process giving the short rate at time $t$ and the bank account $B$ is given by the expression

$$B(t) = \exp\left(\int_0^t r(u)du\right), \quad t \in \mathbb{R}_+.\)$$

In accordance with the notation in [2], we will combine the payoffs to be received at maturity by introducing the random variable $X^d(T) := X1_{\tau > T} + \tilde{X}1_{\tau \leq T}$. We will also denote by $D$ the process representing the cashflows received by the holder of the defaultable claim. We define $D$ formally through an integral representation.

Definition 3.2.3 (Dividend process.) The dividend process $D$ of a defaultable claim $DCT = (X, A, \tilde{X}, Z, \tau)$ with maturity $T$ is specified by

$$D(t) = X^d(T)1_{t \geq T} + \int_0^t (1 - H(u))dA(u) + \int_0^t Z(u)dH(u), \quad t \in [0, T],$$

where we take $\int_0^t$ to mean $\int_{[0,t]}$.

The value $D(t)$ represents the total dividends received up to time $t$.

We can see from this definition that $D$ is a process of finite variation over $[0, T]$ since it is the sum of three terms, each of which is of finite variation:

- The process $X^d(T)1_{t \geq T}$ is non-zero only at the point $t = T$, at which it takes the (random) value $X^d(T)$, and thus it is of finite variation.

- The process defined by the first integral can be re-expressed as

$$\int_0^t (1 - H(u))dA(u) = \int_0^t 1_{\tau > u}dA(u) = A_\tau 1_{\tau \leq t} + A_t 1_{\tau > t},$$

and so the integral process takes the same values as the process $A$ for $t < \tau$ and remains constant for $t \geq \tau$, thus it inherits the finite variation property from $A$.
The process defined by the second integral can also be re-expressed:

$$\int_0^t Z(u)dH(u) = Z(\tau)1_{\tau \leq t},$$

and so the integral process is of finite variation.

As the dividend process is of finite variation, we can consider stochastic integrals relative to its increments.

### 3.2.2 Valuation formula

Our goal is now to produce an analogous formula to the no-arbitrage price established in Section 2.3 for non-defaultable contingent claims that is applicable in the case of securities susceptible to default risk. We begin by defining the price process.

**Definition 3.2.4** (Price process.) The (ex-dividend) price process \( \{X_{d}(t, T)\}_{t \in [0, T]} \) of a defaultable claim \( DCT = (X, A, \tilde{X}, Z, \tau) \), with maturity at time \( T \) represents the time \( t \) value of all future cashflows associated with the claim. According to this definition, we have \( X_{d}(T, T) = 0 \). We propose the following expression for the price process:

$$X_{d}(t, T) = B(t)E_{\mathbb{P}} \left( \int_t^T B(u)^{-1}dD(u) \mid \mathcal{G}_t \right), \quad t \in [0, T], \quad (3.1)$$
in the no-arbitrage setting.

In the usual no arbitrage set-up we would formulate equation (3.1) in a theorem and prove its validity using the standard techniques outlined in Section 2.3. The issue with reduced form models is that the extension of the filtration from \( \mathbb{F} \) to \( \mathbb{G} \) usually leads to an incomplete market, and thus we can’t use replicating portfolios to justify prices. Instead we will show that by taking (3.1) to be the price of a defaultable claim, the market model remains arbitrage free. We will prove this result via a sequence of propositions and lemmata following the presentation given in [2].

The model set-up is the same as we saw in Section 2.3. We let \( S_i, i = 0, 1, \ldots, d \), denote the price processes of \( d + 1 \) primary securities in an arbitrage-free market model. The price process \( S_0 \) is reserved for our numéraire, which in this case will be the bank account \( B \). It is again assumed that the processes \( S_0, \ldots, S_{d-1} \) are adapted, continuous and strictly positive semi-martingales. We assume that \( S_1, \ldots, S_{d-1} \) are non-dividend paying securities, while \( S_d \) is taken to represent the price process of a security which pays dividends according to the process \( D \) we defined earlier (Definition 3.2.3). Our intention is to interpret \( S_d \) as the price process of our defaultable claim and to show that it must satisfy (3.1) from Definition 3.2.4. Note that it is not assumed that \( S_d \) follows a semi-martingale. We again introduce the discounted price processes \( \tilde{S}_i \) by setting \( \tilde{S}_i(t) = S_i(t)/B(t) \).
We now consider a particular trading strategy involving the \(d\)th asset: we purchase one unit of the asset at time 0 and hold it until time \(T\), investing all proceeds from dividends in the bank account \(B\). In the mathematical notation developed in Section 2.3 we are considering the trading strategy \(\psi\), where

\[
\psi(t) = (\psi_0(t), 0, \ldots, 0, 1),
\]

with the associated value process

\[
V_\psi(t) = \psi_0(t)B(t) + S_d(t), \quad t \in [0, T],
\]

having the initial value \(V_\psi(0) = \psi_0(0) + S_d(0)\). If we constrain the strategy \(\psi\) to be self-financing, then we have, for every \(t \in [0, T]\),

\[
V_\psi(t) - V_\psi(0) = D(t) + \int_0^t \psi_0(u)dB(u) + S_d(t) - S_d(0). \tag{3.2}
\]

**Lemma 3.2.1** The discounted value process \(\tilde{V}_\psi(t) = B(t)^{-1}V_\psi(t)\) of the trading strategy \(\psi\) satisfies

\[
\tilde{V}_\psi(t) = \tilde{V}_\psi(0) + \tilde{S}_d(t) - \tilde{S}_d(0) + \int_0^t B(u)^{-1}dD(u) \tag{3.3}
\]

for every \(t \in [0, T]\).

**Proof** We define a new process \(\hat{V}_\psi\) by

\[
\hat{V}_\psi(t) := V_\psi(t) - S_d(t) = \psi_0(t)B(t). \tag{3.4}
\]

Re-expressing (3.2) in terms of the process \(\hat{V}_\psi\), we have

\[
\hat{V}_\psi(t) = \hat{V}_\psi(0) + D(t) + \int_0^t \psi_0(u)dB(u), \tag{3.5}
\]

and so \(\hat{V}_\psi\) follows a semi-martingale as the process \(D\) is of finite variation and the stochastic integral is itself a semi-martingale.

Our goal is to calculate the stochastic differential of \(B(t)^{-1}\hat{V}_\psi(t)\). Before doing so we make the following observations:

- \(B\) is a continuous process of finite variation and thus \(d[B, B^{-1}](t) = 0\), \(d[B, \hat{V}_\psi](t) = 0\) by Lemma 2.1.2.

- We have the equality \(B(t)^{-1}dB(t) + B(t)dB(t)^{-1} = 0\), which arises from applying Itô’s product rule and our first observation:

\[
0 = d(1) = dB(t)B(t)^{-1} = B(t)dB(t)^{-1} + B(t)^{-1}dB(t).
\]
• From (3.5) we obtain that the stochastic differential for \( \hat{V}_\psi \) is given by
\[
d\hat{V}_\psi(t) = dD(t) + \psi_0(t)dB(t).
\] (3.6)

Itô's product rule now gives us
\[
d(B(t)^{-1}\hat{V}_\psi(t)) = B(t)^{-1}d\hat{V}_\psi(t) + \hat{V}_\psi(t)dB(t)^{-1} = B(t)^{-1}(dD(t) + \psi_0(t)dB(t)) + \hat{V}_\psi(t)dB(t)^{-1} = B(t)^{-1}dD(t) + \psi_0(t)(B(t)^{-1}dB(t) + B(t)dB(t)^{-1}) = B(t)^{-1}dD(t),
\]
where the last line follows from our second observation. Integrating the last equality we have
\[
B(t)^{-1}\hat{V}_\psi(t) = B(0)^{-1}\hat{V}_\psi(0) + \int_0^t B(u)dD(u),
\]
and substituting our definition of the process \( \hat{V}_\psi \) yields
\[
B(t)^{-1}(V_\psi(t) - S_d(t)) = B(0)^{-1}(V_\psi(0) - S_d(0)) + \int_0^t B(u)dD(u),
\]
which can be rearranged to give expression (3.3) as was required. \(\Box\)

A straightforward application of Lemma 3.2.1 shows that for every \( t \in [0, T] \) we have:
\[
\hat{V}_\psi(T) - \hat{V}_\psi(t) = S_d(T) - S_d(t) + \int_t^T B(u)^{-1}dD(u).
\] (3.7)

As was mentioned earlier, we have assumed that our market model is arbitrage free, i.e. we have an EMM \( P^* \) (not necessarily unique) that is equivalent to \( P \) and for which the discounted price processes \( \hat{S}_t \) for non-dividend paying primary securities as well as the discounted wealth process \( \hat{V}_\psi \) of admissible self-financing trading strategies \( \phi = (\phi_0, \phi_1, \ldots, \phi_{d-1}, 0) \) follow martingales under \( P^* \). We make the additional assumption that the particular trading strategy \( \psi \) introduced earlier is admissible, so that \( \hat{V}_\psi \) follows a \( P^* \)-martingale.

We now suppose that the time \( t \) market value of the \( d \)th security is derived only from the future dividends produced by the asset (i.e. \( S_d(T) = \hat{S}_d(T) = 0 \)) and we call \( S_d \) the ex-dividend price of the \( d \)th asset (where we have in mind a defaultable claim, but we still maintain generality).

**Proposition 3.2.1** For every \( t \in [0, T] \), the ex-dividend price process \( S_d \) satisfies
\[
S_d(t) = B(t)E_{P^*}\left(\int_t^T B(u)^{-1}dD(u)\big|\mathcal{G}_t\right).
\] (3.8)
CHAPTER 3. DEFAULTABLE CLAIMS

Proof Since $\tilde{V}_\psi$ is assumed to be a $\mathbb{P}^*$-martingale we have, for any $t \in [0,T],
$$
\mathbb{E}_{\mathbb{P}^*}\left(\tilde{V}_\psi(T) - \tilde{V}_\psi(t)\middle|\mathcal{G}_t\right) = 0,
$$
and thus substituting (3.7) into this expression we have
$$
\mathbb{E}_{\mathbb{P}^*}\left(\tilde{S}_d(T) - \tilde{S}_d(t) + \int_t^T B(u)^{-1}dD(u)\middle|\mathcal{G}_t\right) = 0.
$$
(3.9)

Noting that $\tilde{S}_d(t)$ is $\mathcal{G}_t$-measurable, we can rearrange (3.9) to give us
$$
\tilde{S}_d(t) = \mathbb{E}_{\mathbb{P}^*}\left(\tilde{S}_d(T) + \int_t^T B(u)^{-1}dD(u)\middle|\mathcal{G}_t\right),
$$
which reduces to (3.8) when we take into account our assumption that $S_d(T) = \tilde{S}_d(T) = 0$. □

Our aim is now to consider the value processes of general trading strategies $\phi = (\phi_0, \ldots, \phi_d)$ to set up the no-arbitrage framework for our market model. The value process of the strategy $\phi$ is given by $V_\phi(t) = \sum_{i=0}^d \phi_i(t)S_i(t)$. Once again, recalling Definition 2.3.7, a strategy $\phi$ is self-financing if $V_\phi(t) = V_\phi(0) + G_\phi(t)$ for every $t \in [0,T]$, where the gains process $G_\phi$ is defined in this case to be
$$
G_\phi(t) := \int_0^t \phi_d(u)dD(u) + \sum_{i=0}^d \int_0^t \phi_i(u)dS_i(u).
$$

Corollary 3.2.1 For any self-financing trading strategy $\phi$, the discounted value process $\tilde{V}_\phi$ follows a local martingale under $\mathbb{P}^*$.

Proof Since the strategy $\phi$ is self-financing, we have the following expression for the differential of the value process:
$$
dV_\phi(t) = \phi_d(t)dD(t) + \sum_{i=0}^d \phi_i(t)dS_i(t).
$$
(3.10)

As $B$ is a continuous process of finite variation, using Lemma 2.1.2 and Itô’s formula yields
$$
 dB(t)^{-1} = -B(t)^{-2}dB(t) + B(t)^{-3}\underbrace{d[B,B](t)}_{=0}
= -B(t)^{-2}dB(t).
$$

We can use this expression to compute the quadratic covariation of $B^{-1}$ and $S_i$ as follows:
$$
d[B^{-1}, S_i](t) = dB(t)^{-1}dS_i(t)
= -B(t)^{-2}dB(t)dS_i(t)
= -B(t)^{-2}d[B, S_i](t)
= 0,
$$
where the last line follows from Lemma 2.1.2 and the fact that \( B \) is a continuous process of finite variation and the assumption that the \( S_i \)'s follow semi-martingales.

Using the last result and Itô's product rule, the stochastic differentials of the discounted price processes for \( i = 0, 1, \ldots, d \) are given by

\[
d\tilde{S}_i(t) = d(B(t)^{-1}S_i(t)) = S_i(t)dB(t)^{-1} + B(t)^{-1}dS_i(t) + d[B^{-1}, S_i](t) = 0
\]

\[
= S_i(t)dB(t)^{-1} + B(t)^{-1}dS_i(t),
\]

and we can again apply the product rule to find the differential of the discounted value process.

\[
d\tilde{V}_\phi(t) = V_\phi(t)dB(t)^{-1} + B(t)^{-1}dV_\phi(t) = V_\phi(t)dB(t)^{-1} + B(t)^{-1}\left(\phi_d(t)dD(t) + \sum_{i=0}^{d} \phi_i(t)dS_i(t)\right)
\]

using (3.10)

\[
= \left(\sum_{i=0}^{d} \phi_i(t)S_i(t)\right) dB(t)^{-1} + B(t)^{-1}\left(\phi_d(t)dD(t) + \sum_{i=0}^{d} \phi_i(t)dS_i(t)\right)
\]

\[
= \sum_{i=0}^{d} \phi_i(t) \left[S_i(t)dB(t)^{-1} + B(t)^{-1}dS_i(t)\right] + \phi_d(t)B(t)^{-1}dD(t)
\]

\[
= \sum_{i=0}^{d} \phi_i(t)\tilde{S}_i(t) + \phi_d(t)B(t)^{-1}dD(t)
\]

\[
= \sum_{i=1}^{d} \phi_i(t)d\tilde{S}_i(t) + \phi_d(t)B(t)^{-1}dD(t) \quad \text{since } \tilde{S}_0 = B^{-1}B = 1
\]

\[
= \sum_{i=1}^{d-1} \phi_i(t)d\tilde{S}_i(t) + \phi_d(t)\left(d\tilde{S}_d(t) + B(t)^{-1}dD(t)\right)
\]

\[
= \sum_{i=1}^{d-1} \phi_i(t)d\tilde{S}_i(t) + \phi_d(t)d\tilde{S}_d(t),
\]

where we have defined a new process \( \tilde{S}_d \) by the formula

\[
\tilde{S}_d(t) := \tilde{S}_d(t) + \int_0^t B(u)^{-1}dD(u).
\]
Using (3.8) the price process $\hat{S}_d$ can be re-expressed as an integral.

$$\hat{S}_d(t) = \tilde{S}_d(t) + \int_0^t B(u)^{-1} dD(u)$$

$$= B(t)^{-1} S_d(t) + \int_0^t B(u)^{-1} dD(u)$$

$$= \mathbb{E}_{P^*} \left( \int_t^T B(u)^{-1} dD(u) \bigg| G_t \right) + \int_0^t B(u)^{-1} dD(u) \quad \text{using (3.8)}$$

$$= \mathbb{E}_{P^*} \left( \int_t^T B(u)^{-1} dD(u) + \int_0^t B(u)^{-1} dD(u) \bigg| G_t \right)$$

$$= \mathbb{E}_{P^*} \left( \int_0^T B(u)^{-1} dD(u) \bigg| G_t \right),$$

and thus $\hat{S}_d$ follows a martingale under $\mathbb{P}^*$. So we can express the discounted value process as

$$\tilde{V}_\phi(t) = \tilde{V}_\phi(0) + \sum_{i=1}^{d-1} \int_0^t \phi_i(u) d\tilde{S}_i(u) + \int_0^t \phi_d(u) d\hat{S}_d(u),$$

where each of the integrators is a martingale, and thus Theorem 2.1.4 gives us the required result that $\tilde{V}_\phi$ follows a local martingale.

Corollary 3.2.1 shows that if we assume that a market model for securities $S_0, S_1, \ldots, S_{d-1}$ is arbitrage-free and the ex-dividend price process of the additional security $S_d$ is given by Definition 3.2.4, then the market retains its arbitrage-free feature.

3.3 Recovery rules

In the event of default prior to the claim’s maturity, it is often the case that the claim holder will still receive a payment of some sort from the claim issuer. The details of these payments may be outlined in the claim contract or negotiated at the time of default and can be quite complex. For the purposes of formulating a mathematical model, we assume rules that simplify the recovery process. We will illustrate a selection of these rules using default-free and defaultable zero-coupon bonds as references.

Using the notation from Section 2.4, we let $p(t, T)$ denote the arbitrage-free price at $t$ of a zero-coupon default-free bond paying its face value of 1 at maturity $T$, and thus $p(T, T) = 1$. We will introduce new notation for the arbitrage-free price of a defaultable bond: we let $p_d(t, T)$ be the time $t$ price of a zero-coupon bond subject to default risk, with face value 1 and maturity $T$. In this case $p_d(T, T)$ will depend on how we believe the recovery will work.

The recovery payment is often specified in terms of the recovery rate, which we define below.
Definition 3.3.1 (Recovery rate.) The recovery rate $\delta$ is the fraction of the bond’s face value that is paid to the bondholders in the case of default.

We now present two recovery rules that arise naturally from the notion of a recovery rate and differ on the basis of the timing of the recovery payment.

Definition 3.3.2 (Fractional recovery of par value.) If a fixed fraction $\delta$ of the bond’s face value is paid to the bondholder at the time of default $\tau$, the recovery scheme is referred to as the fractional recovery of par value. In this case we would have

$$p_d(T, T) = 1_{\tau > T} + \delta p(\tau, T)^{-1} 1_{\tau \leq T},$$

and in the defaultable claim notation presented in Section 3.2.1 we would model this set-up by taking

$$X = 1,$$
$$A(t) = 0 \text{ for all } t,$$
$$Z(t) = \delta \text{ for all } t,$$
$$\hat{X} = 0.$$

Definition 3.3.3 (Fractional recovery of Treasury value.) If, in the event of default, a fixed fraction $\delta$ of the bond’s face value is paid to the bondholder at maturity $T$, the recovery scheme is referred to as the fractional recovery of Treasury value. We then have

$$p_d(T, T) = 1_{\tau > T} + \delta 1_{\tau \leq T},$$

which would be modeled by taking

$$X = 1,$$
$$A(t) = 0 \text{ for all } t,$$
$$Z(t) = 0 \text{ for all } t,$$
$$\hat{X} = \delta,$$

in our defaultable claim notation.

While apparently simple, these two recovery rules are utilised in the literature. Briys and de Varenne [7] used the fractional recovery of par value rule in their structural approach to corporate bond valuation. Longstaff and Schwartz [19] used the fractional recovery of Treasury value in their set-up for a corporate bond pricing model, where the interest rate and firm value processes are driven by Brownian motion processes. Zhou [24] extended the notion of recovery of Treasury value by postulating that $\delta$ is an increasing function of the ratio of the firm’s value to a given default threshold.

As a matter of interest we will present one final recovery rule known as the fractional recovery of market value.
Definition 3.3.4 (Fractional recovery of market value.) By fractional recovery of market value we mean a situation where, at the time of default, the corporate bondholder receives a fraction of the pre-default market value of the bond. In this case we have

$$p_d(T, T) = 1_{\tau > T} + \delta p_d(\tau-, T)p(\tau, T)^{-1}1_{\tau \leq T},$$

which would be modeled using our defaultable claim notation by

$$X = 1,$$
$$A(t) = 0 \text{ for all } t,$$
$$Z(t) = \delta p_d(t-, T) \text{ for all } t,$$
$$\hat{X} = 0.$$ 

Numerous other recovery rules are used in the literature, all of which are derived from a consideration given to mathematical tractability and interpretability (in the economic sense). The reader is referred to [5] which surveys recent structural and reduced form approaches employed in defaultable claim valuation and includes a table summarising the recovery rules used in a selection of papers produced in the area. Belanger et al. [1] also provide a comparison of recovery rules, extending their survey beyond summary to produce relationships between the prices of defaultable bonds under different recovery schemes.

3.4 Valuing zero-coupon defaultable bonds

We want to determine the time $t$ no-arbitrage price $p_d(t, T)$ of a zero-coupon bond with face value 1, maturing at time $T$ and making no recovery payment in the event of default. In the notation developed in Section 3.2.1, this set-up is represented by the defaultable claim

$$DCT = (1, 0, 0, 0, \tau),$$

which has the simple dividend process

$$D(t) = 1_{\tau > T}1_{t \geq T}.$$ 

Thus we can use Equation (3.1) to find the no-arbitrage price $p_d(t, T)$ as

$$p_d(t, T) = B(t)\mathbb{E}_{\mathbb{P}^*}\left(\int_{t}^{T} B(u)^{-1}dD(u) \bigg| G_t\right)$$
$$= B(t)\mathbb{E}_{\mathbb{P}^*}\left(\int_{t}^{T} B(T)^{-1}1_{\tau > T} \bigg| G_t\right)$$
$$= \mathbb{E}_{\mathbb{P}^*}\left(\int_{t}^{T} B(T)^{-1}1_{\tau > T} \bigg| G_t\right)$$
$$= \mathbb{E}_{\mathbb{P}^*}\left(\exp\left(-\int_{t}^{T} r(u)du\right)1_{\tau > T} \bigg| G_t\right). \quad (3.11)$$
In order to evaluate expectations of the form given by (3.11), we need to consider in more detail how we wish to model the default time $\tau$. There have been a few major approaches that have developed in the literature. Jarrow et al. [14] modeled credit ratings using a continuous time Markov chain and included the default event as a state in the chain. Cathcart and El-Jahel [8] used what they named the ‘signaling approach’. They assumed the existence of a signaling variable which captures the factors that affect an entity’s default probability. The signaling variable was modeled as a stochastic process following a geometric Brownian motion, and the resulting pricing expressions were found to be expressible analytically in terms of the normal distribution function.

The signaling and credit rating models appear in the minority, by far the most utilised approach to modeling default times is the use of stochastic intensities which represent the conditional probability of instantaneous default assuming that default has not yet occurred. These models allow us to make flexible assumptions regarding the nature of the market and often lead to compact closed form expressions for bond prices. The paper by Duffie and Singleton [13] serves as a representative example of how stochastic intensities have been utilised in conventional applications. Lando [18] presents two ways to utilise stochastic intensities. He first demonstrates the use of intensities in conventional reduced-form models and derives the resulting pricing formulae. His second approach is an expansion of the Markov idea used in [14]. In the generalised model presented by Lando, the rates of credit rating transitions are now allowed to depend on state variables, which allows for credit spreads to change between transitions in a lender’s credit rating.

As our goal is to develop a model using the conventional stochastic intensity approach, we will present the required theory in the next section.

### 3.4.1 Hazard process of the default time

This section will follow the presentation given in Ch 5 of [2]. We will develop the concepts of the hazard processes and intensities of random times and derive expressions for expectations that often arise in the pricing of defaultable claims.

As is often the case in mathematics, we will derive our results in more generality than is required as this lends to more compact proofs. To afford us this generality, we will be less specific regarding our underlying probability space and associated random objects.

We assume that we have a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and a non-negative random variable $\tau$ such that $\mathbb{P}(\tau = 0) = 0$ and $\mathbb{P}(\tau > t) > 0$ for any $t \in \mathbb{R}_+$. We introduce the process $H(t) = 1_{\tau \leq t}$ and let $\mathcal{H} = \{H_t\}_{t \geq 0}$ be the filtration generated by $H$, i.e. $\mathcal{H}_t = \sigma(\{H(u) : u \leq t\}) = \sigma(\{\tau \leq u : u \leq t\})$. We let $\mathcal{G} = \{\mathcal{G}_t\}_{t \geq 0}$ be an arbitrary filtration on $(\Omega, \mathcal{G}, \mathbb{P})$. All filtrations defined on our probability space are assumed to satisfy the conditions of right-continuity and completeness.

We now consider an additional filtration $\mathcal{F}$ with the $\sigma$-field $\mathcal{F}_0$ being trivial (i.e. $\mathcal{F}_0 = \{\emptyset, \Omega\}$). We will examine the implications of $\mathcal{F}$ satisfying one of the following conditions:
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Condition 1 The filtration $\mathcal{F}$ is such that $\mathcal{G} = \mathcal{H} \lor \mathcal{F}$, i.e. $\mathcal{G}_t = \mathcal{H}_t \lor \mathcal{F}_t$ for any $t \in \mathbb{R}_+$.  

Condition 2 For every $t \in \mathbb{R}_+$ we have $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{H}_t \lor \mathcal{F}_t$.

Note that $\tau$ is clearly an $\mathcal{H}$-stopping time. We assume that $\tau$ is not an $\mathcal{F}$-stopping time (i.e. we don’t want to consider the case where $\mathcal{F} = \mathcal{G}$).

We are now ready to consider a preliminary result relating to the structure of $\mathcal{G}$.

Lemma 3.4.1 If the filtration $\mathcal{G}$ satisfies $\mathcal{G} \subset \mathcal{H} \lor \mathcal{F}$ (i.e. $\mathcal{G}_t \subset \mathcal{H}_t \lor \mathcal{F}_t$ for each $t \in \mathbb{R}_+$), then $\mathcal{G} \subset \mathcal{G}^*$, where $\mathcal{G}^* = \{ \mathcal{G}^*_t \}_{t \geq 0}$ is the filtration given by

$$\mathcal{G}^*_t := \{ A \in \mathcal{G} : \exists B \in \mathcal{F}_t, A \cap \{ \tau > t \} = B \cap \{ \tau > t \} \}.$$

Proof We first need to show that $\mathcal{G}^*_t$ is a sub-$\sigma$-algebra of $\mathcal{G}$:

- $\Omega \in \mathcal{G}^*_t$ since $\Omega \in \mathcal{G}_t, \mathcal{F}_t$ and $\Omega \cap \{ \tau > t \} = \{ \tau > t \}$.

- Suppose that $A \in \mathcal{G}^*_t$, then there exists a $B \in \mathcal{F}_t$ such that $A \cap \{ \tau > T \} = B \cap \{ \tau > t \}$.

Noting that

$$\begin{align*}
A^c \cap \{ \tau > t \} &= \{ \tau > t \} \cap (A \cap \{ \tau > t \})^c \\
&= \{ \tau > t \} \cap (B \cap \{ \tau > t \})^c \\
&= B^c \cap \{ \tau > t \},
\end{align*}$$

we have $A^c \in \mathcal{G}^*_t$ since $B \in \mathcal{F}_t \Rightarrow B^c \in \mathcal{F}_t$.

- Let $A_n \in \mathcal{G}^*_t$ be such that, for all $n \in \mathbb{N}$, there exists $B_n \in \mathcal{F}_t$ such that $A_n \cap \{ \tau > t \} = B_n \cap \{ \tau > t \}$. Then

$$\begin{align*}
\left( \bigcup_{n=1}^{\infty} A_n \right) \cap \{ \tau > t \} &= \bigcup_{n=1}^{\infty} (A_n \cap \{ \tau > t \}) \\
&= \bigcup_{n=1}^{\infty} (B_n \cap \{ \tau > t \}) \\
&= \left( \bigcup_{n=1}^{\infty} B_n \right) \cap \{ \tau > t \},
\end{align*}$$

and thus $\bigcup_{n=1}^{\infty} A_n \in \mathcal{G}^*_t$ since $\bigcup_{n=1}^{\infty} B_n \in \mathcal{F}_t$.

So we have established that $\mathcal{G}^*_t$ is a sub-$\sigma$-algebra of $\mathcal{G}$, and it now suffices to check that $\mathcal{H}_t \subseteq \mathcal{G}^*_t$ and $\mathcal{F}_t \subseteq \mathcal{G}^*_t$ for all $t \in \mathbb{R}_+$.

- $A \in \mathcal{F}_t \Rightarrow A \cap \{ \tau > t \} = B \cap \{ \tau > t \}$, where $B = A \in \mathcal{F}_t$ and thus $\mathcal{F}_t \subseteq \mathcal{G}^*_t$.  

Suppose that \( A = \{ \tau \leq u \} \) for some \( u \leq t \). Then \( A \cap \{ \tau > t \} = \emptyset = \emptyset \cap \{ \tau > t \} \) and \( \emptyset \in \mathcal{F}_t \), so that \( A \in \mathcal{G}_t^* \). Therefore we have \( \mathcal{H}_t = \sigma(\{ \{ \tau \leq u \}: u \leq t \}) \subseteq \mathcal{G}_t^* \). □

Now we can use this result to derive some useful expressions for expectations of the form \( \mathbb{E}_P(1_{\tau > t} Y | \mathcal{G}_t) \), where \( Y \) is a \( \mathbb{P} \)-integrable random variable. We present these results in the following lemma.

**Lemma 3.4.2**  

i) If Condition 2 holds, then, for any \( \mathcal{G} \)-measurable random variable \( Y \) and any \( t \in \mathbb{R}_+ \), we have

\[
\mathbb{E}_P(1_{\tau > t} Y | \mathcal{G}_t) = \mathbb{P}(\tau > t | \mathcal{G}_t) \frac{\mathbb{E}_P(1_{\tau > t} Y | \mathcal{F}_t)}{\mathbb{P}(\tau > t | \mathcal{F}_t)}. \tag{3.12}
\]

ii) If, in addition, \( \mathcal{H}_t \subseteq \mathcal{G}_t \) (so that Condition 1 holds), then

\[
\mathbb{E}_P(1_{\tau > t} Y | \mathcal{G}_t) = 1_{\tau > t} \mathbb{E}_P(1_{\tau > t} Y | \mathcal{F}_t). \tag{3.13}
\]

**Proof** Since ii) is a direct consequence of i), it is enough to verify the first statement. In order to save space we will let \( C = \{ \tau > t \} \). Rearranging (3.12), we want to show that

\[
\mathbb{E}_P(1_C Y | \mathcal{G}_t) \mathbb{P}(C | \mathcal{F}_t) = \mathbb{P}(C | \mathcal{G}_t) \mathbb{E}_P(1_C Y | \mathcal{F}_t),
\]

and since \( \mathcal{F}_t \subseteq \mathcal{G}_t \), this is equivalent to

\[
\mathbb{E}_P(1_C Y | \mathcal{F}_t) = \mathbb{E}_P(1_C \mathbb{E}_P(1_C Y | \mathcal{F}_t) | \mathcal{G}_t).
\]

To show this we need to verify that, in accordance with the definition of conditional expectation, for any \( A \in \mathcal{G}_t \) we have

\[
\int_A 1_C Y \mathbb{P}(C | \mathcal{F}_t) d\mathbb{P} = \int_A 1_C \mathbb{E}_P(1_C Y | \mathcal{F}_t) d\mathbb{P}.
\]

From Lemma 3.4.1, for any \( A \in \mathcal{G}_t \) we have \( A \cap C = B \cap C \) for some \( B \in \mathcal{F}_t \), so that

\[
\int_A 1_C Y \mathbb{P}(C | \mathcal{F}_t) d\mathbb{P} = \int_{A \cap C} Y \mathbb{P}(C | \mathcal{F}_t) d\mathbb{P} = \int_{B \cap C} Y \mathbb{P}(C | \mathcal{F}_t) d\mathbb{P}
\]

\[
= \int_B 1_C Y \mathbb{P}(C | \mathcal{F}_t) d\mathbb{P} = \int_B \mathbb{E}_P(1_C Y | \mathcal{F}_t) \mathbb{P}(C | \mathcal{F}_t) d\mathbb{P}
\]

\[
= \int_B \mathbb{E}_P(1_C \mathbb{E}_P(1_C Y | \mathcal{F}_t) | \mathcal{F}_t) d\mathbb{P} = \int_B 1_C \mathbb{E}_P(1_C Y | \mathcal{F}_t) d\mathbb{P}
\]

\[
= \int_{B \cap C} \mathbb{E}_P(1_C Y | \mathcal{F}_t) d\mathbb{P} = \int_{A \cap C} \mathbb{E}_P(1_C Y | \mathcal{F}_t) d\mathbb{P}
\]

\[
= \int_A 1_C \mathbb{E}_P(1_C Y | \mathcal{F}_t) d\mathbb{P},
\]

which completes the proof. □
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We now introduce the concepts of the survival and hazard processes, which are useful tools in modelling the default time $\tau$.

**Definition 3.4.1** (Survival process.) For any $t \in \mathbb{R}_+$, we write $F_t = \mathbb{P}(\tau \leq t | \mathcal{F}_t)$, and we let $G$ denote the $\mathbb{F}$-survival process of $\tau$ with respect to the filtration $\mathbb{F}$, where $G$ is defined by

$$G_t := 1 - F_t = \mathbb{P}(\tau > t | \mathcal{F}_t), \quad \forall t \in \mathbb{R}_+.$$

Note that we use the notation $F_t, G_t$ rather than $F(t), G(t)$ to avoid confusion.

**Definition 3.4.2** (Hazard process.) We assume that $F_t < 1$ for all $t \in \mathbb{R}_+$. The $\mathbb{F}$-hazard process of $\tau$ under $\mathbb{P}$, denoted by $\Gamma$, is defined through the formula

$$1 - F_t = e^{-\Gamma_t}$$

or, equivalently,

$$\Gamma_t = -\ln G_t = -\ln(1 - F_t), \quad t \in \mathbb{R}_+.$$

We can use the hazard process to re-express (3.12) and (3.13) as

$$E_{\mathbb{P}}\left(1_{\tau > t} Y | G_t\right) = \mathbb{P}(\tau > t | G_t) E_{\mathbb{P}}\left(1_{\tau > t} e^{\Gamma_t} Y | G_t\right),$$

(3.14)

and

$$E_{\mathbb{P}}\left(1_{\tau > s} Y | G_t\right) = 1_{\tau > t} E_{\mathbb{P}}\left(1_{\tau > s} e^{\Gamma_t} Y | G_t\right),$$

(3.15)

respectively.

We now proceed to develop a further selection of useful expressions.

**Corollary 3.4.1** Let $Y$ be a $\mathcal{G}$-measurable random variable and let $t \leq s$.

i) If Condition 2 holds, then

$$E_{\mathbb{P}}\left(1_{\tau > s} Y | G_t\right) = \mathbb{P}(\tau > t | G_t) E_{\mathbb{P}}\left(1_{\tau > s} e^{\Gamma_t} Y | G_t\right).$$

(3.16)

ii) If Condition 1 holds, then

$$E_{\mathbb{P}}\left(1_{\tau > s} Y | G_t\right) = 1_{\tau > t} E_{\mathbb{P}}\left(1_{\tau > s} e^{\Gamma_t} Y | G_t\right),$$

(3.17)

and if $Y$ is $\mathcal{F}_s$-measurable, then

$$E_{\mathbb{P}}\left(1_{\tau > s} Y | G_t\right) = 1_{\tau > t} E_{\mathbb{P}}\left(e^{\Gamma_t - \Gamma_s} Y | G_t\right).$$

(3.18)

**Proof** We can prove (3.16) by noting that $1_{\tau > s} = 1_{\tau > t} 1_{\tau > s}$, which gives

$$E_{\mathbb{P}}\left(1_{\tau > s} Y | G_t\right) = E_{\mathbb{P}}\left(1_{\tau > t} 1_{\tau > s} Y | G_t\right)$$

$$= \mathbb{P}(\tau > t | G_t) E_{\mathbb{P}}\left(1_{\tau > t} e^{-\Gamma_t} 1_{\tau > s} Y | G_t\right)$$

$$= \mathbb{P}(\tau > t | G_t) E_{\mathbb{P}}\left(1_{\tau > s} e^{-\Gamma_t} Y | G_t\right).$$

We can use the hazard process to re-express (3.12) and (3.13) as

$$E_{\mathbb{P}}\left(1_{\tau > t} Y | G_t\right) = \mathbb{P}(\tau > t | G_t) E_{\mathbb{P}}\left(1_{\tau > t} e^{\Gamma_t} Y | G_t\right),$$

(3.14)

and

$$E_{\mathbb{P}}\left(1_{\tau > s} Y | G_t\right) = 1_{\tau > t} E_{\mathbb{P}}\left(1_{\tau > s} e^{\Gamma_t} Y | G_t\right),$$

(3.15)
where the second line follows from the alternative expression (3.14) we obtained from Lemma 3.4.2.

Relation (3.17) follows directly from (3.16). We can use (3.17) and a standard property of conditional expectations to derive (3.18) as follows:

\[
\mathbb{E}_\mathbb{P}(1_{\tau>s} Y | \mathcal{G}_t) = 1_{\tau>t} \mathbb{E}_\mathbb{P}(1_{\tau>s} e^{\Gamma_t} Y | \mathcal{F}_t) = 1_{\tau>t} \mathbb{E}_\mathbb{P}(1_{\tau>s} | \mathcal{F}_s) e^{\Gamma_t} Y | \mathcal{F}_t) = 1_{\tau>t} \mathbb{E}_\mathbb{P}(\mathbb{P}(\tau > s | \mathcal{F}_s) e^{\Gamma_t} Y | \mathcal{F}_t) = 1_{\tau>t} \mathbb{E}_\mathbb{P}((1 - F_s) e^{\Gamma_t} Y | \mathcal{F}_t) = 1_{\tau>t} \mathbb{E}_\mathbb{P}(e^{\Gamma_t - \Gamma_s} Y | \mathcal{F}_t).
\]

\[\square\]

In most practical applications we make the assumption that we are dealing with an absolutely continuous (with respect to Lebesgue measure) \( \mathcal{F} \)-hazard process \( \Gamma \) in which case we can define the stochastic intensity of the default time as follows.

**Definition 3.4.3** (Stochastic intensity.) An absolutely continuous \( \mathcal{F} \)-hazard process \( \Gamma \) can be expressed in the form

\[
\Gamma_t = \int_0^t \gamma(u) du
\]

for some \( \mathcal{F} \)-progressively measurable, non-negative process \( \gamma \), referred to as the **stochastic intensity** of \( \tau \).

Expressing (3.18) in terms of stochastic intensities we have

\[
\mathbb{E}_\mathbb{P}(1_{\tau>s} Y | \mathcal{G}_t) = 1_{\tau>t} \mathbb{E}_\mathbb{P}(e^{-\int_s^t \gamma(u) du} Y | \mathcal{F}_t).
\]

(3.19)

### 3.4.2 Bond valuation in terms of stochastic intensities

We can now use the concepts presented in Section 3.4.1 to provide an expression for the no-arbitrage price \( p_d(t,T) \) of a zero-coupon defaultable bond that is suitable for practical applications.

**Proposition 3.4.1** If we assume that our default time \( \tau \) admits an absolutely continuous hazard process with stochastic intensity \( \gamma \), then the price of a zero-coupon defaultable bond is given by

\[
p_d(t,T) = 1_{\tau>t} \mathbb{E}_\mathbb{P}^* \left( \exp \left( - \int_t^T r(u) du - \int_t^T \gamma(u) du \right) \right) \bigg| \mathcal{F}_t).
\]

(3.20)

**Proof** Since \( \exp \left( - \int_t^T r(u) du \right) \) is an \( \mathcal{F}_T \)-measurable random variable, we can use expression (3.19) arising from Corollary 3.4.1, which yields (3.20).

\[\square\]
Chapter 4

Claim prices under the MPPP model

The simplest model for risk-free interest rates is to assume that they remain constant over the investment horizon in question. This was utilised in Merton’s early paper [21], but was soon seen to be incompatible with reality, where we see interest rates fluctuate over relatively short time periods. Early models employed to allow for random changes in the risk-free interest rate centered around the use of continuous diffusion processes, which was due to the ease with which they can be implemented. Two major examples of such models were those of Vasicek [23] and Cox, Ingersoll and Ross (CIR) [9]. Letting $W$ represent the standard Brownian motion process, Vasicek’s model can be represented by the stochastic differential equation (SDE)

$$dr(t) = (\alpha - \beta r(t))dr + \eta dW(t)$$

for constants $\alpha, \beta$ and $\eta$. The CIR model introduces an extra level of complexity by allowing for interest rate dependence in the diffusion coefficient. The CIR model is specified by the SDE

$$dr(t) = (\alpha - \beta r(t))dt + \delta \sqrt{r(t)}dW(t)$$

for constants $\alpha, \beta$ and $\delta$. Their relative simplicity has seen these models used widely in applications with modifications being made in the process, the most natural of which has been the allowance for time dependence in the parameters $\alpha, \beta, \eta$ and $\delta$.

The main issue arising from the use of models such as these is that they assume that interest rates vary continuously in time. In reality, we often observe jumps in interest rates. A plot of the overnight interbank cash rate as quoted by the Reserve Bank of Australia for the period starting 4 January 2000 and finishing 31 December 2009 is shown in Figure 4.1. This rate serves as an approximation to the risk-free short rate applicable to borrowing/lending in Australia, and the plot indicates that not only are jumps evident in the short rate, but that a pure jump process may act as a suitable model for short rates. This observation prompted Borovkov et al. [6] to consider using a marked Poisson point process (MPPP) to model the short rate as a pure jump process.
CHAPTER 4. CLAIM PRICES UNDER THE MPPP MODEL

4.1 Short rates under the MPPP model

In this section we will present the set-up for the particular MPPP model developed in [6], including specification of our notation and a result involving an important Laplace functional.

We want to model our short term interest rate \( r \) by specifying a non-random initial value \( r(0) \) and allowing for its value to change by jumps only. Our approach will be to specify a distribution for the jump times and then specify how the jump sizes will behave. Naturally, our first step is to establish our notation.

We will let \( \{S_i\}_{i \in \mathbb{N}} \) denote the random jump times, and \( \{U_i\}_{i \in \mathbb{N}} \) will be the sizes of the random jumps occurring at these times. The number of jumps on the time
interval \([0, t]\) is given by the counting process

\[ N_t = \sum_{j=0}^{\infty} 1_{S_j \leq t}, \]

and we define our short rate by

\[ r(t) = r(0) + \eta(t), \quad t \geq 0, \]

where

\[ \eta(t) = \sum_{j=1}^{N_t} U_j \]

is a process that evolves only through jumps.

### 4.1.1 Marked Poisson point process model

For the purpose of this thesis, we will not require the notion of a marked point process in full generality. Instead we will use a specific case, referred to as the marked Poisson point process (MPPP).

Our goal is to consider a potential set-up to describe the distribution of the points \((S_j, U_j)\). To this end we postulate that the jump times \(S_j\) come from a Poisson point process \(N\) on \(\mathbb{R}_+\) with mean measure \(\Lambda\). We assume that the conditional distribution of the \(j\)th jump size \(U_j\) (a “mark” at \(S_j\)) given the process \(N\) can depend only on the time of the jump, i.e. for any Borel set \(B \subset \mathbb{R}\),

\[ P(U_j \in B | N) = K(S_j, B) \]

for some probability kernel \(K\), where conditioning on the process \(N\) is taken to mean conditioning on all the jump times \(S_i\) of the process. So the statement here is that if a jump occurs at time \(s\), then its size will follow the distribution \(K(s, \cdot)\) independently of what occurred prior to or after the jump.

We denote the process formed by the marked points \((S_j, U_j)\) by \(M\) and refer to it as the Marked Poisson Point Process (MPPP) with mean measure \(\Lambda\) and jump kernel \(K\).

In the notation from Section 4.1, the counting process \(N_t\) is expressible in terms of our Poisson point process \(N\) as \(N_t = N(\mathbf{0}, T]\). We therefore have

\[ N_t \sim \text{Poisson}(\lambda(0, t)) \]

where \(\lambda(0, t) = \Lambda((0, t])\). Under our MPPP set-up, the pure jump process \(\{\eta(t)\}_{t \geq 0}\) is thus a compound Poisson process.
4.1.2 The Laplace functional under the MPPP model

The Laplace functional serves as a useful tool in the calculation of expectations relating to the MPPP. We provide a definition of the functional in this setting as well as a key result we will make frequent use of.

**Definition 4.1.1** (Laplace functional for the MPPP model.) Given a Marked Poisson Point Process $M$ with mean measure $\Lambda$ and jump kernel $K$, we have the following definition for the Laplace functional $L_M(f)$:

$$L_M(f) := \mathbb{E}\left(e^{-M(f)}\right), \quad (4.1)$$

where $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a measurable function and $M(f)$ is the integral of the function $f$ with respect to the process $M$, i.e. $M(f) = \int f \, dM = \sum f(S_j, U_j)$, where the sum is taken over all points $(S_j, U_j)$ in the process.

We can derive an expression for the Laplace functional in the MPPP model in terms of the functional for the underlying Poisson process used to define it.

**Lemma 4.1.1** Given a measurable function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that $f(\cdot, z)$ is bounded and an MPPP $M$ with mean measure $\Lambda$ and jump kernel $K$, we have

$$L_M(f) = L_N(H_f) = \exp\left\{ \int (e^{-H_f(v)} - 1) \, \Lambda(dv) \right\},$$

where

$$H_f(v) = -\ln \int e^{-f(v,z)} K(v,dz),$$

provided that $H_f$ is bounded.

**Proof** Noting that the $U_j$’s are conditionally independent given $N$, we have

$$\mathbb{E}\left(e^{-M(f)}\right) = \mathbb{E}\left(\mathbb{E}\left(e^{-M(f)}\right) \mid N\right) \quad \text{using part ii) of Theorem 2.1.3}$$

$$= \mathbb{E}\left(\mathbb{E}\left(e^{-\sum f(S_j, U_j)}\right) \mid N\right)$$

$$= \mathbb{E}\left(\prod \mathbb{E}\left(e^{-f(S_j, U_j)}\right) \mid N\right)$$

$$= \mathbb{E}\left(\prod \int e^{-f(S_j,z)} K(S_j,dz)\right) \quad \text{using Lemma 2.2.1}$$

$$= \mathbb{E}\left(\prod e^{\ln \int e^{-f(S_j,z)} K(S_j,dz)}\right)$$

$$= \mathbb{E}\left(e^{-\sum \ln \int e^{-f(S_j,z)} K(S_j,dz)}\right)$$

$$= \mathbb{E}\left(e^{-\Sigma H_f(S_j)}\right)$$

$$= \mathbb{E}\left(e^{-N(H_f)}\right) = L_N(H_f) \quad \text{since } H_f \text{ is bounded.} \square$$
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4.2 Default

In order to include default in our model we need to specify the behaviour of the default time $\tau$. As outlined in Section 3.4.1, the conventional way to do this is through stochastic intensities. This is the approach we will take here.

We let $\gamma$ be the intensity for the default time $\tau$. We assume that it depends on time only through the interest rate. Furthermore, we assume that $\gamma(t)$ is a function of the interest rate process to time $t$ inclusive, i.e.

$$\gamma(t) = \gamma(r(t)),$$

where $r(t) = \{r(u): u \leq t\}$. Thus, given the pure jump nature of the interest rate process, $\gamma$ will also follow a jump process with jumps occurring at the same times as interest rate changes. We make the dependence between $\gamma$ and $r$ explicit by imposing the following constraint on the jumps of $\gamma$:

$$\gamma(S_j) - \gamma(S_{j-1}) = g(S_j, r(S_j) - r(S_{j-1})) = g(S_j, U_j) \quad (4.2)$$

for some measurable function $g$.

The constraint given by (4.2) is chosen to facilitate calculations involving expectations of the hazard function associated with $\gamma$.

4.3 Pricing bonds under the MPPP model

Now we are in a position to consider pricing formulae under the MPPP model. We assume that we are in a no-arbitrage setting, i.e. there exists an EMM $\mathbb{P}^*$. We assume that under $\mathbb{P}^*$ the risk-free interest rate follows the dynamics established in Section 4.1 and that the default time is governed by the stochastic intensity formulated in Section 4.2. We again let $\mathcal{H}_t = \sigma(\{\tau \leq u: u \leq t\})$, and we let $\mathcal{F}_t = \sigma(\{r(u): u \leq t\})$.

Borovkov et al. [6] showed that, under this model, the no-arbitrage price $p(t,T)$ at time $t$ for a zero-coupon default free bond with maturity $T$ is given by

$$p(t,T) = \mathbb{E}_{\mathbb{P}^*}\left(\exp\left\{-\int_t^T r(u)du\right\}\right|\mathcal{F}_t)$$

$$= \exp\left\{-(T-t)r(t) + \hat{\nu}(t,T) - \lambda(t,T)\right\}, \quad (4.3)$$

where

$$\hat{\nu}(t,T) := \int_t^T \left(\int_{v \in \mathbb{R}} e^{-\left(T-v\right)z} K(v, dz)\right) \Lambda(dv).$$

We can use our result from Section 3.4.2 to arrive at an analogous result for bonds subject to default risk.
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Theorem 4.3.1 Under our MPPP model, the time $t$ no-arbitrage price $p_d(t, T)$ of a zero-coupon defaultable bond maturing at $T$ is given by

$$p_d(t, T) = 1_{t>T} \exp \left\{ - (T-t) \left( r(t) + \gamma(t) \right) + \nu(t, T) - \lambda(t, T) \right\}, \quad (4.4)$$

where

$$\nu(t, T) = \int_t^T \left( \int_{v \in \mathbb{R}} e^{-(T-v)(z+g(v,z))} \mathcal{K}(v, dz) \right) \Lambda(dv). \quad (4.5)$$

Proof The step nature of the processes $r$ and $\gamma$ allows us to express their integrals in the particular fashion shown below.

$$\int_t^T r(u) du = (T-t)r(t) + \sum_{j=N_t+1}^{N_T} U_j(T-S_j), \quad (4.6)$$

$$\int_t^T \gamma(u) du = (T-t)\gamma(t) + \sum_{j=N_t+1}^{N_T} g(S_j, U_j)(T-S_j). \quad (4.7)$$

We can apply expression (3.20) from Section 3.4.2 directly and use (4.6) and (4.7) to obtain

$$p_d(t, T) = 1_{t>T} \mathbb{E}_{\mathbb{P}^*} \left( e^{-\int_t^T r(u) du - \int_t^T \gamma(u) du} \left| \mathcal{F}_t \right. \right)$$

$$= 1_{t>T} \mathbb{E}_{\mathbb{P}^*} \left( e^{-(T-t)r(t) - \sum_{j=N_t+1}^{N_T} U_j(T-S_j) - (T-t)\gamma(t) - \sum_{j=N_t+1}^{N_T} g(S_j, U_j)(T-S_j)} \left| \mathcal{F}_t \right. \right)$$

$$= 1_{t>T} e^{-(T-t)(r(t)+\gamma(T))} \mathbb{E}_{\mathbb{P}^*} \left( e^{-\sum_{j=N_t+1}^{N_T} [(U_j+g(S_j, U_j))(T-S_j)]} \left| \mathcal{F}_t \right. \right)$$

$$= 1_{t>T} e^{-(T-t)(r(t)+\gamma(T))} \mathbb{E}_{\mathbb{P}^*} \left( e^{-M(f)} \right), \quad (4.8)$$

where $f$ is given by

$$f(v, z) = (T-v)(z+g(v, z))1_{t<v\leq T}.$$ 

So we are required to evaluate the Laplace functional at the function $f$. From Lemma 4.1.1 we have

$$\mathbb{E}_{\mathbb{P}^*} \left( e^{-M(f)} \right) = L_M(f) = \exp \left\{ \int_{v \in \mathbb{R}_+} \left( e^{-H_f(v)} - 1 \right) \Lambda(dv), \right\}$$

where

$$H_f(v) = - \ln \int_{z \in \mathbb{R}} e^{-f(v, z)} \mathcal{K}(v, dz)$$

$$= - \ln \int_{z \in \mathbb{R}} e^{-(T-v)(z+g(v,z))1_{t<v\leq T}} \mathcal{K}(v, dz)$$

$$= -1_{t<v\leq T} \ln \int_{z \in \mathbb{R}} e^{-(T-v)(z+g(v,z))} \mathcal{K}(v, dz).$$
Therefore
\[
L_M(f) = \exp \left\{ \int_{v \in \mathbb{R}_+} \left( e^{1_{t < v \leq T} \ln \int_{z \in \mathbb{R}} e^{-(T-v)(z+g(v,z))} \mathcal{K}(v,dz)} - 1 \right) \Lambda(dv) \right\}
\]
\[
= \exp \left\{ \int_{t}^{T} \left( e^{\ln \int_{z \in \mathbb{R}} e^{-(T-v)(z+g(v,z))} \mathcal{K}(v,dz)} - 1 \right) \Lambda(dv) \right\}
\]
\[
= \exp \left\{ \int_{t}^{T} \left( \int_{z \in \mathbb{R}} e^{-(T-v)(z+g(v,z))} \mathcal{K}(v,dz) \right) \Lambda(dv) \right\}
\]
\[
= \exp \left\{ \nu(t,T) - \lambda(t,T) \right\}.
\]
(4.9)

Now substituting the expression given by (4.9) into (4.8) gives
\[
p_d(t,T) = 1_{t > T} e^{-(T-t)(r(t) + \gamma(T)) + \nu(t,T) - \lambda(t,T)},
\]
which is (4.4), as required. \qed

### 4.4 Pricing bond derivatives

Derivatives are financial contracts whose value is wholly determined by the value of some underlying securities. In this section we will show that, under the MPPP model, the no-arbitrage prices of standard call and put options on defaultable bonds can be expressed in terms of compound Poisson probabilities.

#### 4.4.1 Call options on defaultable bonds

**Definition 4.4.1 (Call option.)** A call option is a contract that gives the holder the right, but not the obligation, to purchase an asset at a later date at a price agreed to now.

We will consider European call options on zero-coupon defaultable bonds, which means that we enter into a contract that gives us the right to pay a price \( K \) (the strike price) for the underlying bond at a fixed future time \( s \). We write the price of this option at time \( t \) as \( \text{Call}(t,s,K,T) \), where \( T \) is the maturity date of the underlying bond, \( K \) is the strike price, agreed to at the option’s inception, that will be paid to purchase the bond at time \( s \) and \( t \leq s \leq T \). Thus the contract will generate a payoff at time \( s \) given by
\[
(x^+(t,T) - K)^+,
\]
where \( x^+ = \max(x,0) \).

Recalling that we denote by
\[
\mathcal{P}_{\lambda,L}(B) = e^{-\lambda} \sum_{k=0}^{\infty} L^k(B)
\]
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the compound Poisson distribution with parameter \( \lambda \) and compounded law \( L \), we have the following result regarding the price of a European call option.

**Theorem 4.4.1** Assume our MPPP model set-up. Then the time \( t \) price of a European call option with strike price \( K \) and maturity \( s \) on an underlying zero-coupon defaultable bond with maturity \( T \) is given by

\[
\text{Call}(t, s, K, T) = p_d(t, T)\mathcal{P}_{\nu(t, s), L_2}(A) - Kp_d(t, s)\mathcal{P}_{\nu(t, s), L_1}(A),
\]

(4.10)

where \( A = \{(x, x') \in \mathbb{R}^2 : x + x' < y\} \), the prices \( p_d(\cdot, \cdot) \) are given by (4.4), \( \nu \) is defined as in (4.5) and the compounded measures \( L_1 \) and \( L_2 \) are given by

\[
L_1(dx, dx') = \frac{e^{-s(x+x')}}{\nu(t, s)} \int_s^t e^{v(x+g(v,x))}\mathcal{K}(v, dx)1_{x'=g(v,x)}\Lambda(dv),
\]

and

\[
L_2(dx, dx') = \frac{e^{-T(x+x')}}{\tilde{\nu}(t, s)} \int_s^t e^{v(x+g(v,x'))}\mathcal{K}(v, dx)1_{x'=g(v,x)}\Lambda(dv),
\]

\[
\tilde{\nu}(t, s) = \int_{t}^{s} \left( \int_{z \in \mathbb{R}} e^{-(T-t)(z+g(v,z))}\mathcal{K}(v, dz) \right) \Lambda(dv),
\]

respectively, while

\[
y = \frac{\nu(s, T) - \lambda(s, T) - \ln K}{T - s} - r(t) - \gamma(t).
\]

**Proof** The only cashflow generated by this call option is the payoff of \((p_d(s, T) - K)^+\) received by the option holder at the fixed maturity date \( s \). From Theorem 4.3.1 we can see that the bond price can be expressed in the form

\[
p_d(s, T) = 1_{\tau\geq s}Y,
\]

where

\[
Y = \exp\{-(T-s)(r(s) + \gamma(s)) + \nu(s, T) - \lambda(s, T)\}
\]

is an \( F_s \)-measurable random variable. Noting that \((1_{\tau\geq s}Y - K)^+ = 1_{\tau\geq s}(Y - K)^+\) and that \((Y - K)^+\) is also \( F_s \)-measurable, we can conclude that this derivative contract falls within the defaultable claim pricing framework we presented in Chapter 3. Therefore the no-arbitrage price of our call is given by

\[
\text{Call}(t, s, K, T) = \mathbb{E}_P^* \left( e^{-\int_{t}^{s} r(u)du}1_{\tau\geq s}(Y - K)^+ \bigg| \mathcal{G}_t \right).
\]

Recalling the results of Corollary 3.4.1, we can use expression (3.19) (the stochastic intensity alternative) to obtain

\[
\mathbb{E}_P^* \left( 1_{\tau\geq s}e^{-\int_{t}^{s} r(u)du}(Y - K)^+ \bigg| \mathcal{G}_t \right) = 1_{\tau\geq t}\mathbb{E}_P^* \left( e^{-\int_{t}^{s} r(u)du-\int_{t}^{s} \gamma(u)du}(Y - K)^+ \bigg| \mathcal{F}_t \right).
\]

(4.11)
We now define the following variables, which will be convenient in simplifying our calculations:

\[
Y_1 = Y_1(t, s) := \sum_{j=N_t+1}^{N_s} U_j(s - S_j),
\]

\[
Y_2 = Y_2(t, s) := \sum_{j=N_t+1}^{N_s} U_j,
\]

\[
Y_3 = Y_3(t, s) := \sum_{j=N_t+1}^{N_s} g(S_j, U_j)(s - S_j),
\]

\[
Y_4 = Y_4(t, s) := \sum_{j=N_t+1}^{N_s} g(S_j, U_j).
\]

It is important to note that, due to the independent increments of the Poisson process and the dependence of the \( U_j \)'s only on the corresponding jump times \( S_j \), all the \( Y_j \)'s are independent of \( \mathcal{F}_t \).

Recalling (4.6) and (4.7), we can formulate the following expressions in terms of our \( Y_j \)'s:

\[
\int_t^s r(u)du = (s - t)r(t) + Y_1,
\]

\[
r(s) - r(t) = \eta(s) - \eta(t) = Y_2,
\]

\[
\int_t^s \gamma(u)du = (s - t)\gamma(t) + Y_3,
\]

\[
\gamma(s) - \gamma(t) = Y_4.
\]

We can now return to (4.11) and proceed by substituting these expressions:

\[
Call(t, s, K, T) = 1_{r>t} \mathbb{E}_{\mathbb{P}^*} \left( e^{-\int_t^s r(u)du - \int_t^s \gamma(u)du} (Y - K)^+ \big| \mathcal{F}_t \right)
\]

\[
= 1_{r>t} \mathbb{E}_{\mathbb{P}^*} \left( e^{-(s-t)(r(t)+\gamma(t)) - Y_1 - Y_3 (Y - K)^+} \big| \mathcal{F}_t \right)
\]

\[
= 1_{r>t} e^{-(s-t)(r(t)+\gamma(t))} \mathbb{E}_{\mathbb{P}^*} \left( e^{-Y_1 - Y_3 (Y - K)^+} \big| \mathcal{F}_t \right). \tag{4.12}
\]

Now focus on the expectation in (4.12) and the expression \((Y - K)^+\) in particular, which is non-zero when

\[
e^{-(T-s)(r(s)+\gamma(s))+\nu(s,T)-\lambda(s,T)} > K
\]

\[
\iff r(s) + \gamma(s) < \frac{\nu(s,T) - \lambda(s,T) - \ln K}{T-s}
\]

\[
\iff r(s) - r(t) + \gamma(s) - \gamma(t) < \frac{\nu(s,T) - \lambda(s,T) - \ln K}{T-s} - r(t) - \gamma(t)
\]

\[
\iff Y_2 + Y_4 < \frac{\nu(s,T) - \lambda(s,T) - \ln K}{T-s} - r(t) - \gamma(t).
\]
So \((Y - K)^+ = (Y - K)1_{Y_2 + Y_4 < y}\) according to our definition of \(y\) in the theorem statement. We can now expand the expectation in (4.12) as follows:

\[
\begin{align*}
\mathbb{E}_{\mathbb{P}^*} (e^{-Y_1 - Y_3} (Y - K)^+ | \mathcal{F}_t) &= \mathbb{E}_{\mathbb{P}^*} (e^{-Y_1 - Y_3} (Y - K)1_{Y_2 + Y_4 < y} | \mathcal{F}_t) \\
&= \mathbb{E}_{\mathbb{P}^*} (e^{-Y_1 - Y_3} Y 1_{Y_2 + Y_4 < y} | \mathcal{F}_t) - K\mathbb{E}_{\mathbb{P}^*} (e^{-Y_1 - Y_3} 1_{Y_2 + Y_4 < y} | \mathcal{F}_t) \\
&= \mathbb{E}_{\mathbb{P}^*} (e^{-Y_1 - Y_3} Y 1_{Y_2 + Y_4 < y} | \mathcal{F}_t) - K\mathbb{E}_{\mathbb{P}^*} (e^{-Y_1 - Y_3} 1_{Y_2 + Y_4 < y}) ,
\end{align*}
\]

where the last line follows from the independence of \(\mathcal{F}_t\) and the \(Y_j\)'s.

Our next step is to substitute \(Y\) into the first expectation in (4.13) and simplify the result:

\[
\begin{align*}
\mathbb{E}_{\mathbb{P}^*} (e^{-Y_1 - Y_3} Y 1_{Y_2 + Y_4 < y} | \mathcal{F}_t) &= \mathbb{E}_{\mathbb{P}^*} (e^{-Y_1 - Y_3 - (T-s)(r(t) + \gamma(t)) + \nu(s,t) - \lambda(s,t)} Y 1_{Y_2 + Y_4 < y} | \mathcal{F}_t) \\
&= \mathbb{E}_{\mathbb{P}^*} (e^{-Y_1 - Y_3 - (T-s)(Y_2 + Y_4 + r(t) + \gamma(t)) + \nu(s,t) - \lambda(s,t)} Y 1_{Y_2 + Y_4 < y} | \mathcal{F}_t) \\
&= e^{-(T-s)(r(t) + \gamma(t)) + \nu(s,t) - \lambda(s,t)} \mathbb{E}_{\mathbb{P}^*} (e^{-Y_1 - Y_3 - (T-s)(Y_2 + Y_4)} Y 1_{Y_2 + Y_4 < y} | \mathcal{F}_t) \\
&= e^{-(T-s)(r(t) + \gamma(t)) + \nu(s,t) - \lambda(s,t)} \mathbb{E}_{\mathbb{P}^*} (e^{-Y_1 - Y_3 - (T-s)(Y_2 + Y_4)} 1_{Y_2 + Y_4 < y}) .
\end{align*}
\]

Substituting (4.14) into (4.13) gives

\[
\begin{align*}
\mathbb{E}_{\mathbb{P}^*} (e^{-Y_1 - Y_3} (Y - K)^+ | \mathcal{F}_t) &= e^{-(T-s)(r(t) + \gamma(t)) + \nu(s,t) - \lambda(s,t)} \times \\
&\quad \mathbb{E}_{\mathbb{P}^*} (e^{-Y_1 - Y_3 - (T-s)(Y_2 + Y_4)} 1_{Y_2 + Y_4 < y}) \\
&\quad - K\mathbb{E}_{\mathbb{P}^*} (e^{-Y_1 - Y_3} 1_{Y_2 + Y_4 < y}) ,
\end{align*}
\]

and, finally, substituting this into (4.12) gives

\[
\begin{align*}
\text{Call}(t, s, K, T) \\
&= 1_{r > e^{-(T-t)(r(t) + \gamma(t)) + \nu(s,t) - \lambda(s,t)}} \mathbb{E}_{\mathbb{P}^*} (e^{-Y_1 - Y_3 - (T-s)(Y_2 + Y_4)} 1_{Y_2 + Y_4 < y}) \\
&\quad - 1_{r > e^{-(s-t)(r(t) + \gamma(t))}} K\mathbb{E}_{\mathbb{P}^*} (e^{-Y_1 - Y_3} 1_{Y_2 + Y_4 < y}) .
\end{align*}
\]

To evaluate (4.15), we will find a general formula for expectations of the form

\[
\begin{align*}
\mathbb{E}_{\mathbb{P}^*} (e^{-w_1 Y_1 - w_2 Y_2 - w_3 Y_3 - w_4 Y_4} (Y_1, Y_2, Y_3, Y_4) \in B) ,
\end{align*}
\]

where \(B\) is a Borel subset of \(\mathbb{R}^4\) (i.e. \(B \in \mathcal{B}(\mathbb{R}^4)\)).

\textbf{Claim 1} The vector \(Y = Y(t, s) := (Y_1, Y_2, Y_3, Y_4)\) has the compound Poisson distribution given by

\[
\mathbb{P}^*(Y \in A) = \mathcal{P}_{\lambda, W}(A) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} W^*k(A), \quad A \in \mathcal{B}(\mathbb{R}^4),
\]

where \(\lambda = \lambda(t, s)\) and the compounded law \(W = W_{t, s}\) has distribution given by

\[
W(A) = \frac{1}{\lambda} \int_t^s \mathbb{P}^* [((s - v)X_v, X_v, (s - v)g(v, X_v), g(v, X_v)) \in A] \Lambda(dv)
\]

with \(X_v\) being a random variable following the distribution \(\mathcal{K}(v, \cdot)\).
Proof The Laplace transform \( \mu(w_1, w_2, w_3, w_4) = \mu_{t,s}(w_1, w_2, w_3, w_4) \) of \( W \), is given by

\[
\mu(w_1, w_2, w_3, w_4) = \int_{\mathbb{R}^4} e^{-w_1x_1-w_2x_2-w_3x_3-w_4x_4} W(dx_1, dx_2, dx_3, dx_4) = \frac{1}{\lambda} \int_t^s \int_{z \in \mathbb{R}} e^{z(w_1(s-v)+w_2)-g(v,z)(w_3(s-v)+w_4)} \mathcal{K}(v, dz) \Lambda(dv).
\]

To show that \( Y \) indeed follows the claimed compound Poisson distribution, we can compute its Laplace transform and show that it equals \( e^{\lambda(\mu-1)} \), the Laplace transform of the compound Poisson process. To perform this calculation, we can use the Laplace functional by noting that

\[
w_1Y_1 + w_2Y_2 + w_3Y_3 + w_4Y_4 = M(f)
\]

where

\[
f(v, z) = (z(w_1(s-v) + w_2) + g(v, z)(w_3(s-v) + w_4)) 1_{t<s}.
\]

For this \( f \), the function \( H_f \) from Lemma 4.1.1 is given by

\[
H_f(v) = -\ln \int_{z \in \mathbb{R}} e^{-f(v, z)} \mathcal{K}(v, dz)
\]

\[
= -\ln \int_{z \in \mathbb{R}} e^{-(z(w_1(s-v)+w_2)-g(v,z)(w_3(s-v)+w_4))1_{t<s}} \mathcal{K}(v, dz)
\]

\[
= -1_{t<s} \ln \int_{z \in \mathbb{R}} e^{-(z(w_1(s-v)+w_2)-g(v,z)(w_3(s-v)+w_4))} \mathcal{K}(v, dz),
\]

and so we can find the Laplace transform of \( Y \) accordingly:

\[
L_M(f) = \mathbb{E} \left( e^{-M(f)} \right) = L_N(H_f)
\]

\[
= \exp \left\{ \int_{v \in \mathbb{R}_+} \left( e^{-H_f(v)} - 1 \right) \Lambda(dv) \right\}
\]

\[
= \exp \left\{ \int_{v \in \mathbb{R}_+} 1_{t<s} \ln f(v) e^{-(z(w_1(s-v)+w_2)-g(v,z)(w_3(s-v)+w_4))} \mathcal{K}(v, dz) - 1 \right\} \Lambda(dv)
\]

\[
= \exp \left\{ \int_t^s \left( \int_{z \in \mathbb{R}} e^{-(z(w_1(s-v)+w_2)-g(v,z)(w_3(s-v)+w_4))} \mathcal{K}(v, dz) - 1 \right) \Lambda(dv) \right\}
\]

\[
= e^{\lambda(\mu_{t,s}-1)}.
\]

Thus we have confirmed the particular compound Poisson nature of \( Y \). \( \square \)

Claim 2 We now return to (4.16) and claim that

\[
\mathbb{E}_{P^s} \left( e^{-w_1Y_1-w_2Y_2-w_3Y_3-w_4Y_4}; (Y_1, Y_2, Y_3, Y_4) \in B \right) = e^{\lambda(\mu_{w_1,w_2,w_3,w_4}-1)} \times \mathcal{P}_{\lambda(\mu_{w_1,w_2,w_3,w_4}),W[w_1,w_2,w_3,w_4]}(B),
\]

(4.17)

where \( W[w_1, w_2, w_3, w_4] = W_{t,s}[w_1, w_2, w_3, w_4] \) is the conjugate distribution

\[
W[w_1, w_2, w_3, w_4](dx_1, dx_2, dx_3, dx_4) := e^{-w_1x_1-w_2x_2-w_3x_3-w_4x_4} \mu(w_1, w_2, w_3, w_4) W(dx_1, dx_2, dx_3, dx_4).
\]
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Proof Noting that
\[ W^{*k}[w_1, w_2, w_3, w_4](dx_1, dx_2, dx_3, dx_4) = \frac{e^{-w_1 x_1 - w_2 x_2 - w_3 x_3 - w_4 x_4}}{\mu^k(w_1, w_2, w_3, w_4)} W^k(dx_1, dx_2, dx_3, dx_4) \]
and using Claim 1, we have
\[
\mathbb{E}_{\pi^*} \left( e^{-w_1 Y_1 - w_2 Y_2 - w_3 Y_3 - w_4 Y_4}; (Y_1, Y_2, Y_3, Y_4) \in B \right) \\
= \int_B e^{-w_1 x_1 - w_2 x_2 - w_3 x_3 - w_4 x_4} \mathcal{P}_{\lambda,W}(dx_1, dx_2, dx_3, dx_4) \\
= \int_B e^{-w_1 x_1 - w_2 x_2 - w_3 x_3 - w_4 x_4} e^{-\lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} W^k(dx_1, dx_2, dx_3, dx_4)} \\
= \int_B e^{-w_1 x_1 - w_2 x_2 - w_3 x_3 - w_4 x_4} e^{-\lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \mu^k(w_1, w_2, w_3, w_4)} W^{*k}[w_1, w_2, w_3, w_4](dx_1, dx_2, dx_3, dx_4) \\
= \int_B e^{\lambda \mu(w_1, w_2, w_3, w_4) - 1} e^{-\lambda \mu(w_1, w_2, w_3, w_4)} \left( \lambda \mu(w_1, w_2, w_3, w_4) \right)^k \\
W^{*k}[w_1, w_2, w_3, w_4](dx_1, dx_2, dx_3, dx_4) \\
= \int_B e^{\lambda \mu(w_1, w_2, w_3, w_4) - 1} \mathcal{P}_{\Lambda \mu(w_1, w_2, w_3, w_4), W}[w_1, w_2, w_3, w_4](dx_1, dx_2, dx_3, dx_4) \\
= e^{\lambda \mu(w_1, w_2, w_3, w_4) - 1} \mathcal{P}_{\Lambda \mu(w_1, w_2, w_3, w_4), W}[w_1, w_2, w_3, w_4](B),
\]
giving expression (4.17). \qed

Now we can return to the question of the expectations that appear in (4.15).

Claim 3 We have

\[
\mathbb{E}_{\pi^*} \left( e^{-w_1 Y_1 - w_2 Y_2 - w_3 Y_3 - w_4 Y_4} 1_{Y_2+Y_4<y} \right) = e^{\lambda \mu(w_1, w_2, w_3, w_4) - 1} \mathcal{P}_{\Lambda \mu(w_1, w_2, w_3, w_4), L}(A),
\]
where
\[
L(dx_2, dx_4) = \frac{e^{-x_2(w_2+w_3) - x_4(w_4+w_3)}}{\mu(w_1, w_2, w_3, w_4)} \int_t^s e^{v(x_2+w_3 g(v, x_2))} \mathcal{K}(v, dx_2) 1_{x_4=g(v,x_2)} \Lambda(dv)
\]
and
\[
A = \{(x_2, x_4) \in \mathbb{R}^2 : x_2 + x_4 < y \}.
\]

Proof Noting that we can find \( \mathbb{E}_{\pi^*} \left( e^{-w_1 Y_1 - w_2 Y_2 - w_3 Y_3 - w_4 Y_4} 1_{Y_2+Y_4<y} \right) \) by taking the set \( B \) in Claim 2 to be:
\[
B = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_2 + x_4 < y \}.
\]
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We have

\[
\mathbb{E}_s^* \left( e^{-w_1 Y_1 - w_2 Y_2 - w_3 Y_3 - w_4 Y_4} 1_{Y_2 + Y_4 < y} \right)
\]

\[
= \int_{(x_2,x_4) \in A} \int_{x_1 \in \mathbb{R}} \int_{x_3 \in \mathbb{R}} \mathcal{P}_{\lambda(w_1, w_2, w_3, w_4), W(w_1, w_2, w_3, w_4)}(dx_1, dx_2, dx_3, dx_4)
\]

\[
= \int_{(x_2,x_4) \in A} \int_{x_1 \in \mathbb{R}} \int_{x_3 \in \mathbb{R}} e^{-\lambda(w_1, w_2, w_3, w_4)} \sum_{k=0}^{\infty} W^*[w_1, w_2, w_3, w_4](dx_1, dx_2, dx_3, dx_4)
\]

\[
= \int_{(x_2,x_4) \in A} \mathcal{P}_{\lambda(w_1, w_2, w_3, w_4), L}(dx_2, dx_4)
\]

where

\[
L(dx_2, dx_4)
\]

\[
= \int_{(x_1,x_3) \in \mathbb{R}^2} W[w_1, w_2, w_3, w_4](dx_1, dx_2, dx_3, dx_4)
\]

\[
= \int_{(x_1,x_3) \in \mathbb{R}^2} e^{-w_1 x_1 - w_2 x_2 - w_3 x_3 - w_4 x_4} W(dx_1, dx_2, dx_3, dx_4)
\]

\[
= \frac{e^{-w_2 x_2 - w_4 x_4}}{\mu(w_1, w_2, w_3, w_4)} \int_{(x_1,x_3) \in \mathbb{R}^2} e^{-w_1 x_1 - w_3 x_3} W(dx_1, dx_2, dx_3, dx_4)
\]

\[
= \frac{\lambda \mu(w_1, w_2, w_3, w_4)}{\lambda \mu(w_1, w_2, w_3, w_4)} \int_{(x_1,x_3) \in \mathbb{R}^2} e^{-w_1 x_1 - w_3 x_3} \times
\]

\[
\int_t^s \mathbb{P}^* \left( \{(s - v) X_v, X_v, (s - v) g(v, X_v), g(v, X_v) \in (dx_1, dx_2, dx_3, dx_4) \} \right) \Lambda(dv)
\]

\[
= \frac{e^{-w_2 x_2 - w_4 x_4}}{\lambda \mu(w_1, w_2, w_3, w_4)} \int_{x_3 \in \mathbb{R}} \int_t^s e^{-w_1 (s-v) x_2 - w_3 x_3} \times
\]

\[
\mathbb{P}^* \left( \{(s - v) g(v, X_v), g(v, X_v) \in (dx_2, dx_3, dx_4) \} \right) \Lambda(dv)
\]

and the distribution used for \( W \) followed from Claim 1.

Now we can use Claim 3 to find the expectations from (4.15) by taking appropriate values for the \( w_j \)'s.

\[\square\]
By taking \( w_1 = 1, w_2 = 0, w_3 = 1, w_4 = 0 \) in Claim 3, we find an expression for the second expectation in (4.15):

\[
E_{P^*} \left( e^{-Y_1 - Y_3} 1_{Y_2 + Y_4 < y} \right) = e^{\nu(t,s) - \lambda(t,s)} P_{\nu(t,s), L_1}(A),
\]

where

\[
L_1(dx_2, dx_4) = \frac{e^{-s(x_2 + x_4)}}{\nu(t,s)} \int_t^s e^{r(x_2 + g(v,x_2))} K(v, dx_2) 1_{x_4 = g(v,x_2)} \Lambda(dv)
\]

and

\[
A = \{(x_2, x_4) \in \mathbb{R}^2 : x_2 + x_4 < y\}.
\]

Which follows from the observation that

\[
\lambda \mu(1, 0, 1, 0) = \nu(t,s).
\]

By taking \( w_1 = 1, w_2 = T - s, w_3 = 1, w_4 = T - s \) in Claim 3, we find an expression for the first expectation in (4.15):

\[
E_{P^*} \left( e^{-Y_1 - Y_3 - (T-s)(Y_2 + Y_4)} 1_{Y_2 + Y_4 < y} \right) = e^{\tilde{\nu}(t,s) - \lambda(t,s)} P_{\tilde{\nu}(t,s), L_2}(A),
\]

where the set \( A \) is the same as above and

\[
\tilde{\nu}(t,s) := \lambda \mu(1, T-s, 1, T-s) = \int_t^s \left( \int_{z \in \mathbb{R}} e^{-(T-v)(z + g(v,z))} K(v, dz) \right) \Lambda(dv).
\]

Before proceeding, note the following identity which will be useful in simplifying our call expression:

\[
\nu(s,T) + \tilde{\nu}(t,s) = \nu(t,T).
\]

This relation follows directly from the definitions of \( \nu \) and \( \tilde{\nu} \):

\[
\nu(s,T) + \tilde{\nu}(t,s) = \int_s^T \left( \int_{z \in \mathbb{R}} e^{-(T-v)(z + g(v,z))} K(v, dz) \right) \Lambda(dv) + \\
\int_t^s \left( \int_{z \in \mathbb{R}} e^{-(T-v)(z + g(v,z))} K(v, dz) \right) \Lambda(dv)
\]

\[
= \int_t^T \left( \int_{z \in \mathbb{R}} e^{-(T-v)(z + g(v,z))} K(v, dz) \right) \Lambda(dv) \\
= \nu(t,T).
\]

Now we are finally in a position to return to (4.15) and verify the call price given by (4.10). Substituting (4.18) and (4.19) into (4.15) and making use of the identity
given by (4.20), we have

\[
\text{Call}(t, s, K, T) = 1_{\tau > s} e^{-(T-t)(r(t)+\gamma(t)) + \nu(t,s) - \lambda(t,s)} \mathbb{E}_{\mathbb{P}^*} \left( e^{-Y_1 - Y_2 - (T-s) (Y_2 + Y_4)} 1_{Y_2 + Y_4 < y} \right) \\
- 1_{\tau > t} e^{-(s-t)(r(t)+\gamma(t))} K \mathbb{E}_{\mathbb{P}^*} \left( e^{-Y_1 - Y_2} 1_{Y_2 + Y_4 < y} \right) \\
= 1_{\tau > t} e^{-(T-t)(r(t)+\gamma(t)) + \nu(t,s) - \lambda(t,s)} K \mathbb{P}_{\mathbb{P}_{\nu(t,s), L_1}(A)} \\
- 1_{\tau > t} e^{-(s-t)(r(t)+\gamma(t)) + \nu(t,s) - \lambda(t,s)} K \mathbb{P}_{\mathbb{P}_{\nu(t,s), L_2}(A)} \\
= 1_{\tau > t} e^{-(T-t)(r(t)+\gamma(t)) + \nu(t,s) - \lambda(t,s)} K \mathbb{P}_{\mathbb{P}_{\nu(t,s), L_1}(A)} \\
- 1_{\tau > t} e^{-(s-t)(r(t)+\gamma(t)) + \nu(t,s) - \lambda(t,s)} K \mathbb{P}_{\mathbb{P}_{\nu(t,s), L_2}(A)} \\
= p_d(t, T) \mathbb{P}_{\mathbb{P}_{\nu(t,s), L_2}(A)} - K p_d(t, s) \mathbb{P}_{\mathbb{P}_{\nu(t,s), L_1}(A)} \\
\]

where the last line follows from comparison with (4.4) from Theorem 4.3.1.

\[
\square
\]

4.4.2 Put options on defaultable bonds

**Definition 4.4.2** (Put option.) A put option is a contract that gives the holder the right, but not the obligation, to sell an asset at a later date at a price agreed to now.

We will consider European put options on zero-coupon defaultable bonds, which means that we enter into a contract that gives us the right to sell the underlying bond at a price \( K \) (the strike price) at a fixed future time \( s \). We write the price of this option at time \( t \) as \( \text{Put}(t, s, K, T) \), where \( T \) is the maturity date of the underlying bond, \( K \) is the strike price agreed to at the option’s inception, that will be received for the underlying bond at time \( s \).

In particular, we will find prices for two put options which differ in the coverage they offer in the case of default. We have postulated that the underlying bond becomes worthless in the event of default and can’t recover value. Given the nature of financial markets, it is natural to question whether it is reasonable to expect to be able to sell a worthless asset at a price \( K > 0 \). For this reason we consider two types of options which we refer to as type A and B put options.

**Definition 4.4.3** (Type A put option.) A type A put option refers to the case where our option provides no default coverage. It generates a payoff at its maturity time \( s \) of

\[
(K - p_d(s, T))^+ 1_{\tau > s},
\]

which represents the fact that we are unable to sell the underlying bond if it defaults in the period \([t, s]\) for which our option is valid for.

We denote the price of such an option by \( \text{Put}_A(t, s, K, T) \).

**Definition 4.4.4** (Type B put option.) A type B put option generates a payoff at its maturity time \( s \) given by

\[
(K - p_d(s, T))^+,
\]

which reflects an assumption that despite the underlying asset becoming worthless, if default occurs in the period \([t, s]\), the option holder is still able to receive the amount \( K \) at the option’s maturity.
Using the techniques we have developed for pricing defaultable claims, we can derive expressions for the no-arbitrage prices of both of these put options.

**Theorem 4.4.2** Assume our MPPP model set-up. Then the time \( t \) price of a type \( A \) European put option with strike price \( K \) and maturity \( s \) on an underlying zero-coupon defaultable bond with maturity \( T \) is given by

\[
\text{Put}_A(t, s, K, T) = K p_d(t, s) \mathcal{P}_{\nu(t, s), L_1}(A') - p_d(t, T) \mathcal{P}_{\nu(t, s), L_2}(A'),
\]

(4.21)

where \( A' = \{(x, x') \in \mathbb{R}^2 : x + x' \geq y\} \), the prices \( p_d(\cdot, \cdot) \) are given by (4.4) and \( \nu \) is defined as in (4.5), whereas \( \tilde{\nu}, y \) and the compounded measures \( L_1 \) and \( L_2 \) are defined as in Theorem 4.4.1.

**Proof** We proceed in the same way as the proof for Theorem 4.4.1. Expressing the bond price as

\[
p_d(s, T) = 1_{\tau > s} Y,
\]

where

\[
Y = e^{-(T-s)(r(s)+\gamma(s))+\nu(s, T)-\lambda(s, T)},
\]

our payoff for this put option becomes

\[
(K - Y)^+ 1_{\tau > s}.
\]

Noting that \( K - Y \geq 0 \iff Y_2 + Y_1 \geq y \), we can proceed through a virtually identical sequence of arguments as those presented in the proof of Theorem 4.4.1 to confirm the expression given by (4.21).

**Theorem 4.4.3** Assume our MPPP model set-up. Then the time \( t \) price of a type \( B \) European put option with strike price \( K \) and maturity \( s \) on an underlying zero-coupon defaultable bond with maturity \( T \) is given by

\[
\text{Put}_B(t, s, K, T) = \text{Put}_A(t, s, K, T) + K(1_{\tau > t} p(t, s) - p_d(t, s)),
\]

(4.22)

where the prices \( p(\cdot, \cdot) \) and \( p_d(\cdot, \cdot) \) are given by (4.3) and (4.4), respectively, and \( \text{Put}_A(t, s, K, T) \) is the no-arbitrage price of a type \( A \) put option as specified by Theorem 4.4.2.

**Proof** As in the proof of Theorem 4.4.2, we can express our payoff at time \( s \) as

\[
(K - p_d(s, T))^+ = (K - Y)^+ 1_{\tau > s} + K 1_{\tau \leq s} 1_{\tau > t},
\]

and so our put price is given by

\[
\text{Put}_B(t, s, K, T) = \mathbb{E}_{\mathbb{P}^*} \left( e^{-\int_t^s r(u) du} (K - Y)^+ 1_{\tau > s} + e^{-\int_t^s r(u) du} K 1_{\tau > t} 1_{\tau \leq s} \big| G_t \right) \]

\[
= \text{Put}_A(t, s, K, T) + 1_{\tau > t} K \mathbb{E}_{\mathbb{P}^*} \left( e^{-\int_t^s r(u) du} 1_{\tau \leq s} \big| G_t \right) \]

\[
= \text{Put}_A(t, s, K, T) + 1_{\tau > t} K \mathbb{E}_{\mathbb{P}^*} \left( e^{-\int_t^s r(u) du} (1 - 1_{\tau > s}) \big| G_t \right) \]

\[
= \text{Put}_A(t, s, K, T) + K(1_{\tau > t} p(t, s) - p_d(t, s)).
\]

\( \square \)
4.4.3 Put-call parity

We have the following relation between the price of a call option and a price of a type B put option under our MPPP model.

**Theorem 4.4.4 (Put-call parity)** For the call and put prices given by Theorems 4.4.1 and 4.4.3 respectively we have the relation

\[
\text{Call}(t, s, K, T) + K \mathbf{1}_{\tau > t} p(t, s) = \text{Put}_B(t, s, K, T) + p_d(t, T).
\]

**Proof** Combining (4.21) and (4.22) and using the definitions from Theorems 4.4.1 and 4.4.3, we have

\[
\begin{align*}
\text{Put}_B(t, s, K, T) + p_d(t, T) &= K p_d(t, s) \mathcal{P}_{\nu(t, s), L_1}(A') - p_d(t, T) \mathcal{P}_{\nu(t, s), L_2}(A') + \\
&\quad K (\mathbf{1}_{\tau > t} p(t, s) - p_d(t, s)) + p_d(t, T) \\
&= K p_d(t, s) \left(1 - \mathcal{P}_{\nu(t, s), L_1}(A') \right) - p_d(t, T) \left(1 - \mathcal{P}_{\nu(t, s), L_2}(A) \right) + \\
&\quad K (\mathbf{1}_{\tau > t} p(t, s) - p_d(t, s)) + p_d(t, T) \\
&= p_d(t, T) \mathcal{P}_{\nu(t, s), L_2}(A) - K p_d(t, s) \mathcal{P}_{\nu(t, s), L_1}(A) + K (\mathbf{1}_{\tau > t} p(t, s) - p_d(t, s)) + \\
&\quad K p_d(t, s) - p_d(t, T) + p_d(t, T) \\
&= \text{Call}(t, s, K, T) + K \mathbf{1}_{\tau > t} p(t, s).
\end{align*}
\]

We could have also argued this relation based on a simple no-arbitrage argument. We consider two portfolios, one consisting of a type B put option maturing at time \( s \) on an underlying defaultable bond maturing at time \( T \) and a defaultable bond maturing at time \( T \), and the other portfolio being made up of a call option on the same underlying bond and \( K \) default free bonds maturing at time \( s \). This set-up is shown in Table 4.1.

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Value at ( t )</th>
<th>Value at ( s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \text{Put}_B(t, s, K, T) + p_d(t, T) )</td>
<td>( (K - p_d(s, T))^+ + p_d(s, T) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( = \max(K, p_d(s, t)) )</td>
</tr>
<tr>
<td>2</td>
<td>( \text{Call}(t, s, K, T) + K p(t, s) )</td>
<td>( (p_d(s, T) - K)^+ + K )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( = \max(p_d(s, t), K) )</td>
</tr>
</tbody>
</table>

As both portfolios produce the same payoff at maturity, we require their values at time \( t \) to be equal to eliminate arbitrage opportunities, and thus our parity relationship must hold.
Chapter 5
Simulations

In this chapter we attempt to confirm the expression for the price of a defaultable bond that was established in Section 4.3 and find the prices of call and put options using simulation techniques. To aid in computation, we will limit ourselves to a specific case of the MPPP model. We will compute the theoretical price of a defaultable bond under this model and compare it to the estimated result produced by a Monte Carlo type simulation method.

5.1 A simplified model

Within the MPPP framework we assume that the jumps in the interest rate can only occur at the fixed points \( \{t_j \}_{j=1}^m \) i.e. we postulate that the mean measure \( \Lambda \) is concentrated at these points: \( \Lambda(\{t_j\}) = \lambda_j \geq 0 \). We assume that the jumps have a fixed magnitude \( a \), with positive jumps occurring at time \( t_j \) with probability \( p_j \) and negative jumps with probability \( 1 - p_j \).

In recent times, monetary policy management by the Reserve Bank of Australia has seen interest rates change by multiples of 2.5\% (where we include no change as a multiple by 0) at roughly monthly intervals. Thus, this simplified model seems reasonable in the Australian setting.

Borovkov et al. [6] describe a method for calibrating (i.e. determining values of \( \lambda_j \) and \( p_j \)) this type of model from yield curve data. As we are interested in assessing the validity of our bond price expression and less concerned with realistic parameter estimates, we will specify these parameters ourselves rather than calibrate the model.

5.1.1 Parameter specification

In accordance with our observations regarding monetary policy in Australia, we postulate that interest rates can change by multiples of 2.5\% at the end of each month, i.e. \( a = 0.0025 \) and \( t_j = j/12 \) for \( j = 1,\ldots,12 \). To further simplify our model, we assume that the probability of increases/decreases in the interest rate are constant at each time point, i.e. \( p_j = p \) for all \( j \).
The final component of our model that requires specification is the function $g$ that gives the relation between increments in the default intensity and the time and size of jumps in the interest rate. We again apply a simplifying assumption by supposing that $g$ is independent of the time variable $v$, i.e. the sizes of jumps in the stochastic intensity are independent of the jump times. Because our interest rate process can only jump by $\pm 0.0025$ we only need to specify $g$ at $z = \pm 0.0025$.

If our belief is that default risk increases when interest rates rise and decreases when they fall, we would let $g(v, 0.0025) = c_1 > 0$ and $g(v, -0.0025) = c_2 < 0$. If we believe the opposite to be true, then we would set $g(v, 0.0025) = c_1 < 0$ and $g(v, -0.0025) = c_2 > 0$.

Empirical analyses conducted by Longstaff et al. [19] and later Collin-Dufresne et al. have found negative correlations between short term interest rates and credit rates in the American market. This indicates that the market generally believes that the probability of default increases (decreases) when interest rates fall (rise). If we are to assume that these correlations also hold in the Australian market, we should take $c_1 < 0$ and $c_2 > 0$.

We choose to set (somewhat arbitrarily) $c_1 = -0.005$ and $c_2 = 0.01$ to minimise the likelihood of the stochastic intensity falling below 0 (as the intensity only makes sense for positive values).

### 5.2 Bond price results

We will consider the price of a zero-coupon defaultable bond maturing in 1 year from the present date. Under our simplified model, the time 0 price of our bond as given by (4.4) is

$$p_d(0, 1) = e^{-r(0) - \gamma(0) + \nu(0, 1) - \sum_{j=1}^{12} \lambda_j}$$

where

$$\nu(0, 1) = \int_0^1 \left( \int_{z \in \mathbb{R}} e^{-(1-v)(z+g(v,z))} \mathcal{K}(v, dz) \right) \Lambda(dv)$$

$$= \int_0^1 (pe^{-(1-v)(0.0025+g(v,0.0025))} + (1-p)e^{-(1-v)(-0.0025+g(v,-0.0025))}) \Lambda(dv)$$

$$= \sum_{j=1}^{12} \lambda_j (pe^{-1-j/12}(0.0025+g(j/12,0.0025)) + (1-p)e^{1-j/12}(-0.0025+g(j/12,-0.0025)))$$

$$= \sum_{j=1}^{12} \lambda_j (pe^{(1-j/12)(0.0025+0.005)} + (1-p)e^{(1-j/12)(-0.0025+0.01)})$$

$$= \sum_{j=1}^{12} \lambda_j (pe^{0.0025(1-j/12)} + (1-p)e^{-0.0075(1-j/12)}).$$
We use the code included in Appendix A to generate 10000 simulations of the discounted payoff of a defaultable bond maturing in one year. As is standard practice with Monte Carlo techniques, we appeal to the Law of Large Numbers and take the average of these figures to give an estimate of the price of the bond, which we can compare with the theoretical price given by (5.1).

We perform these simulations for different combinations of values for \( r(0), \gamma(0) \) and \( p \), recording the difference between the theoretical bond price and the simulated price as well as the standard error of the simulated price. The results are summarised in the tables included in Appendix B.

In all cases the difference between the theoretical bond price and the simulated result was less than twice the standard error, supporting our derivation of the theoretical bond price.

5.3 Option price results

In this section we will examine the simulated behaviour of option prices under our simplified model. We consider the current prices of call and type B put options maturing in one year, where the underlying asset is a defaultable bond maturing in 2 year’s time, i.e. we are interested in \( \text{Call}(0, 1, K, 2) \) and \( \text{Put}_B(0, 1, K, 2) \).

The payoffs produced by the call and put options at maturity (time 1) are given by \((p_d(1, 2) - K)^+ \) and \((K - p_d(1, 2))^+ \) respectively. We use the code in Appendix C to generate 1000 simulations of the discounted payoffs of these options for a uniform mesh of strike prices between 0 and 1. The estimated option prices were plotted against the strike price for different values of \( p \) (the probability of an upward jump in the interest rate). The results are shown in Figures 5.1 and 5.2. It is clear from the figures that there is some fluctuation in the simulated option prices. We could minimise this by increasing the number of simulations, but this comes at a significant cost with regards to computation time. Tables displaying the simulated prices and their standard errors are included in Appendix D. From these tables we can see that the standard errors aren’t so large as to preclude us from making general observations regarding the behaviour of option prices.

Given the payoffs for call and put options, we would expect the call price to be positively related to the strike price and the put price to display a negative relation with \( K \). These expected relations are clearly observed in Figures 5.1 and 5.2. It is also clear that the likelihood of a positive jump in the interest rate seems to have an affect on the relationship between the option price and the strike price \( K \). In Figure 5.1 we can see that a higher value of \( p \) leads to a higher option price at low strikes, but a lower price at high strikes. The relationship for put options is more straightforward, Figure 5.2 shows that a higher value of \( p \) leads to a consistently lower put option price. As \( p \) governs the probability of jumps in both the interest rate and the stochastic intensity of default, we constructed the same plots (shown in Figures 5.3 and 5.4) for the case when the jumps in the stochastic intensity are much smaller (-0.0005 and 0.001 cf. -0.005 and 0.01 respectively). In this case, the relationship that was initially observed for the call price has vanished and the
relationship seen for put options has diminished in significance. Thus we can see that the interplay between the sizes of jumps in the interest rate and stochastic intensity is important in determining the type of behaviour that will be observed in option prices.

![Plot of the call price vs strike](image_url)

Figure 5.1: Call option prices: $g(v, 0.0025) = -0.005, g(v, -0.0025) = 0.01$
Figure 5.2: Put option prices: $g(v, 0.0025) = -0.005, g(v, -0.0025) = 0.01$
Figure 5.3: Call option prices: $g(v, 0.0025) = -0.0005, g(v, -0.0025) = 0.001$
Figure 5.4: Put option prices: $g(v, 0.0025) = -0.0005, g(v, -0.0025) = 0.001$
Chapter 6

Concluding remarks

The purpose of this thesis was to review the general framework for the pricing of defaultable securities and to extend an existing model for interest rates to include default risk. We focused on reduced form models and directed our attention towards the pricing framework for defaultable bonds. We formulated a specific reduced form model for default risk and investigated some closed-form pricing formulae that can be derived. We proposed a model where the risk-free interest rate is modeled as an MPPP process. We modeled the default time through the use of a stochastic intensity whose evolution was entangled with that of the interest rate process. We found that by imposing restrictions on the relationship between the default intensity and the risk-free interest rate, we could derive a simple expression for the no-arbitrage price of a defaultable bond. Under the same assumptions, we were also able to derive expressions for the prices of European call and put options on an underlying defaultable bond. Finally, we were able to present a specific case of our model and use simulations to verify our bond price and investigate the behaviour of call and put option prices.
Appendix A

Matlab bond price simulation code

function [] = simulation()
format long

%Initialise parameters
r0 = 0.04;
gamma0 = 0.02;
jump_up = 0.0025;
prob_up = 0.5;
jump_down = 0.0025;
prob_down = 1 - prob_up;
lambda = ones(1,12);

%Calculate the value of nu(0,1)
a = zeros(1,12);
for j = 1:12
    a(j) = lambda(j)*(prob_up*exp(-(1 - j/12)*(jump_up + ...
                     gfunc(j/12,jump_up))) + prob_down*exp(-(1 - j/12)*... 
                     (-jump_down + gfunc(j/12,-jump_down))));
end
nu = sum(a);

%Calculate the theoretical price of the bond

price = exp(-r0 - gamma0 + nu - sum(lambda))

%Simulate n observations of the discounted defaultable claim

n = 1e4;
p = zeros(1,n);
for i = 1:n

% simulate the jump times

jumps = poissrnd(lambda);

% this step is used to allow for the case where we have multiple jumps at a single time point

mx = max(jumps);
J = zeros(mx,12);
tmp = zeros(mx,1);
for k = 1:12
    tmp = [ones(jumps(k),1);zeros(mx-jumps(k),1)];
    J(:,k) = tmp;
end

% simulate the interest rate jump sizes

u = rand(mx,12);
updown = (u < prob_up) - (u > prob_up);
temp = updown.*J;
r_increments = jump_up*(temp > 0) - jump_down*(temp < 0);

% set the default intensity jump sizes

gamma_incr = zeros(1,mx*12);
r_incr = r_increments(:)';
for j = 1:(mx*12)
    gamma_incr(j) = gfunc((ceil(j/mx)/12),r_incr(j));
end

% simulate the default event

prob_default = 1 - exp(-gamma0 - sum(gamma_incr.*(1 - ...
    ceil((1:(mx*12))/mx)/12)));
default = (rand<prob_default);

% record the discounted payoff for each simulation

p(i) = exp(- r0 - sum(r_incr.*(1 - ...
    ceil((1:(mx*12))/mx)/12)))*(1 - default);
end

% Estimate the MC price and s.e. error

MCprice = sum(p)/n
MCerror = (sum((p - MCprice).^2)/(n*(n-1)))^0.5
diff = MCprice - price

function g = gfunc(u,z)
    if z == 0.0025
        g = -0.005;
    elseif z == 0
        g = 0;
    else
        g = 0.01;
    end
Appendix B

Tables of bond price results

Table B.1: \( r(0) = 0.09, \gamma(0) = 0.03 \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>Theoretical Price</th>
<th>Theoretical price - MCprice</th>
<th>MC s.e.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>0.889388</td>
<td>0.000669</td>
<td>0.001254</td>
</tr>
<tr>
<td>0.5</td>
<td>0.874857</td>
<td>0.000896</td>
<td>0.001869</td>
</tr>
<tr>
<td>0.2</td>
<td>0.860563</td>
<td>-0.002603</td>
<td>0.002244</td>
</tr>
</tbody>
</table>

Table B.2: \( r(0) = 0.04, \gamma(0) = 0.03 \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>Theoretical Price</th>
<th>Theoretical price - MCprice</th>
<th>MC s.e.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>0.934988</td>
<td>0.000832</td>
<td>0.001322</td>
</tr>
<tr>
<td>0.5</td>
<td>0.919712</td>
<td>0.001857</td>
<td>0.001984</td>
</tr>
<tr>
<td>0.2</td>
<td>0.904685</td>
<td>-0.001972</td>
<td>0.002371</td>
</tr>
</tbody>
</table>

Table B.3: \( r(0) = 0.09, \gamma(0) = 0.07 \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>Theoretical Price</th>
<th>Theoretical price - MCprice</th>
<th>MC s.e.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>0.854515</td>
<td>-0.001606</td>
<td>0.002076</td>
</tr>
<tr>
<td>0.5</td>
<td>0.840264</td>
<td>0.003744</td>
<td>0.002541</td>
</tr>
<tr>
<td>0.2</td>
<td>0.826819</td>
<td>0.001916</td>
<td>0.002821</td>
</tr>
</tbody>
</table>
APPENDIX B. TABLES OF BOND PRICE RESULTS

Table B.4: \( r(0) = 0.04, \gamma(0) = 0.07 \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>Theoretical Price</th>
<th>Theoretical price - MCprice</th>
<th>MC s.e.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>0.898327</td>
<td>-0.001898</td>
<td>0.002177</td>
</tr>
<tr>
<td>0.5</td>
<td>0.883649</td>
<td>-0.003892</td>
<td>0.002550</td>
</tr>
<tr>
<td>0.2</td>
<td>0.869211</td>
<td>0.001186</td>
<td>0.002956</td>
</tr>
</tbody>
</table>
Appendix C

Matlab option price simulation code

function [] = Callgraph()

K = 0:0.01:1;
callprice1 = zeros(1, length(K));
callprice2 = zeros(1, length(K));
callprice3 = zeros(1, length(K));

putprice1 = zeros(1, length(K));
putprice2 = zeros(1, length(K));
putprice3 = zeros(1, length(K));

for i = 1:101
    [callprice1(i),putprice1(i)] = optionsim((i-1)/100,0.8);
    [callprice2(i),putprice2(i)] = optionsim((i-1)/100,0.5);
    [callprice3(i),putprice3(i)] = optionsim((i-1)/100,0.2);
end

plot(K,callprice1,K,callprice2,'--',K,callprice3,':')
xlabel('Strike price (K)')
ylabel('Call price')
title('Plot of the call price vs strike')
legend('p = 0.8','p = 0.5', 'p = 0.2')

figure

plot(K,putprice1,K,putprice2,'--',K,putprice3,':')
xlabel('Strike price (K)')
ylabel('Put price')
title('Plot of the put price vs strike')
legend('p = 0.8','p = 0.5', 'p = 0.2')
end

%---------------------------------------------------
function [CallMCprice, PutMCprice, CallMCerr, PutMCerr] = optionsim(K, p)

% Initialise parameters
r0 = 0.04;
gamma0 = 0.02;
jump_up = 0.0025;
prob_up = p;
jump_down = 0.0025;
prob_down = 1 - prob_up;
lambda = ones(1,24);

% Simulate n observations of the discounted defaultable claim
n = 1e3;
pcall = zeros(1,n);
pput = zeros(1,n);
p_d = 0;
r1 = 0;
for i = 1:n
    % Simulate the jump times
    jumps = poissrnd(lambda);
    % This step is used to allow for the case where we have multiple jumps at a single time point
    mx = max(jumps);
    J = zeros(mx,24);
tmp = zeros(mx,1);
    for k = 1:12
        tmp = [ones(jumps(k),1);zeros(mx-jumps(k),1)];
        J(:,k) = tmp;
    end
    % Simulate the interest rate jump sizes
    u = rand(mx,24);
    updown = (u < prob_up) - (u > prob_up);
    temp = updown.*J;
    r_increments = jump_up*(temp > 0) - jump_down*(temp < 0);
    % Set the default intensity jump sizes
    gamma_incr = zeros(1,mx*24);
    r_incr = r_increments(:)';
    for j = 1:(mx*24)
gamma_incr(j) = gfunc((ceil(j/mx)/12), r_incr(j));
end

% simulate the default event

prob_default = 1 - exp(-2*gamma0 - sum(gamma_incr.*(2 - ...  
    ceil((1:(mx*24))/mx)/12)));
default = (rand<prob_default);

% record the discounted payoff for each simulation

r_incr2 = r_incr(1:(12*mx));
r_incr3 = r_incr((12*mx+1):24*mx);
r1 = r0 + sum(r_incr2);
p_d = exp(- r1 - sum(r_incr3.*(2 - ...  
    ceil(((mx*12+1):(mx*24))/mx)/12)))*(1 - default);
pcall(i) = exp(- r0 - sum(r_incr2.*(1 - ...  
    ceil((1:(mx*12))/mx)/12)))*max(p_d - K,0);
pput(i) = exp(- r0 - sum(r_incr2.*(1 - ...  
    ceil((1:(mx*12))/mx)/12)))*max(K - p_d,0);

% Estimate the MC price and s.e. error for the call option

CallMCprice = sum(pcall)/n;
CallMCerr = (sum((pcall - CallMCprice).^2)/(n*(n-1)))^0.5;

% Estimate the MC price and s.e. error for the put option

PutMCprice = sum(pput)/n;
PutMCerr = (sum((pput - PutMCprice).^2)/(n*(n-1)))^0.5;

end

%----------------------------------------------------
% The function g, which determines the increments in 
% the default intensity is defined here.

function g = gfunc(u,z)
    if z == 0.0025
        g = -0.005;
    elseif z == 0
        g = 0;
    else
        g = 0.01;
    end
end
Appendix D

Tables of option price results

Table D.1: $K = 0.3$ and $g(v, 0.0025) = -0.005, g(v, -0.0025) = 0.01$

<table>
<thead>
<tr>
<th>$p$</th>
<th>Call MC price</th>
<th>Call MC s.e.</th>
<th>Put MC price</th>
<th>Put MC s.e.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>0.604928</td>
<td>0.002304</td>
<td>0.004022</td>
<td>0.001068</td>
</tr>
<tr>
<td>0.5</td>
<td>0.580964</td>
<td>0.005611</td>
<td>0.024538</td>
<td>0.002547</td>
</tr>
<tr>
<td>0.2</td>
<td>0.576332</td>
<td>0.006805</td>
<td>0.035491</td>
<td>0.003012</td>
</tr>
</tbody>
</table>

Table D.2: $K = 0.6$ and $g(v, 0.0025) = -0.005, g(v, -0.0025) = 0.01$

<table>
<thead>
<tr>
<th>$p$</th>
<th>Call MC price</th>
<th>Call MC s.e.</th>
<th>Put MC price</th>
<th>Put MC s.e.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>0.321151</td>
<td>0.001443</td>
<td>0.010912</td>
<td>0.002481</td>
</tr>
<tr>
<td>0.5</td>
<td>0.318831</td>
<td>0.002948</td>
<td>0.045040</td>
<td>0.004899</td>
</tr>
<tr>
<td>0.2</td>
<td>0.318754</td>
<td>0.003875</td>
<td>0.074460</td>
<td>0.006149</td>
</tr>
</tbody>
</table>

Table D.3: $K = 0.9$ and $g(v, 0.0025) = -0.005, g(v, -0.0025) = 0.01$

<table>
<thead>
<tr>
<th>$p$</th>
<th>Call MC price</th>
<th>Call MC s.e.</th>
<th>Put MC price</th>
<th>Put MC s.e.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>0.040368</td>
<td>0.000288</td>
<td>0.012918</td>
<td>0.003312</td>
</tr>
<tr>
<td>0.5</td>
<td>0.052776</td>
<td>0.000578</td>
<td>0.077097</td>
<td>0.007804</td>
</tr>
<tr>
<td>0.2</td>
<td>0.065180</td>
<td>0.000857</td>
<td>0.118634</td>
<td>0.009461</td>
</tr>
</tbody>
</table>
### Table D.4: $K = 0.3$ and $g(v, 0.0025) = -0.0005, g(v, -0.0025) = 0.001$

<table>
<thead>
<tr>
<th>$p$</th>
<th>Call MC price</th>
<th>Call MC s.e.</th>
<th>Put MC price</th>
<th>Put MC s.e.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>0.586812</td>
<td>0.003996</td>
<td>0.012568</td>
<td>0.001853</td>
</tr>
<tr>
<td>0.5</td>
<td>0.607251</td>
<td>0.004135</td>
<td>0.012693</td>
<td>0.001872</td>
</tr>
<tr>
<td>0.2</td>
<td>0.623516</td>
<td>0.004586</td>
<td>0.014819</td>
<td>0.002023</td>
</tr>
</tbody>
</table>

### Table D.5: $K = 0.6$ and $g(v, 0.0025) = -0.0005, g(v, -0.0025) = 0.001$

<table>
<thead>
<tr>
<th>$p$</th>
<th>Call MC price</th>
<th>Call MC s.e.</th>
<th>Put MC price</th>
<th>Put MC s.e.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>0.312154</td>
<td>0.002187</td>
<td>0.026321</td>
<td>0.003792</td>
</tr>
<tr>
<td>0.5</td>
<td>0.330187</td>
<td>0.002311</td>
<td>0.026521</td>
<td>0.003821</td>
</tr>
<tr>
<td>0.2</td>
<td>0.347797</td>
<td>0.002543</td>
<td>0.029093</td>
<td>0.004012</td>
</tr>
</tbody>
</table>

### Table D.6: $K = 0.9$ and $g(v, 0.0025) = -0.0005, g(v, -0.0025) = 0.001$

<table>
<thead>
<tr>
<th>$p$</th>
<th>Call MC price</th>
<th>Call MC s.e.</th>
<th>Put MC price</th>
<th>Put MC s.e.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>0.040692</td>
<td>0.000337</td>
<td>0.027446</td>
<td>0.004776</td>
</tr>
<tr>
<td>0.5</td>
<td>0.055650</td>
<td>0.000462</td>
<td>0.040670</td>
<td>0.005794</td>
</tr>
<tr>
<td>0.2</td>
<td>0.071207</td>
<td>0.000629</td>
<td>0.053254</td>
<td>0.006611</td>
</tr>
</tbody>
</table>
Bibliography


